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On Short Wave-Long Wave Interactions in the Relativistic Context: Application to the Relativistic Euler Equations

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Abstract. In this paper we introduce a model of relativistic short wave-long wave interaction where the short waves are described by the massless $(1+3)$-dimensional Thirring model of nonlinear Dirac equation and the long waves are described by the $(1+3)$-dimensional relativistic Euler equations. The interaction coupling terms are modeled by a potential proportional to the relativistic specific volume in the Dirac equation and an external force proportional to the square modulus of the Dirac wave function in the relativistic Euler equation. An important feature of the model is that the Dirac equations are based on the Lagrangian coordinates of the relativistic fluid flow. In particular, an important contribution of this paper is a clear formulation of the relativistic Lagrangian transformation. As far as the authors know the definition of the Lagrangian transformation given in this paper is new. Finally, we establish the short-time existence and uniqueness of a smooth solution of the Cauchy problem for the regularized model. This follows through the symmetrization of the relativistic Euler equation introduced by Makino and Ukai (1995) and requires a slight extension of a well known theorem of T. Kato (1975) on quasi-linear symmetric hyperbolic systems.

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1 Introduction

We consider the short wave-long wave interaction for a relativistic fluid described by the relativistic Euler equations in $\mathbb{R}^3$. The latter, when no external forces are acting, is given by (see, e.g. [18–21])

$$\begin{aligned}&\partial_t\left(\frac{\rho+\epsilon^2 p}{1-\epsilon^2|u|^2}-\epsilon^2 p\right)+\sum_{k=1}^{3}\partial_{x_k}\left(\frac{\rho+\epsilon^2 p}{1-\epsilon^2|u|^2}u_k\right)=0,\\&\partial_t\left(\frac{\rho+\epsilon^2 p}{1-\epsilon^2|u|^2}u_j\right)+\sum_{k=1}^{3}\partial_{x_k}\left(\frac{\rho+\epsilon^2 p}{1-\epsilon^2|u|^2}u_j u_k+p\delta_{jk}\right)=0,\quad j=1,2,3.\end{aligned}\tag{1.1}$$

Here, ρ is the mass density and $u=(u_1,u_2,u_3)$ is the velocity vector of the fluid, both functions of $(t,x)\in\mathbb{R}_+\times\mathbb{R}^3$. The parameter ϵ represents the inverse of the light speed [18], δ_{jk} is the Kronecker symbol, so that the $d\times d$ identity matrix is $I_d:=(\delta_{jk})_{j,k=1}^{d}$, and $p=p(\rho)$ is the pressure. The physical domain for the unknown (ρ,u) is

$$\rho\geq 0,\quad |u|^2:=\sum_{j=1}^{3}u_j^2<\epsilon^{-2},$$

while the pressure $p=p(\rho)$ satisfies $0\leq p'(\rho)<\epsilon^{-2}$.

In the relativistic context, the short waves are described by the Dirac equation proposed by Dirac [10] in search of compatibility between relativity and quantum theories. On the other hand as a replacement for the nonlinear cubic Schrödinger equation there are different models of the nonlinear cubic Dirac equation (see, e.g. [1,3–6,16,22]). Here, as in [8], we will be concerned with the Thirring model proposed by Thirring [22] whose mathematical study has been considered in several papers (see, e.g. [3,6,9,16]). More specifically, here we only consider the zero mass case.

In [8] two examples of models of short wave-long wave interactions in the relativistic context were addressed. Namely, the case of a one-dimensional scalar conservation law in the relativistic context such as the one introduced by LeFloch *et al.* [17] (see also [14]), and the interaction with long waves described by the augmented Born-Infeld (ABI) equations in electromagnetism, introduced by Brenier [2]. Both of these examples were in one spatial variable.

In this paper we are concerned first with establishing a model of relativistic short wave-long wave interaction where the short waves are described by a massless $(1+3)$-dimensional extension of the Thirring model of nonlinear Dirac equation, with an interaction term representing the potential of an external force. On