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On Short Wave-Long Wave Interactions in the Relativistic Context

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Abstract. In this paper we introduce models of short wave-long wave interactions in the relativistic setting. In this context the nonlinear Schrödinger equation is no longer adequate for describing short waves and is replaced by a nonlinear Dirac equation. Two specific examples are considered: the case where the long waves are governed by a scalar conservation law; and the case where the long waves are governed by the augmented Born-Infeld equations in electromagnetism.

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1 Introduction

In [3], Benney proposed a general theory describing interactions between short waves and long waves, in the classical non-relativistic context. More specifically, in Benney's model short waves are described by a non-linear Schrödinger equation. As for the long waves, in [3] two examples are given: a linear transport

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equation, and the Burgers equation, namely, with some simplifications, these examples are

$$\begin{aligned} &iu_t+u_{xx}=|u|^2u+\alpha vu,\\ &v_t+c_1v_x=\left(\alpha|u|^2\right)_x, \end{aligned}$$

and

$$\begin{aligned} &iu_t+u_{xx}=|u|^2u+\alpha vu,\\ &v_t+\left(\frac{v^2}{2}\right)_x=\left(\alpha|u|^2\right)_x, \end{aligned}$$

where $\alpha>0$ is a constant. We recall, among works dedicated to the study of this original model in [3], that the well-posedness for the linear case was addressed in [31], while the case of the Burgers equation with dispersion, that is, the KdV equation, was addressed in [2]. In [11], global existence for the Burgers flux with a cubic perturbation, av^2-bv^3, $b>0$, was obtained. We denote the coupling prescribed in [3] by $\binom{vu}{|u|^2}$. An important improvement in the model set forth in [3] was achieved in [12] where the coupling, in the case where long waves are described by scalar conservation laws, was prescribed as $\binom{g(v)u}{g'(v)|u|^2}$, where $\operatorname{supp} g'$ may be suitably chosen so as to guarantee the preservation of the physical domain. Moreover, the improvement proposed in [12] also enabled the study of interactions with long waves governed by systems of conservation laws such as elasticity, electromagnetism, symmetric systems, etc. It also opened the way for the study of interactions with compressible fluids in [13], followed by extensions to heat conductive fluids and magnetohydrodynamics equations (see, e.g., [15–18, 24]). An important feature in the latter references for interactions with fluids is that the nonlinear Schrödinger equation governing the short waves is based on the Lagrangian coordinates of the fluid. Also, the coupling in these works on interactions involving fluids has the form $\binom{g(v)h'(|u|^2)u}{g'(v)h(|u|^2)}$, with $\operatorname{supp} h'$ compact in $[0,\infty)$.

In the relativistic context, the short waves can no longer be described by a nonlinear Schrödinger equation since this type of equation yields infinite speed of propagation, which violates the relativity principle that no signal can propagate with speed higher than the speed of light. The natural substitute for the Schrödinger equation is the Dirac equation proposed by Dirac [14] in search of compatibility between relativity and quantum theories. On the other hand as a replacement for the nonlinear cubic Schrödiger equation there are different models