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# $L^2$ Stability and Weak-BV Uniqueness for Nonisentropic Euler Equations

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Dedicated to Professor Gui-Qiang Chen on the occasion of his 60th birthday

**Abstract.** We prove the $L^2$ stability for weak solutions of non-isentropic Euler equations in one space dimension whose initial data are perturbed from a small BV data under the $L^2$ distance. Using this result, we can show the uniqueness of small BV solutions among a large family of weak solutions.

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**Key words**: Compressible Euler system, uniqueness, stability, relative entropy, conservation law.

## 1 Introduction

The 1-d compressible Euler equations, widely used for compressible inviscid flow such as gas dynamics, can be written in the Eulerian coordinates, as

$$\begin{aligned}&\rho_t+(\rho w)_{x'}=0,\\&(\rho w)_t+(\rho w^2+p)_{x'}=0,\\&\left(\frac{1}{2}\rho w^2+\rho\mathcal{E}\right)_t+(\rho w^3+vp)_{x'}=0,\end{aligned}\tag{1.1}$$

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where three equations represent conservation of mass, momentum and energy, respectively. When we use a Lagrangian frame, co-moving with the fluid, given by $x=\int \rho dx'$, the equations become

$$\begin{aligned} &\tau_t - w_x = 0, \\ &w_t + p_x = 0, \\ &\left(\frac{1}{2}w^2 + \mathcal{E}\right)_t + (wp)_x = 0, \end{aligned} \tag{1.2}$$

whose solution is equivalent to (1.1) [36]. Here, $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ are time and space, $\tau=1/\rho$ is the specific volume, $p$ is pressure, $w$ is fluid velocity, and $\mathcal{E}$ is the specific internal energy. For convenience, let us use the Lagrangian coordinates.

The system is closed by specifying a constitutive law. For convenience, we consider a polytropic ideal $\gamma$-law gas, with equation of state

$$\mathcal{E} = c_v\theta = \frac{p\tau}{\gamma-1}, \quad p\tau = \bar{R}\theta, \tag{1.3}$$

so that

$$p = Ke^{\frac{S}{c_v}}\tau^{-\gamma}. \tag{1.4}$$

Here $S$ is the entropy, $\theta$ is the temperature, $\bar{R},K,c_v$ are positive constants, and $\gamma>1$ is the adiabatic gas constant. These state variables satisfy the Gibbs relation

$$\theta dS = d\mathcal{E} + pd\tau. \tag{1.5}$$

The Lagrangian sound speed is given by

$$c = \sqrt{-p_\tau} = \sqrt{K\gamma}\tau^{-\frac{\gamma+1}{2}}e^{\frac{S}{2c_v}}. \tag{1.6}$$

Euler equations (1.2) can be written in the form of hyperbolic conservation laws

$$u_t + \big(f(u)\big)_x = 0, \quad t>0, \quad x\in\mathbb{R} \tag{1.7}$$

with

$$(u_1,u_2,u_3) = \left(\tau, w, \frac{1}{2}w^2 + \mathcal{E}\right), \tag{1.8}$$

and by (1.3),

$$(f_1,f_2,f_3) = \left(-u_2, \frac{(\gamma-1)\big(u_3 - u_2^2/2\big)}{u_1}, \frac{(\gamma-1)u_2\big(u_3-u_2^2/2\big)}{u_1}\right). \tag{1.9}$$

Our result also holds for system (1.1) in the Eulerian coordinates.