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On the Brachistochrone Problem

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Abstract. As a problem that dates back to the end of the seventieth century, the brachistochrone problem is one of the oldest problems in the calculus of variations and as such, has generated a myriad of publications. However, in most classical texts and in most papers, the favored way to solve the problem is to make two a priori assumptions, viz., that the brachistochrone lies in a vertical plane, and that it can be represented as the graph of a function in this plane; besides, with few exceptions, the existence of a solution is not rigorously established: instead it is sometimes even taken for granted that the solution is that of the associated Euler-Lagrange equations, even though these are well known to be only necessary conditions for the existence of a minimizer. The objective of this article is to show how all these shortcomings can be very simply, and rigorously, overcome, by means of arguments that do not need any a priori assumptions and that otherwise require only a modicum of basic notions from calculus, so as to rigorously establish the existence and uniqueness of the brachistochrone in full generality. One originality of our approach is that from the outset we seek the brachistochrone as a parameterized curve in the three-dimensional space, i.e., that can be represented by means of three parametric equations, instead of by means of a single graph in a vertical plane. Contrary to expectations, this increase of generality renders the ongoing analysis much simpler. Our objective is thus to show how the direct method of the calculus of variations based on the Euler-Lagrange equations can be used to solve the brachistochrone problem. Otherwise, there are other ways to solve this problem, for instance by means of convex optimization or optimal control; such methods are briefly described at the end of the paper.

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1 Introduction

Consider the following mechanical problem: A material point with mass m is subjected to uniform gravity and slides without friction along a curve joining a point $\boldsymbol{P}_0$ in the "horizontal plane" to a different point $\boldsymbol{P}_1$ situated below, or in, the horizontal plane.

Under the assumption that the velocity at $\boldsymbol{P}_0$ at the initial time vanishes, the brachistochrone problem consists in seeking whether there exists a smooth curve along which the time for the material point to go from $\boldsymbol{P}_0$ to $\boldsymbol{P}_1$ is the shortest (Fig. 1). If such a curve exists, it is called a brachistochrone ("brákhistos" means "shortest" and "khrónos" means "time" in ancient Greek).

This minimization problem was first proposed as an open one to his contemporaries by Johann Bernoulli in 1696 (cf. [4]). Answers were then quickly proposed by several outstanding mathematicians: Isaac Newton, Gottfried Wilhelm Leibniz, Johann Bernoulli, Jacob Bernoulli (brother of Johann), and Guillaume de l'Hôpital. An account of the history of the brachistochrone can be found in, e.g., Shafer [10], or Sussman & Willems [11].

This problem constitutes one of the oldest ones of the calculus of variations and has of course generated a myriad of publications. So, why write an additional one?

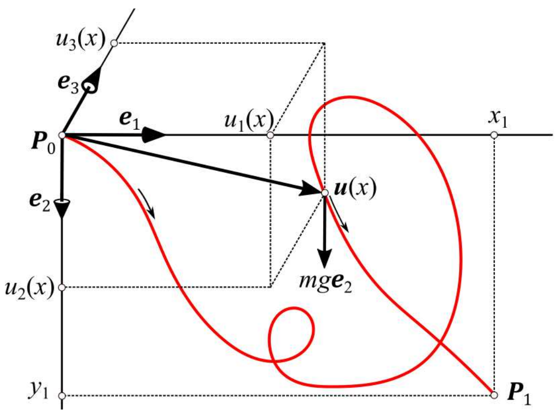

Figure 1: Given a point $\boldsymbol{P}_0$ chosen as the origin of the "horizontal plane" and a point $\boldsymbol{P}_1 = (x_1, y_1, 0)$ below $(y_1 > 0)$, or in $(y_1 = 0)$, the horizontal plane, the brachistochrone, if it exists, is a smooth curve along which the time for a material point with mass m sliding without friction along this curve under the influence of gravity, and with a zero velocity at $\boldsymbol{P}_0$, is the shortest for joining $\boldsymbol{P}_0$ to $\boldsymbol{P}_1$.