Decay of the Compressible Navier-Stokes Equations with Hyperbolic Heat Conduction

Zhigang Wu*, Wenyue Zhou and Yujie Li

Department of Mathematics, Donghua University, 201620, China.

Received 3 November 2022; Accepted 31 January 2023

Abstract. The global solution to the Cauchy problem of the compressible Navier-Stokes equations with hyperbolic heat conduction in dimension three is constructed when the initial data in $H^3$ norm is small. By using several elaborate energy functionals together with the interpolation trick, we simultaneously obtain the optimal $L^2$-decay estimate of the solution and its derivatives when the initial data is bounded in negative Sobolev (Besov) space or $L^1(\mathbb{R}^3)$. Specially speaking, the fluid density, the fluid velocity and the fluid temperature in $L^2$-norm have the same decay rate as the Navier-Stokes-Fourier equations, while the flux $q$ has faster $L^2$-decay rate as $(1+t)^{-2}$. Our proof is based on a family of scaled energy estimates with minimum derivative counts and interpolations among them without linear decay analysis for a $8 \times 8$ Green matrix of the system. To the best of our knowledge, it is the first result on the large time behavior of this system.

AMS subject classifications: 35B40, 35A09, 35Q35

Key words: Decay rate, Navier-Stokes equations, hyperbolic heat conduction, energy method.

1 Introduction

The compressible Navier-Stokes equations with hyperbolic heat conduction [13] takes the following form:

$$\partial_t \rho + \text{div}(\rho u) = 0, \quad (1.1a)$$

*Corresponding author. Email address: zgwu@dhue.edu.cn (Z. Wu)
\[
\begin{align*}
\partial_t (\rho u) + \text{div} (\rho u \otimes u) + \nabla P &= \text{div} S, \quad (1.1b) \\
\partial_t \left( \rho \left( e + \frac{1}{2} u^2 \right) \right) + \text{div} \left( \rho u \left( e + \frac{1}{2} u^2 \right) + uP \right) + \text{div} q &= \text{div} (uS), \quad (1.1c) \\
\tau \partial_t q + q + \kappa \nabla \theta &= 0, \quad (1.1d)
\end{align*}
\]

where the unknown functions \(\rho, u = (u_1, u_2, \cdots, u_n), P, S, e, q\) represent fluid density, velocity, pressure, stress tensor, specific internal energy per unit mass and flux, respectively. The Eq. (1.1d) represents Cattaneo law (Maxwell law, etc.), and \(\tau > 0\) is the constant relaxation time and \(\kappa > 0\) is the constant heat conductivity. Here we assume the fluid to be a Newtonian fluid, that is, \(S = \mu (\nabla u + (\nabla u)^T) + \mu' \text{div } u I\), where \(\mu\) and \(\mu'\) are the coefficient of viscosity and the second coefficient of viscosity, respectively, satisfying \(\mu > 0, \mu' + 2\mu/n \geq 0\).

In this paper, we will study the global existence and large time behavior of the smooth solutions for the system (1.1) with the following initial data:

\[
\rho(x,0) = \rho_0(x) > 0, \quad u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x) > 0, \quad q(x,0) = q_0(x). \quad (1.2)
\]

Here we consider the general equations of state and assume that the pressure \(P(\rho, \theta)\) and \(e = e(\rho, \theta)\) are smooth function of \((\rho, \theta)\) satisfying

\[
\rho^2 c_p(\rho, \theta) = P(\rho, \theta) - \theta P_\theta(\rho, \theta), \quad (1.3)
\]

where \(\theta\) is the absolute temperature. Obviously, our assumption includes the case of a polytropic gas \(P = R \rho \theta, e = c_v \theta\).

When \(\tau = 0\), the system (1.1) is the classical full compressible Navier-Stokes equations, in which the relation between the heat flux and the temperature represents Fourier law, \(q = -\kappa \nabla \theta\). Due to its importance for both physical and mathematical applications, the well-posed theory has been widely studied for the system combined with Fourier law, or the isentropic case, see [1, 7, 8, 10, 14–18] and references therein.

In the following, we mainly review some results on the decay rate of the closely related models. A lot of works have been done on the existence, stability and \(L^p\)-decay rates with \(p \geq 2\) for either isentropic or non-isentropic (heat-conductive) cases, cf. [5, 6, 21–23] in various settings by using (weighted) energy method together with spectrum analysis. Recently, Danchin and Xu [2] developed optimal decay rate in general critical spaces and any dimension \(n \geq 2\) under a mild additional decay assumption that is satisfied if the low frequencies of the initial data. On the other hand, Liu and Zeng [20] first studied the pointwise estimates of solution to general hyperbolic-parabolic systems in dimension one by using the method of Green function. Hoff and Zumbrun [11] investi-