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Asymptotic Stability of Shock Wave for the Outflow Problem Governed by the One-Dimensional Radiative Euler Equations

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Abstract. This paper is devoted to the study of the asymptotic stability of the shock wave of the outflow problem governed by the one-dimensional radiative Euler equations, which are a fundamental system in the radiative hydrodynamics with many practical applications in astrophysical and nuclear phenomena. The outflow problem means that the flow velocity on the boundary is negative. Comparing with our previous work on the asymptotic stability of the rarefaction wave of the outflow problem for the radiative Euler equations in [6], two points should be pointed out. On one hand, boundary condition on velocity is considered instead of boundary condition on temperature, which induces a perfect boundary condition on anti-derivative perturbations so that boundary estimates on perturbed unknowns are trickily and smoothly established. On the other hand, the rarefaction wave is an expansive wave, while the shock wave is a compressive wave. So we need take good advantages of properties of the shock wave instead. Our investigation on the outflow problem provides a good understanding on the radiative effect and boundary effect in the setting of shock wave.

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1 Introduction

In this paper, we will continue to study the outflow problem governed by the one-dimensional radiative Euler equations, which is the second one of our series of papers on such kind of outflow problem, actually, the first one on the asymptotic stability of the shock wave for the radiative Euler equations with a boundary. The radiative Euler equations are a fundamental system to describe the motion of the compressible gas with the radiative heat transfer phenomena, which has many applications in astrophysics and nuclear explosions. Mathematically, the one-dimensional radiative Euler equations in the Eulerian coordinates can be modelled as a hyperbolic-elliptic coupled system of the following form:

$$
\begin{cases}
\rho_t+(\rho u)_x=0,\\
(\rho u)_t+\left(\rho u^2+p\right)_x=0,\\
\left\{\rho\left(e+\dfrac{u^2}{2}\right)\right\}_t+\left\{\rho u\left(e+\dfrac{u^2}{2}\right)+pu\right\}_x+q_x=0,\\
-q_{xx}+aq+b\left(\theta^4\right)_x=0,
\end{cases}
\tag{1.1}
$$

where ρ, u, p, e and θ are respectively the density, velocity, pressure, internal energy and absolute temperature of the gas, and q is the radiative heat flux. Positive constants a and b depend only on the gas itself. Like the classic compressible Euler equations, the first three equations in (1.1) stand for the conservation of the mass, momentum and energy respectively. The fourth equation in (1.1) is related to the radiative heat transfer phenomenon, and one can refer [1, 9, 23, 28, 39, 44] for more details. System (1.1) can also be derived by the non-relativistic limit (speed of light tending to $+\infty$) from a hyperbolic-kinetic system, and rigorous mathematical derivation can be found in [15].

Precisely speaking, we will investigate the initial-boundary value problem of system (1.1) in the half space $\{(x,t)|0\le x,t<\infty\}$ with the initial data

$$
(\rho,u,\theta)(x,0)=(\rho_0,u_0,\theta_0)(x) \quad \text{for} \quad x\ge 0,
\tag{1.2}
$$

satisfying

$$
\inf_{x\in[0,+\infty)}(\rho_0,\theta_0)(x)>0
\tag{1.3}
$$

and the asymptotic boundary condition at the far field $x=+\infty$

$$(\rho,u,\theta,q)(+\infty,t)=(\rho_+,u_+,\theta_+,0), \quad t\geq 0. \tag{1.4}$$

In what follows, we give a roughly classification of the time-asymptotic states of the solution $(\rho,u,\theta,q)(x,t)$ based on the boundary data $(\rho,u,\theta,q)(0,t)$. It is expected that as time tends to the infinity, the solution is asymptotically described by one of the following waves, such as viscous shock wave, stationary wave, rarefaction wave or the superposition of stationary wave and rarefaction wave. As shown in the Fig. 1, we state some cases which have been solved already or will possibly be solved in the future.

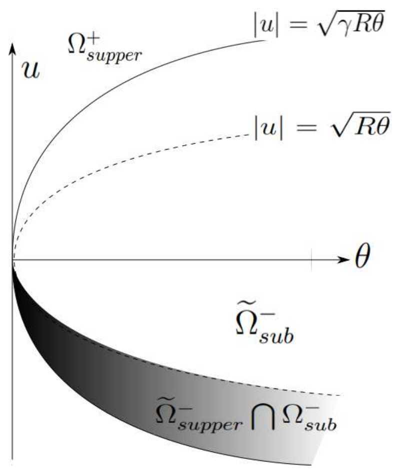

Figure 1:

Case I: inflow problems ($u(0,t)>0$).

If

$$(\rho_+,u_+,\theta_+)\in\Omega^+_{supper}:=\left\{(\rho,u,\theta);\ u>\sqrt{\gamma R\theta}\right\},$$

the boundary condition is given as

$$(\rho,u,\theta,q)(0,t)=(\rho_-,u_-,\theta_-,0). \tag{1.5}$$

Then

(i) If (ρ_+,u_+,θ_+) locates on the 3-rarefaction curve past through (ρ_-,u_-,θ_-), then there exists a 3-rarefaction wave to the corresponding Riemann problem which connects (ρ_-,u_-,θ_-) and (ρ_+,u_+,θ_+) and the 3-rarefaction wave solution is asymptotic stable. It has been proved in [5].

(ii) If (ρ_+,u_+,θ_+) locates on the contact discontinuity curve past through (ρ_-,u_-,θ_-), then the asymptotic stability of the contact discontinuity wave, under the restriction of the smallness of the wave strength, can be obtained. It has been proved in [7].

Case II: outflow problems ($u(0,t)<0$).

(i) If

$$(\rho_+,u_+,\theta_+)\in\widetilde{\Omega}^-_{supper}\bigcap\Omega^-_{sub}:=\big\{(\rho,u,\theta);\ -\sqrt{\gamma R\theta}<u<-\sqrt{R\theta}\big\},$$

the boundary condition can be given as

$$(\theta,q)(0,t)=(\theta_-,0). \tag{1.6}$$

Under this boundary condition, we can find proper numbers $(\rho_-,u_-)=(\rho,u)(0,t)$ to show the asymptotic stability of the rarefaction wave without restrictions on the smallness of the wave strength, which has been proved in [6].

(ii) If

$$(\rho_+,u_+,\theta_+)\in\widetilde{\Omega}^-_{sub}:=\big\{(\rho,u,\theta);\ -\sqrt{R\theta}<u<0\big\},$$

the boundary condition can be given as

$$u(0,t)=u_-,\quad q(0,t)=0. \tag{1.7}$$

Moreover, the initial condition (1.2) and the boundary condition (1.7) satisfy the compatibility condition that $u_0(0)=u_-$ at the origin $(0,0)$. Under this boundary condition, we can find proper numbers $(\rho_-,\theta_-)=(\rho,\theta)(0,t)$ to show the asymptotic stability of the 3-shock wave under some small assumptions, which is the theme of this manuscript.

We need to emphasize that the classification of asymptotic states of the radiative Euler equations is very complicated. For the other cases, especially on the transonic region $\widetilde{\Gamma}^-_{sub}:=\{(\rho,u,\theta);|u|=\sqrt{R\theta}\}$, the situation is more complicated. They will be investigated in the future.

In this paper, we are interested in studying the asymptotic stability of the viscous shock wave of the outflow problem (1.1)-(1.4) and (1.7). Our main results will be stated in Section 2. Comparing with our previous work on the asymptotic stability of the rarefaction wave to the outflow problem for the radiative Euler equations in [6], two points should be pointed out. On one hand, the rarefaction wave is an expansive wave, while the shock wave is a compressive wave. They are different such that we have to develop a different approach to take good advantage of properties of the shock wave instead. On the other hand, we impose a different boundary condition, that concerns velocity instead of temperature, which is a perfect boundary condition on the anti-derivative perturbations so that boundary estimates on perturbed unknowns are trickily and smoothly established. In particular, the boundary condition on temperature $\theta(0,t)=\theta_-$ is frequently considered for the outflow problem governed by the Navier-Stokes equations such as [14,26]. However, different from that boundary condition concerns on temperature, boundary condition (1.7) is reasonable since system (1.1) is of Euler-type. Finally, the determination of the shift is based on tedious and tricky calculations.

Now let us review some related work. As far as we know, so far most of the existing results concern the analysis of the global-in-time existence and stability of the elementary waves for the one-dimensional case.

For the Cauchy problem, the global-in-time existence of solutions around a constant state was shown in [16]. For the analysis of the rarefaction wave, if the initial data is a small perturbation of a given rarefaction wave with small strength, it was proved in [20] that the solutions converge to the rarefaction wave as $t\to+\infty$. Then the authors in [11] showed that when the absorption coefficient α tends to $+\infty$, the solutions converge to the rarefaction wave with the convergence rate $\alpha^{-\frac{1}{3}}|\ln\alpha|^2$, where the absorption coefficient α is defined by the relationship $a=3\alpha^2$ and $b=4\alpha\sigma$ for positive constants a,b and the Stefan-Boltzmann constant σ. The asymptotic stability of a single viscous contact wave was proved in [41,42]. The existence and stability for zero mass perturbation of the small amplitude shock profile were respectively studied in [21,22]. The authors in [29] showed the nonlinear orbital asymptotic stability of small amplitude shock profiles for general hyperbolic-elliptic coupled systems of the type modeling the radiative gas. Analysis of large amplitude shock profiles was given in [2,24]. Finally, for the case of composite waves, the stability of the composite wave of rarefaction waves and a viscous contact wave was investigated in [33,43]. Recently the authors in [4] studied the unique global-in-time existence and the asymptotic stability of the composite wave of two viscous shock waves by employing the anti-derivative method.

For the initial-boundary value problem, the study of the inflow and outflow problem is initiated by us recently and systematically to deal with the weak dissipation of the radiative Euler system and the difficulties from boundary effect. Our series of results on the inflow and outflow problem in [5–7] provide a good understanding to radiative effect and boundary effect in the setting of elementary waves such as rarefaction wave and contact discontinuity wave. It is natural that shock wave with boundary effect is considered in present paper, which will provide a good understanding to radiative effect and boundary effect in the setting of shock wave. For the introduction of the inflow and outflow problems, one can refer to the paper by Matsumura [25] for more details. There are also many results on the study of the inflow or outflow problem governed by other systems such as the Navier-Stokes system (see [12, 13, 18, 26, 30–32, 38, 40]).

We need to mention that we are also motivated by the related investigations on the simplified model (Hamer model), which gives a good approximation to the radiative Euler equations in a certain physical situation, c.f. [10,19]. The investigations on the simplified model provide a good understanding on the radiative effect. The exhaustive literature list is beyond the scope of the paper, and thus only a few closely related results on wave patterns are mentioned, c.f. [3, 8, 17, 34–37]. Interested readers can refer to them and references therein.

The rest of the paper is organized as follows. In Section 2, the viscous shock wave is constructed based on the Riemann problem of the full Euler equations. Properties of viscous shock wave which will be frequently used in this paper and the main theorem of this paper are given. Then series of a priori estimates are established in Sections 3-5 so that our main theorem are proved by combing the local wellposedness theorem.

2 Mathematical formulation and main results

In this section, we construct the viscous shock wave for (1.1) and then state our main results.

2.1 Viscous shock wave

It is well-known that for any given $(\rho_+,u_+,\theta_+)\in\widetilde{\Omega}_{sub}$, there is a curved surface in a small neighbourhood of (ρ_+,u_+,θ_+) for the solution (ρ_-,u_-,θ_-) of a shock structure with the shock speed $s_3>0$, which satisfies the following Rankine-Hugoniot conditions:

$$\begin{cases} -s_3(\rho_- - \rho_+) + (\rho_- u_- - \rho_+ u_+) = 0, \\ -s_3(\rho_- u_- - \rho_+ u_+) + (\rho_- u_-^2 + p_- - \rho_+ u_+^2 - p_+) = 0, \\ -s_3\left[\rho_-\left(e_- + \frac{1}{2}u_-^2\right) - \rho_+\left(e_+ + \frac{1}{2}u_+^2\right)\right] \\ \quad +\left\{\rho_- u_-\left[e_- + \frac{(u_-)^2}{2}\right] + p_- u_- - \rho_+ u_+\left[e_+ + \frac{(u_+)^2}{2}\right] - p_+ u_+\right\} = 0 \end{cases} \tag{2.1}$$

and the entropy condition

$$u_+ + \sqrt{R\gamma\theta_+} < s_3 < u_- + \sqrt{R\gamma\theta_-}. \tag{2.2}$$

Let

$$\Omega(z_+) := \big\{(\rho,u,\theta)\big|\,|(\rho-\rho_+,u-u_+,\theta-\theta_+)| \le \bar{\delta}\big\}, \tag{2.3}$$

where $\bar{\delta}$ is a positive constant depending only on (ρ_+,u_+,θ_+). The corresponding viscous 3-shock wave of (1.1) is expressed by $z^s := (\rho^s,u^s,\theta^s,q^s)(x-s_3t)$, connecting the far field states $(\rho^s,u^s,\theta^s,q^s)(\pm\infty) = (\rho_\pm,u_\pm,\theta_\pm,0)$. Substituting $z^s(\xi) = (\rho^s,u^s,\theta^s,q^s)(\xi)(\xi = x-s_3t)$ into (1.1), one has

$$\begin{cases} -s_3(\rho^s)' + (\rho^s u^s)' = 0, \\ -s_3(\rho^s u^s)' + \big(\rho^s (u^s)^2 + p^s\big)' = 0, \\ -s_3\left\{\rho^s\left(e^s + \dfrac{(u^s)^2}{2}\right)\right\}' + \left\{\rho^s u^s\left(e^s + \dfrac{(u^s)^2}{2}\right) + p^s u^s\right\}' + (q^s)' = 0, \\ aq^s + b(\theta^{s4})' = 0, \\ u^s(-\infty) = u_-, \quad (\rho^s,u^s,\theta^s)(+\infty) = (\rho_+,u_+,\theta_+), \end{cases} \tag{2.4}$$

where $' = \frac{d}{d\xi}$, $p^s = p(\rho^s,\theta^s)$, $e^s = e(\theta^s)$, $p_\pm = p(\rho_\pm,\theta_\pm)$ and $e_\pm = e(\theta_\pm)$. Integrating (2.4) with respect to ξ over $[\xi,+\infty)$, we obtain

$$\begin{cases} -s_3(\rho^s-\rho_+) + \rho^s u^s - \rho_+ u_+ = 0, \\ -s_3(\rho^s u^s - \rho_+ u_+) + \rho^s (u^s)^2 + p^s - \rho_+ u_+^2 - p_+ = 0, \\ -s_3\left\{\rho^s\left[e^s + \dfrac{(u^s)^2}{2}\right] - \rho_+\left(e_+ + \dfrac{u_+^2}{2}\right)\right\} \\ \quad + \rho^s u^s\left[e^s + \dfrac{(u^s)^2}{2}\right] - \rho_+ u_+\left(e_+ + \dfrac{u_+^2}{2}\right) + p^s u^s - p_+ u_+ = \dfrac{b}{a}\left[(\theta^s)^4\right]', \end{cases} \tag{2.5}$$

which implies

$$\begin{cases} \rho^s - \rho_+ = -\dfrac{\rho_+}{u^s - s_3}(u^s - u_+), \\ R(\theta^s - \theta_+) = -\left(u^s - s_3 - \dfrac{R\theta_+}{u_+ - s_3}\right)(u^s - u_+), \end{cases} \tag{2.6}$$

$$\begin{cases} (\theta^s)' = H_1(\theta^s), \\ \theta^s(-\infty) = \theta_-, \quad \theta^s(+\infty) = \theta_+, \end{cases} \tag{2.7}$$

where

$$
\begin{aligned}
H_1(\theta^s) := & \frac{a}{4b(\theta^s)^3}\left[C_v\rho_+ - \frac{Rp_+}{(u^s-s_3)(u_+-s_3)-R\theta_+}\right](u_+-s_3)(\theta^s-\theta_+) \\
& - \frac{a}{4b(\theta^s)^3}\frac{R^2\rho_+(u_+-s_3)^3}{2[(u^s-s_3)(u_+-s_3)-R\theta_+]^2}(\theta^s-\theta_+)^2
\end{aligned} \tag{2.8}
$$

and

$$
(\rho^s)' = -\frac{\rho^s}{u^s-s_3}(u^s)', \quad (u^s)' = -\frac{R(u^s-s_3)}{(u^s-s_3)^2-R\theta^s}(\theta^s)'. \tag{2.9}
$$

By a straightforward computation, we get the following information on $H_1'(\theta_+)$ from the entropy condition (2.2) and subsonic condition $-\sqrt{R\theta_\pm} < u_\pm < 0$:

$$
\begin{aligned}
H_1'(\theta_+) &= \frac{a}{4b\theta_+^3}\left[C_v\rho_+ - \frac{Rp_+}{(u_+-s_3)^2-R\theta_+}\right](u_+-s_3) \\
&= \frac{a}{4b\theta_+^3}\left[\frac{R\rho_+}{\gamma-1} - \frac{R^2\rho_+\theta_+}{(u_+-s_3)^2-R\theta_+}\right](u_+-s_3) \\
&< -\frac{aR\rho_+}{4b\theta_+^3}\sqrt{R\gamma\theta_+}\left[\frac{1}{\gamma-1} - \frac{R\theta_+}{(u_+-s_3)^2-R\theta_+}\right] < 0.
\end{aligned} \tag{2.10}
$$

Therefore, by the theory of ordinary differential system, there exists a unique solution to system (2.7). Therefore, we find the unique solution to system (2.4). Now we introduce some properties of the solution to system (2.4) which will be used later.

Lemma 2.1. *For any fixed* $(\rho_+,u_+,\theta_+)\in\widetilde{\Omega}_{sub}^-$*, suppose* $\gamma>1$, $\theta_+<\theta_-$ *and* $(\rho_-,u_-,\theta_-)\in\Omega(z_+)\cap\widetilde{\Omega}_{sub}^-$ *satisfies Rankine-Hugoniot condition* (2.1). *Then system* (2.4) *admits a smooth solution* $(\rho^s,u^s,\theta^s,q^s)(x-s_3t)$*, which is unique up to the spatial shift and satisfies the following properties:*

(1) $(\rho^s)'<0$, $(u^s)'<0$, $(\theta^s)'<0$;

(2) There exists a positive constant c such that

$$
\begin{aligned}
&\left|(\rho^s-\rho_\pm,u^s-u_\pm,\theta^s-\theta_\pm)(x-s_3t)\right| \lesssim \delta e^{-c\delta|x-s_3t|}, \\
&\left|\left(q^s,(\rho^s)',(u^s)',(\theta^s)',(\rho^s)'',(u^s)'',(\theta^s)''\right)(x-s_3t)\right| \lesssim \delta^2 e^{-c\delta|x-s_3t|},
\end{aligned} \tag{2.11}
$$

where $\delta := |(u_+-u_-,\theta_+-\theta_-)|$.

The proof of Lemma 2.1 is omitted since the argument is standard. Actually, one can follow the argument for the proof of [27, Lemma 1, pp. 85].

2.2 Anti-derivative perturbation

The solution of (1.1) is expected to converge to the 3-viscous shock wave as $t\to+\infty$. Let us consider the case that the initial data $(\rho_0,u_0,\theta_0)(x)$ is given in a neighborhood of $(\rho^s,u^s,\theta^s)(x+\alpha-\beta)$ for a large constant $\beta>0$, where the shift α will be determined later. Actually, we require that the viscous shock wave is far from the boundary when $t=0$. Then we should determine the shift α such that the solution (ρ,u,θ) is expected to tend to $(\rho^s,u^s,\theta^s)(x-s_3t+\alpha-\beta)$. To do this, we denote the anti-derivative perturbations around the 3-viscous shock wave by $(\Phi,\widetilde{\Psi},\widetilde{W},\Gamma)$ as

$$\begin{aligned}
&\Phi(x,t)=\int_x^\infty\left[\rho^s(y-s_3t+\alpha-\beta)-\rho(t,y)\right]dy,\\
&\widetilde{\Psi}(x,t)=\int_x^\infty\left[(\rho^su^s)(y-s_3t+\alpha-\beta)-(\rho u)(t,y)\right]dy,\\
&\widetilde{W}(x,t)=\int_x^\infty\left\{\rho^s\left[C_v\theta^s+\frac{(u^s)^2}{2}\right](y-s_3t+\alpha-\beta)-\rho\left(C_v\theta+\frac{u^2}{2}\right)(t,y)\right\}dy,\\
&\Gamma(x,t)=\int_x^\infty[q^s(y-s_3t+\alpha-\beta)-q(t,y)]\,dy
\end{aligned}\tag{2.12}$$

and denote the perturbation of (ρ,u,θ,q) around the 3-viscous shock wave by (ϕ,ψ,ζ,ω), which satisfies

$$\begin{aligned}
&\phi:=\rho-\rho^s=\Phi_x,\quad \psi:=u-u^s=\frac{1}{\rho}\left(\widetilde{\Psi}_x-u^s\Phi_x\right),\\
&\zeta:=\theta-\theta^s=\frac{1}{C_v\rho}\left(\widetilde{W}_x-\frac{1}{2}\rho\psi^2-\rho u^s\psi-E^s\phi\right),\\
&\omega:=q-q^s=\Gamma_x,
\end{aligned}\tag{2.13}$$

where $E^s=:C_v\theta^s+\frac{1}{2}(u^s)^2$. It is easy to see that (ϕ,ψ,ζ,ω) satisfies

$$\begin{cases}
\phi_t+u\phi_x+\rho\psi_x=Q_1, & (2.14a)\\
\psi_t+u\psi_x+R\zeta_x+\dfrac{R\theta}{\rho}\phi_x=Q_2, & (2.14b)\\
C_v\zeta_t+C_vu\zeta_x+R\theta\psi_x+\omega_x=Q_3, & (2.14c)\\
-w_{xx}+aw+4b\theta^3\zeta_x+4b\theta_x^s\zeta\left(\theta^2+\theta\theta^s+(\theta^s)^2\right)=q_{xx}^s, & (2.14d)
\end{cases}$$

where

$$Q_1:=-\left(\rho_x^s\psi+u_x^s\phi\right)=O(1)\left|(\rho_x^s,u_x^s)\right|\left|(\phi,\psi)\right|,\tag{2.15a}$$

$$Q_2 := -u_x^s\psi - \rho_x^s\left(\frac{R\zeta}{\rho} - \frac{R\theta^s}{\rho\rho^s}\phi\right) = O(1)|(\rho_x^s, u_x^s, \theta_x^s)||(\phi,\psi,\zeta)|, \tag{2.15b}$$

$$Q_3 := -C_v\theta_x^s\psi - Ru_x^s\zeta = O(1)|\theta_x^s\psi + u_x^s\zeta| \tag{2.15c}$$

with

$$\begin{aligned}&(\phi,\psi,\zeta)(x,0) = (\phi_0,\psi_0,\zeta_0)(x) \to (0,0,0) \quad \text{as} \quad x\to+\infty,\\ &\psi(0,t) = u_- - u^s(-st+\alpha-\beta),\\ &\omega(0,t) = -q^s(-st+\alpha-\beta).\end{aligned} \tag{2.16}$$

Introducing the new good unknowns

$$\Psi = \tilde{\Psi} - u^s\Phi, \quad W = \frac{\gamma-1}{R}\big(\tilde{W} - E^s\Phi - u^s\Psi\big), \tag{2.17}$$

we get

$$\begin{aligned}&\rho\psi = \Psi_x + u_x^s\Phi,\\ &C_v\rho\zeta = C_vW_x + C_v\theta_x^s\Phi + u_x^s\Psi - \frac{1}{2}\rho\psi^2,\\ &C_vW_x = C_v\rho\zeta - C_v\theta_x^s\Phi - u_x^s\Psi + \frac{1}{2}\rho\psi^2.\end{aligned} \tag{2.18}$$

By a straightforward computation, we have

$$\begin{cases}\Phi_t + (\rho u - \rho^s u^s) = 0,\\ \tilde{\Psi}_t + \left(\rho u^2 - \rho^s u^{s2}\right) + (p - p^s) = 0,\\ \tilde{W}_t + \left\{\rho u\left(e + \dfrac{u^2}{2}\right) - \rho^s u^s\left[e^s + \dfrac{(u^s)^2}{2}\right]\right\} + (pu - p^s u^s) = q^s - q,\end{cases} \tag{2.19}$$

which particularly implies that on the boundary $x=0$

$$\Phi_t(0,t) + u_-\phi(0,t) + (\rho^s\psi)(0,t) = 0, \tag{2.20}$$

$$\tilde{\Psi}_t(0,t) + u_-^2\phi(0,t) + \rho^s(u^s+u_-)\psi(0,t) + (p-p^s)(0,t) = 0. \tag{2.21}$$

Combining (2.20) and (2.21), we further get

$$(\tilde{\Psi} - u_-\Phi)_t(0,t) + \rho^s u^s\psi(0,t) + (p-p^s)(0,t) = 0. \tag{2.22}$$

It follows from the third equation in (2.19) that

$$\tilde{W}_t(0,t) + \frac{1}{2}\left[\rho u^3 - \rho^s(u^s)^3\right](0,t) + \frac{\gamma}{\gamma-1}(pu - p^s u^s)(0,t) = q^s(0,t). \tag{2.23}$$

By the straightforward computation, it implies

$$\begin{aligned}&\tilde{W}_t(0,t)+\frac{1}{2}u_-^3\phi(0,t)+\left[\frac{1}{2}\rho^s\left(u_-^2+u_-u^s+(u^s)^2\right)+\frac{\gamma p^s}{\gamma-1}\right]\psi(0,t)\\&\quad+\frac{\gamma}{\gamma-1}u_-\left(p-p^s\right)(0,t)=q^s(0,t).\end{aligned}\tag{2.24}$$

Therefore,

$$\begin{aligned}&\left(\tilde{W}-\frac{u_-^2}{2}\Phi\right)_t(0,t)+\left[\frac{1}{2}\rho^s u^s(u_-+u^s)+\frac{\gamma p^s}{\gamma-1}\right]\psi(0,t)\\&\quad+\frac{\gamma u_-}{\gamma-1}(p-p^s)(0,t)=q^s(0,t).\end{aligned}\tag{2.25}$$

Thus, we get

$$\left[\tilde{W}-\frac{\gamma u_-}{\gamma-1}\tilde{\Psi}+\left(\frac{\gamma}{\gamma-1}-\frac{1}{2}\right)u_-^2\Phi\right]_t(0,t)=-(J\psi)(0,t)+q^s(0,t),\tag{2.26}$$

where

$$J:=\frac{1}{2}\rho^s(u^s)^2+\frac{\gamma}{\gamma-1}p^s+\left(\frac{1}{2}-\frac{\gamma}{\gamma-1}\right)\rho^s u^s u_-.\tag{2.27}$$

Integrating (2.26) over $[0,t]$, we immediately obtain

$$\begin{aligned}&\left[\tilde{W}-\frac{\gamma u_-}{\gamma-1}\tilde{\Psi}+\left(\frac{\gamma}{\gamma-1}-\frac{1}{2}\right)u_-^2\Phi\right](0,t)\\=&\left[\tilde{W}-\frac{\gamma u_-}{\gamma-1}\tilde{\Psi}+\left(\frac{\gamma}{\gamma-1}-\frac{1}{2}\right)u_-^2\Phi\right](0,0)-\int_0^t(J\psi-q^s)(0,\tau)d\tau.\end{aligned}\tag{2.28}$$

We expect that as $t\to+\infty$,

$$\left[\tilde{W}-\frac{\gamma u_-}{\gamma-1}\tilde{\Psi}+\left(\frac{\gamma}{\gamma-1}-\frac{1}{2}\right)u_-^2\Phi\right](0,t)\to 0.$$

That is, if let

$$\begin{aligned}I(\alpha):=&\tilde{W}(0,0)-\frac{\gamma u_-}{\gamma-1}\tilde{\Psi}(0,0)+\left(\frac{\gamma}{\gamma-1}-\frac{1}{2}\right)u_-^2\Phi(0,0)\\&-\int_0^\infty(J\psi-q^s)(0,\tau,\alpha,\beta)d\tau,\end{aligned}\tag{2.29}$$

then we require that

$$I(\alpha)=0.\tag{2.30}$$

If (2.30) holds, then by (2.21)-(2.30), we can get the following boundary condition:

$$\tilde{W}(0,t)-\frac{\gamma u_-}{\gamma-1}\tilde{\Psi}(0,t)+\left(\frac{\gamma}{\gamma-1}-\frac{1}{2}\right)u_-^2\Phi(0,t)=A(t), \tag{2.31}$$

where

$$A(t):=\int_t^\infty (J\psi-q^s)(0,\tau,\alpha,\beta)d\tau=O(1)\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta t}.$$

Now we will determine α such that (2.30) holds. In fact, we will compute the derivative $I'(\alpha)$ of $I(\alpha)$ such that the shift α can be determined. Up to now, we have gotten that

$$\begin{aligned}
I(\alpha)=&\int_0^\infty\left\{\rho^s\left[C_v\theta^s+\frac{(u^s)^2}{2}\right](y+\alpha-\beta)-\rho_0\left(C_v\theta_0+\frac{u_0^2}{2}\right)(y)\right\}dy\\
&-\frac{\gamma u_-}{\gamma-1}\int_0^\infty\left[(\rho^s u^s)(y+\alpha-\beta)-(\rho_0 u_0)(y)\right]dy\\
&+\left(\frac{\gamma}{\gamma-1}-\frac{1}{2}\right)u_-^2\int_0^\infty\left[\rho^s(y+\alpha-\beta)-\rho_0(y)\right]dy\\
&-\int_0^\infty (J\psi-q^s)(-s_3\tau+\alpha-\beta)d\tau.
\end{aligned}\tag{2.32}$$

Then it follows from direct calculations that,

$$\begin{aligned}
I'(\alpha)=&\left[\rho_+\left(C_v\theta_++\frac{u_+^2}{2}\right)-\rho^s\left(C_v\theta^s+\frac{(u^s)^2}{2}\right)(\alpha-\beta)\right]\\
&-\frac{\gamma u_-}{\gamma-1}\left[\rho_+u_+-(\rho^s u^s)(\alpha-\beta)\right]+\left(\frac{\gamma}{\gamma-1}-\frac{1}{2}\right)u_-^2\left[\rho_+-\rho^s(\alpha-\beta)\right]\\
&-\frac{1}{s_3}\left[J(u_--u^s)-q^s\right](\alpha-\beta).
\end{aligned}\tag{2.33}$$

In order to simplify and obtain a good and useful expression on $I'(\alpha)$, let us integrate (2.4) over $[0,\xi]$ to get

$$\begin{cases}
\rho^s u^s-\rho_-u_-=s_3(\rho^s-\rho_-),\\
p^s-p_-=s_3(\rho^s u^s-\rho_-u_-)-\left[\rho^s(u^s)^2-\rho_-u_-^2\right],\\
\left\{\rho_\pm\left(e_-+\frac{u_-^2}{2}\right)-\rho^s\left[e^s+\frac{(u^s)^2}{2}\right]\right\}+\frac{1}{s_3}q^s\\
\quad=\frac{1}{s_3}\left\{\rho_-u_-\left(e_-+\frac{u_-^2}{2}\right)-\rho^s u^s\left[e^s+\frac{(u^s)^2}{2}\right]+p_-u_--p^s u^s\right\}.
\end{cases}\tag{2.34}$$

Substituting (2.34) into (2.33), we further obtain

$$\begin{aligned} I'(\alpha) = & \left[\rho_+\left(C_v\theta_+ + \frac{u_+^2}{2}\right) - \rho_-\left(C_v\theta_- + \frac{u_-^2}{2}\right)\right] \\ & -\frac{\gamma u_-}{\gamma-1}(\rho_+u_+ - \rho_-u_-) + \left(\frac{\gamma}{\gamma-1} - \frac{1}{2}\right)u_-^2(\rho_+ - \rho_-) \\ & + \left\{\rho_-\left(C_v\theta_- + \frac{u_-^2}{2}\right) - \rho^s\left[C_v\theta^s + \frac{(u^s)^2}{2}\right](\alpha-\beta)\right\} + \frac{1}{s_3}q^s(\alpha-\beta) \\ & -\frac{\gamma u_-}{\gamma-1}\left[\rho_-u_- - (\rho^s u^s)(\alpha-\beta)\right] + \left(\frac{\gamma}{\gamma-1} - \frac{1}{2}\right)u_-^2\left[\rho_- - \rho^s(\alpha-\beta)\right] \\ & -\frac{1}{s_3}\left\{\left[\frac{1}{2}\rho^s(u^s)^2 + \frac{\gamma p^s}{\gamma-1} + \left(\frac{1}{2} - \frac{\gamma}{\gamma-1}\right)\rho^s u^s u_-\right](u_- - u^s)\right\}(\alpha-\beta), \end{aligned} \tag{2.35}$$

which is reduced to the following perfect equality by tedious and tricky calculations:

$$\begin{aligned} I'(\alpha) = & \left[\rho_+\left(C_v\theta_+ + \frac{u_+^2}{2}\right) - \rho_-\left(C_v\theta_- + \frac{u_-^2}{2}\right)\right] \\ & -\frac{\gamma u_-}{\gamma-1}(\rho_+u_+ - \rho_-u_-) + \left(\frac{\gamma}{\gamma-1} - \frac{1}{2}\right)u_-^2(\rho_+ - \rho_-) := m. \end{aligned} \tag{2.36}$$

Therefore, we get

$$I(\alpha) - I(0) = m\alpha.$$

Hence, (2.30) holds if and only if

$$\alpha = -\frac{I(0)}{m}. \tag{2.37}$$

2.3 Main theorem

Now, we are ready to introduce the main results of this paper. Define

$$\begin{aligned} \Phi_0(x) &= -\int_x^\infty \left[\rho_0(y) - \rho^s(y,0,0,\beta)\right]dy, \\ \Psi_0(x) &= -\int_x^\infty \left[(\rho_0u_0)(y) - (\rho^s u^s)(y,0,0,\beta)\right]dy, \\ W_0(x) &= -\int_x^\infty \left[\rho_0\left(C_v\theta_0 + \frac{u_0^2}{2}\right)(y) - \rho^s\left(C_v\theta^s + \frac{u^{s2}}{2}\right)(y,0,0,\beta)\right]dy. \end{aligned} \tag{2.38}$$

We derive the system satisfied by the anti-derivative unknowns (Φ,Ψ,W,Γ) from the definitions in (2.12) and (2.17) that

$$
\begin{cases}
\Phi_t + u^s\Phi_x + \Psi_x + u^s_x\Phi = 0, & \text{(2.39a)}\\
\Psi_t + u^s\Psi_x + R\theta^s\Phi_x + RW_x + (\gamma-1)u^s_x\Psi - \dfrac{R\theta^s}{\rho^s}\rho^s_x\Phi = G_1, & \text{(2.39b)}\\
C_vW_t + C_vu^sW_x + R\theta^s\Psi_x - \dfrac{q^s_x}{\rho^s}\Phi - \dfrac{p^s_x}{\rho^s}\Psi + \Gamma_x = G_2, & \text{(2.39c)}\\
-\Gamma_{xx} + a\Gamma + \dfrac{4b(\theta^s)^3}{\rho}W_x + \dfrac{4b(\theta^s)^3}{\rho}\theta^s_x\Phi + \dfrac{4b(\theta^s)^3}{C_v\rho}u^s_x\Psi = G_3 & \text{(2.39d)}
\end{cases}
$$

with initial data

$$
(\Phi,\Psi,W)(x,0) = (\Phi_0,\Psi_0,W_0)(x), \tag{2.40}
$$

and the boundary condition obtained from (2.31) that

$$
W(0,t) - \frac{u_-}{R}\Psi(0,t) + \theta_-\Phi(0,t) = A(t) = O(1)\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta t}. \tag{2.41}
$$

Hereinafter, $G_i\,(i=1,2,3)$ are expressed by

$$
\begin{aligned}
G_1 &:= \frac{\gamma-1}{2}\rho\psi^2 - u^s_x\Phi\psi - \psi\Psi_x = O(1)\left(|\psi|^2 + |u^s_x\Phi|^2 + |\Psi_x|^2\right),\\
G_2 &:= -E^s\Phi_x\psi - u^s_x\psi\Psi - u^su^s_x\psi\Phi - C_v\psi W_x - R\zeta\Psi_x\\
&\quad = O(1)\big(|u^s_x||(\Phi,\Psi)||\psi| + |(\Phi_x,\Psi_x,W_x)||\psi| + |\zeta\Psi_x|\big),\\
G_3 &:= q^s_x - b\zeta^2\left(\zeta^2 + 4\theta^s\zeta + 6(\theta^s)^2\right) + \frac{4b(\theta^s)^3}{C_v}\psi^2 = q^s_x + O(1)\left(|\psi|^2 + |\zeta|^2\right).
\end{aligned} \tag{2.42}
$$

Define the solution space $X_{m_1,m_2,M}(0,t)$ as

$$
\begin{aligned}
X_{m_1,m_2,M}(0,t) := \Bigg\{ &(\Phi,\Psi,W,\Gamma)\,|\,(\Phi,\Psi,W) \in \sum_{k=0}^{3} C^k\big([0,t];H^{3-k}(\mathbb{R}_+)\big),\\
&\Gamma \in \sum_{k=0}^{4} C^k\big([0,t];H^{4-k}(\mathbb{R}_+)\big)\partial_t^i(\phi,\psi,\zeta) \in L^2\big(0,t;H^{2-i}(\mathbb{R}_+)\big),\\
&\partial_t^i\omega \in L^2\big(0,t;H^{3-i}(\mathbb{R}_+)\big)\,(i=0,1,2),\\
&\inf_{[0,t]\times\mathbb{R}_+}\rho(x,t) \geq m_1,\ \inf_{[0,t]\times\mathbb{R}_+}\theta(x,t) \geq m_2,\\
&\sup_{[0,t]\times\mathbb{R}_+}\left(\sum_{k=0}^{3}\left\|\left(\partial_t^k\Phi,\partial_t^k\Psi\right)\right\|_{3-k} + \sum_{k=0}^{2}\left\|\partial_t^kW\right\|_{3-k}\right) \leq M \Bigg\},
\end{aligned} \tag{2.43}
$$

where m_1,m_2,M are positive constants and $\|\cdot\|_k := \|\cdot\|_{H^k(\mathbb{R}_+)}$. Hereafter, we denote $\|\cdot\| := \|\cdot\|_0$.

Then the main theorem of this paper is stated as follows.

Theorem 2.1. *Assume the adiabatic exponent* $\gamma\in(1,2)$ *and the states* (ρ_+,u_+,θ_+) *and* (ρ_-,u_-,θ_-) *are in* $\widetilde{\Omega}_{sub}^-$ *with* $u_+<u_-<0$ *and* (ρ_-,u_-,θ_-) *being at the 3-shock curve passing through* (ρ_+,u_+,θ_+) *with shock speed* s_3*,* i.e., (2.1)-(2.3) *hold. If the initial data satisfies*

$$(\Phi_0,\Psi_0,W_0)(x)\in H^3(\mathbb{R}_+),\qquad\qquad (2.44)$$
$$(\rho_0,u_0,\theta_0)(x)-(\rho^s,u^s,\theta^s)(x,0,\alpha,\beta)\in(H^2\cap L^1)(\mathbb{R}_+)$$

and if there exist suitably small constants $\delta_0>0,\epsilon_0>0$ *such that*

$$\delta\lesssim\delta_0,\quad \|(\Phi_0,\Psi_0,W_0)\|_3+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1+\mathrm{e}^{-\delta\beta}\lesssim\epsilon_0, \tag{2.45}$$

then there exist positive constants m_1*,* m_2 *and* M *such that, the outflow problem* (2.39)-(2.41) *admits a unique solution* (Φ,Ψ,W,Γ)*, so that problem* (1.1), (1.2) *and* (1.7) *admits a unique solution* $(\rho,u,\theta,q)(x,t)$*, such that*

$$(\Phi,\Psi,W,\Gamma)\in X_{m_1,m_2,M}(0,+\infty).$$

Here $(\rho^s,u^s,\theta^s,q^s)(t,x)$ *is the shock wave defined in* (2.4)*. Furthermore, it holds*

$$\sup_{x\geq 0}|(\rho,u,\theta,q)(t,x)-(\rho^s,u^s,\theta^s,q^s)(x,t,\alpha,\beta)|\to 0\quad as\quad t\to+\infty, \tag{2.46}$$

where $\alpha=\alpha(\beta)$ *is defined in* (2.34).

We will employ the anti-derivative unknowns (Φ,Ψ,W,Γ) to show the main theorem by presenting two propositions concerned with local existence and a priori estimates respectively. First, the local existence of the solution to system (2.39)-(2.41) is stated as follows.

Proposition 2.1 (Local existence)**.** *Under the conditions of Theorem* 2.1*, there exist positive constants* $\overline{\delta}_2$*,* $\bar{\epsilon}_2$ *and* $\overline{C}$ $(\overline{C}\bar{\epsilon}_2\leq\epsilon_2)$ *such that the following statements hold: For any* $\tau>0$*, let*

$$(\Phi_\tau,\Psi_\tau,W_\tau)(x):=(\Phi,\Psi,W)(x,\tau)\in H^3(\mathbb{R}_+) \tag{2.47}$$

for any $M\in(0,\bar{\epsilon}_2)$*,* $\delta\leq\overline{\delta}_2(<\overline{\delta})$ *and* $\beta>1$*. Assume that*

$$\|(\Phi_\tau,\Psi_\tau,W_\tau)\|_3\lesssim M,\quad \sup_{t\geq 0}\{|A(t)|+|A'(t)|\}\lesssim\mathrm{e}^{-\delta\beta},$$

then there exists a positive constant $t_0=t_0(M,\beta)$ *independent on* τ *such that, problem* (2.39), (2.41) *and* (2.47) *admits a unique solution* $(\Phi,\Psi,W,\Gamma)(x,t)\in\mathbb{X}_{m_1,m_2,M}(\tau,\tau+t_0)$*.*

The proof of Proposition 2.1 is standard. Actually, one can follow the argument in the proof of [5, Proposition 3.1, pp.601-603] to show Proposition 2.1 similarly. Therefore, we omit the proof for the shortness. Based on Proposition 2.1, we can show Theorem 2.1 if the following proposition is proved.

Proposition 2.2 (A priori estimate)**.** *Under the conditions of Proposition* 2.1*, there are constants* $\overline{\delta}_3(\leq\overline{\delta}_2)$ *and* $\overline{\epsilon}_3(\leq\overline{\epsilon}_2)$ *such that, if* $(\Phi,\Psi,W,\Gamma)(x,t)\in\mathbb{X}_{m_1,m_2,M}(0,T)$ *for some* $T>0$ *is a solution to the initial-boundary value problem* (2.39)-(2.41)*. If* $\delta\leq\overline{\delta}_3$ *and for* $t\in[0,T]$

$$\sup_{\tau\in[0,t]}\left\{\|(\Phi,\Psi,W,\Gamma)(\tau)\|_3+\|(\phi_t,\psi_t,\zeta_t,\omega_t)(\tau)\|_1\right\}\leq\overline{\epsilon}_3,$$

then it holds that for $t\in[0,T]$*,*

$$\begin{aligned}
&\|(\Phi,\Psi,W,\Gamma)(t)\|^2+\|(\phi,\psi,\zeta)(t)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(t)\|_1^2 \\
&\quad+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(t)\|^2+\|\omega(t)\|_3^2+\|\omega_t(t)\|_2^2 \\
&\quad+\int_0^t|(\Phi,\Psi,W)|^2(0,\tau)d\tau+\int_0^t\int_{\mathbb{R}_+}\left(|u_x^s||(\Psi,W)|^2+\Gamma^2\right)dxd\tau \\
&\quad+\int_0^t\left(\|(\phi,\psi,\zeta)(\tau)\|_2^2+\|w(\tau)\|_3^2+\|w_t(\tau)\|_2^2\right)d\tau \\
&\quad+\int_0^t|(\phi,\phi_x,\psi_x,\zeta_x,w_x,\phi_{xx},\psi_{xx},\zeta_{xx},w_{xx},w_{xxx},\phi_{tx},\psi_{tx},\zeta_{tx},w_{tx})|^2(0,\tau)d\tau \\
&\lesssim\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_1^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\mathrm{e}^{-\delta\beta}.
\end{aligned} \tag{2.48}$$

The a priori estimate (2.48) will be shown in the rest sections. Actually, once Proposition 2.2 is obtained, the local solution $(\phi,\psi,\zeta,\omega)(x,t)$ obtained in Proposition 2.1 can be extended to $t=\infty$. Moreover, estimate (2.48) implies that

$$\int_0^\infty\left(\|(\phi,\psi,\zeta,\omega)(\tau)\|^2+\frac{d}{dt}\|(\phi,\psi,\zeta,\omega)(\tau)\|^2\right)d\tau<+\infty, \tag{2.49}$$

which together with the Sobolev inequality implies the asymptotic behavior (2.46). This concludes the proof of Theorem 2.1.

3 Basic energy estimate on fluid dynamic perturbations

In this and next two sections, we focus on proving Proposition 2.2. Firstly, we prove basic energy estimates. Set

$$N(t):=\sup_{\tau\in[0,t]}\left\{\|(\Phi,\Psi,W,\Gamma)(\tau)\|_3+\|(\phi_t,\psi_t,\zeta_t,\omega_t)(\tau)\|_1\right\}. \tag{3.1}$$

For system (2.14) with (2.15) and (2.16), we get

Lemma 3.1. *Under the assumption of Proposition* 2.2*, if* $N(t)$ *and* δ *are suitably small, it holds that*

$$\begin{aligned}&\|(\phi,\psi,\zeta)(t)\|^2+\int_0^t\left(\phi^2+\zeta^2\right)(0,\tau)d\tau+\|(\omega,\omega_x)(\tau)\|^2d\tau \\ \lesssim&\|(\phi_0,\psi_0,\zeta_0)\|^2+\mathrm{e}^{-\delta\beta}+\left(\delta+N(t)\right)\int_0^t\|(\phi,\psi,\zeta,\phi_x,\psi_x,\zeta_x,\zeta_{xx},\omega_{xx})(\tau)\|^2d\tau.\end{aligned} \tag{3.2}$$

Proof. Multiplying the equations in (2.14) by $\frac{R\theta}{\rho}\phi$, $\rho\psi$, $\frac{\rho\zeta}{\theta}$, $\frac{R\omega}{4b\theta^4}$ respectively, we have

$$\left(\frac{R\theta}{2\rho}\phi^2\right)_t+\left(\frac{R\theta u}{2\rho}\phi^2\right)_x+R\theta\phi\psi_x=\left\{\left(\frac{R\theta}{2\rho}\right)_t+\left(\frac{R\theta u}{2\rho}\right)_x\right\}\phi^2+\frac{R\theta}{\rho}\phi Q_1, \tag{3.3}$$

$$\left(\frac{\rho\psi^2}{2}\right)_t+\left(\frac{\rho u\psi^2}{2}\right)_x+R\rho\psi\zeta_x+R\theta\phi_x\psi=\left\{\left(\frac{\rho}{2}\right)_t+\left(\frac{\rho u}{2}\right)_x\right\}\psi^2+\rho\psi Q_2, \tag{3.4}$$

$$\begin{aligned}&\left(\frac{C_v\rho}{2\theta}\zeta^2\right)_t+\left(\frac{C_v\rho u}{2\theta}\zeta^2\right)_x+R\rho\zeta\psi_x+\frac{\rho\zeta}{\theta}\omega_x \\ =&\left\{\left(\frac{C_v\rho}{2\theta}\right)_t+\left(\frac{C_v\rho u}{2\theta}\right)_x\right\}\zeta^2+\frac{\rho\zeta}{\theta}Q_3,\end{aligned} \tag{3.5}$$

$$\begin{aligned}&-\left(\frac{R\omega\omega_x}{4b\theta^4}\right)_x+\left(\frac{R}{4b\theta^4}\right)_x\omega\omega_x+\frac{R\omega_x^2}{4b\theta^4}+\frac{aR\omega^2}{4b\theta^4}+\frac{\rho\zeta_x}{\theta}\omega \\ =&-\frac{R\theta^s_x\zeta\omega}{\theta^4}\left\{\theta^2+\theta\theta^s+(\theta^s)^2\right\}+\frac{R\omega q^s_{xx}}{4b\theta^4}.\end{aligned} \tag{3.6}$$

Summing them up, we have

$$\left(\frac{R\theta}{2\rho}\phi^2+\frac{\rho\psi^2}{2}+\frac{C_v\rho}{2\theta}\zeta^2\right)_t+\left(\frac{R}{4b\theta^4}\right)_x\omega\omega_x+\frac{R\omega_x^2}{4b\theta^4}+\frac{aR\omega^2}{4b\theta^4}+I_{1x}=I_2, \tag{3.7}$$

where

$$\begin{aligned}I_1:=&\frac{R\theta u}{2\rho}\phi^2+\frac{\rho u\psi^2}{2}+\frac{C_v\rho u\zeta^2}{2\theta}+R\rho\psi\zeta+R\theta\phi\psi-\frac{R\omega\omega_x}{4b\theta^4}+\frac{\rho\zeta}{\theta}\omega, \\ I_2:=&\left\{\left(\frac{R\theta}{2\rho}\right)_t+\left(\frac{R\theta u}{2\rho}\right)_x\right\}\phi^2+\left\{\left(\frac{C_v\rho}{2\theta}\right)_t+\left(\frac{C_v\rho u}{2\theta}\right)_x\right\}\zeta^2+R\rho_x\psi\zeta\end{aligned}$$

$$
\begin{aligned}
&+R\theta_x\phi\psi+\left(\frac{\rho}{\theta}\right)_x\zeta\omega+\frac{R\theta}{\rho}\phi Q_1+\rho\psi Q_2+\frac{\rho\zeta}{\theta}Q_3\\
&-\frac{R\theta^s_x\zeta\omega}{\theta^4}\left\{\theta^2+\theta\theta^s+(\theta^s)^2\right\}+\frac{R\omega q^s_{xx}}{4b\theta^4}.
\end{aligned}\tag{3.8}
$$

It is easy to see that

$$
|I_2|\lesssim(\delta+N(t))|(\phi,\psi,\zeta,\phi_x,\psi_x,\zeta_x,\zeta_{xx})|^2+|\omega q^s_{xx}|.\tag{3.9}
$$

Moreover, it follows from the boundary condition $u(0,t)=u_-<0$ that

$$
\begin{aligned}
-I_1(0,t)=&-\left(\frac{R\theta u}{2\rho}\phi^2+\frac{\rho u\psi^2}{2}+\frac{C_v\rho u\zeta^2}{2\theta}+R\rho\psi\zeta+R\theta\phi\psi-\frac{R\omega\omega_x}{4b\theta^4}+\frac{\rho\zeta}{\theta}\omega\right)(0,t)\\
&\geq c\left(\phi^2+\zeta^2\right)(0,t)-C\left(\psi^2+\omega^2+\omega\omega_x\right)(0,t)\\
&\geq c\left(\phi^2+\zeta^2\right)(0,t)-C\delta e^{-c\delta\beta}e^{-c\delta t},
\end{aligned}\tag{3.10}
$$

where positive constants c and C do not depend on (ϕ,ψ,ζ,ω). Integrating (3.7) over $\mathbb{R}_+\times[0,t]$ and using (3.9)-(3.10), we get (3.2). This completes the proof of Lemma 3.1. □

Next, in order to estimate $\int_0^t\|(\phi,\psi,\zeta)(\tau)\|^2d\tau$, we will derive the following estimate on the anti-derivative functions (Φ,Ψ,W,Γ).

Lemma 3.2. *Under the assumptions listed in Proposition* 2.2, *if $N(t)$ and δ are suitably small, and if $\gamma\in(1,2)$, it holds that*

$$
\begin{aligned}
&\|(\Phi,\Psi,W)(t)\|^2+\int_0^t|(\Psi,\Psi,W)|^2(0,\tau)d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}\left(|u^s_x||(\Psi,W)|^2+\Gamma^2+\Gamma_x^2\right)(x,\tau)dxd\tau\\
&\lesssim\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|^2+e^{-\delta\beta}\\
&\quad+(\delta+N(t))\int_0^t\|(\phi,\psi,\zeta,\phi_x,\psi_x,\zeta_x,\zeta_{xx},\omega_{xx})(\tau)\|^2d\tau.
\end{aligned}\tag{3.11}
$$

Proof. Multiplying the first three equations in (2.39) by $\frac{1}{\rho^s}\Phi$, $\frac{\Psi}{p^s}$ and $\frac{W}{\theta^s p^s}$ respectively, we obtain the identities

$$
\left(\frac{\Phi^2}{2\rho^s}\right)_t+\left(\frac{u^s}{2\rho^s}\Phi^2\right)_x+\frac{1}{\rho^s}\Phi\Psi_x=0,\tag{3.12}
$$

$$\begin{aligned}&\left(\frac{\Psi^2}{2p^s}\right)_t+\left(\frac{u^s}{2p^s}\Psi^2\right)_x+\left(\frac{1}{\rho^s}\Phi\right)_x\Psi+\frac{R\Psi}{p^s}W_x\\&\quad+\left(\frac{(\gamma-2)u_x^s}{2p^s}-\frac{(\gamma-1)q_x^s}{2(p^s)^2}\right)\Psi^2=\frac{\Psi G_1}{p^s},\end{aligned}\tag{3.13}$$

$$\begin{aligned}&\left(\frac{C_vW^2}{2p^s\theta^s}\right)_t+\left(\frac{C_vu^sW^2}{2p^s\theta^s}\right)_x+\left(\frac{R}{p^s}\Psi\right)_xW+\frac{W\Gamma_x}{\theta^sp^s}\\&\quad+\frac{C_v\gamma(-u_x^s)}{\theta^sp^s}W^2-\frac{Rq_x^s}{\theta^s(p^s)^2}W^2-\frac{Rq_x^s}{(p^s)^2}\Psi W=\frac{WG_2}{\theta^sp^s}.\end{aligned}\tag{3.14}$$

Summing (3.12)-(3.14) together, one has

$$\begin{aligned}&\left(\frac{\Phi^2}{2\rho^s}+\frac{\Psi^2}{2p^s}+\frac{C_vW^2}{2p^s\theta^s}\right)_t+I_{3x}+\frac{(\gamma-2)u_x^s}{2p^s}\Psi^2+\frac{C_v\gamma(-u_x^s)}{\theta^sp^s}W^2+\frac{W\Gamma_x}{\theta^sp^s}\\=&\frac{\Psi G_1}{\rho^s}+\frac{WG_2}{\theta^sp^s}+\frac{(\gamma-1)q_x^s}{2(p^s)^2}\Psi^2+\frac{Rq_x^s}{\theta^s(p^s)^2}W^2+\frac{Rq_x^s}{(p^s)^2}\Psi W,\end{aligned}\tag{3.15}$$

where

$$I_3:=\frac{u^s\Phi^2}{2\rho^s}+\frac{u^s\Psi^2}{2p^s}+\frac{C_vu^sW^2}{2\theta^sp^s}+\frac{\Phi\Psi}{\rho^s}+\frac{R}{p^s}\Psi W.\tag{3.16}$$

Finally, multiplying (2.39d) by $\frac{\rho}{4b(\theta^s)^2}\frac{\Gamma}{\theta^sp^s}$ yields

$$\begin{aligned}&-\left[\frac{\rho}{4b(\theta^s)^3}\frac{\Gamma\Gamma_x}{p^s}\right]_x+\left[\frac{\rho}{4b(\theta^s)^3p^s}\right]_x\Gamma\Gamma_x+\frac{\rho(\Gamma_x^2+a\Gamma^2)}{4b(\theta^s)^3p^s}\\&\quad+\frac{\Gamma W_x}{\theta^sp^s}+\frac{\theta_x^s\Phi\Gamma}{\theta^sp^s}+\frac{u_x^s\Psi\Gamma}{C_v\theta^sp^s}=\frac{\rho\Gamma G_3}{4b(\theta^s)^3p^s}.\end{aligned}\tag{3.17}$$

Then it follows from (3.15) and (3.17) that

$$\begin{aligned}&\left(\frac{\Phi^2}{2\rho^s}+\frac{\Psi^2}{2p^s}+\frac{C_vW^2}{2p^s\theta^s}\right)_t+\frac{\rho(\Gamma_x^2+a\Gamma^2)}{4b(\theta^s)^3p^s}\\&\quad+\left[I_3-\frac{\rho\Gamma\Gamma_x}{4b(\theta^s)^3p^s}+\frac{W\Gamma}{\theta^sp^s}\right]_x+\left[\frac{\rho}{4b(\theta^s)^3p^s}\right]_x\Gamma\Gamma_x-\left[\frac{1}{\theta^sp^s}\right]_xW\Gamma\\&\quad+\frac{(\gamma-2)u_x^s}{2p^s}\Psi^2+\frac{C_v\gamma(-u_x^s)}{\theta^sp^s}W^2+\frac{\theta_x^s\Phi\Gamma}{\theta^sp^s}+\frac{u_x^s\Psi\Gamma}{C_v\theta^sp^s}\\&=O(1)|\Psi G_1+WG_2+\Gamma G_3|+O(1)|q_x^s||\Psi^2+W^2+\Psi W|.\end{aligned}\tag{3.18}$$

Note that by boundary condition (2.41), we have

$$\Psi(0,t)=\frac{R}{u_-}(W+\theta_-\Phi)(0,t)+O(1)\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta t}.$$

So

$$\begin{aligned}
-I_3(0,t)=&\left(\frac{-u_-\Phi^2}{2\rho_-}+\frac{(-u_-)\Psi^2}{2p_-}+\frac{C_v(-u_-)W^2}{2p_-\theta_-}\right)(0,t)\\
&-\frac{\Psi}{\rho_-\theta_-}(\theta_-\Phi+W)(0,t)+O(1)\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta t}\\
=&\frac{-u_-}{p_-}\left(\frac{\Phi^2}{2}+\frac{\Psi^2}{2}+\frac{C_vW^2}{2\theta_-}\right)(0,t)-\frac{\Psi}{\theta_-p_-}\frac{u_-}{R}\Psi(0,t)+O(1)\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta t}\\
=&\frac{(-u_-)}{p_-}\left(\frac{R\theta_-\Phi^2}{2}+\frac{3\Psi^2}{2}+\frac{C_vW^2}{2\theta_-}\right)(0,t)+O(1)\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta t}. \qquad (3.19)
\end{aligned}$$

So (3.19) implies that the boundary term $-I_3(0,t)$ can be controlled.

Suppose $1<\gamma<2$, then $\gamma-2<0$. Then integrating (3.18) over $\mathbb{R}_+\times(0,t)$, integrating by part, and using (3.19) and the Sobolev inequality, we obtain that

$$\begin{aligned}
&\|(\Phi,\Psi,W)(t)\|^2+\int_0^t\Big(|(\Phi,\Psi,W)|^2(0,\tau)+\|\Gamma(\tau)\|_1^2\Big)d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}|u_x^s|(\Psi^2+W^2)dxd\tau\\
\lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\int_0^t|(\Gamma\Gamma_x+W\Gamma)|(0,\tau)d\tau\\
&+\int_0^t\int_{\mathbb{R}_+}|(u_x^s,\theta_x^s)||\Gamma(\Gamma_x+\Phi+\Psi+W)|dxd\tau+\delta\int_0^t\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta\tau}d\tau\\
&+\int_0^t\int_{\mathbb{R}_+}\Big(|q_x^s||\Psi^2+W^2+\Psi W|+|\Psi G_1+WG_2+\Gamma G_3|\Big)dxd\tau. \qquad (3.20)
\end{aligned}$$

Note that

$$q^s=-\frac{4b}{a}(\theta^s)^3\theta_x^s,\quad \Gamma_x(0,t)=-q^s(0,t)=O(1)|\theta_x^s(0,t)|,\quad \Gamma_x=\omega.$$

So we have

$$\begin{aligned}
\int_0^t\Gamma^2(0,\tau)d\tau\lesssim&\int_0^t\|\Gamma(\tau)\|_\infty^2d\tau\lesssim\int_0^t\|\Gamma(\tau)\|\|\Gamma_x(\tau)\|d\tau\\
\lesssim&\frac{1}{4}\int_0^t\|\Gamma(\tau)\|^2d\tau+\int_0^t\|\omega(\tau)\|^2d\tau, \qquad (3.21)
\end{aligned}$$

and then

$$\begin{aligned}
&\int_0^t\int_0^t |(\Gamma\Gamma_x+W\Gamma)|(0,\tau)d\tau\\
\lesssim&\frac{1}{4}\int_0^t W^2(0,\tau)d\tau+\int_0^t \Gamma_x^2(0,\tau)d\tau+\int_0^t \Gamma^2(0,\tau)d\tau\\
\lesssim&\frac{1}{4}\int_0^t W^2(0,\tau)d\tau+\int_0^t (\theta_x^s)^2(0,t)(0,\tau)d\tau+\int_0^t \Gamma^2(0,\tau)d\tau\\
\lesssim&\frac{1}{4}\int_0^t \left(W^2(0,\tau)+\|\Gamma(\tau)\|^2\right)d\tau+\int_0^t \|\omega(\tau)\|^2d\tau+\mathrm{e}^{-\delta\beta}.
\end{aligned}\tag{3.22}$$

Therefore, it follows from (3.20) that

$$\begin{aligned}
&\|(\Phi,\Psi,W)(t)\|^2+\int_0^t\Big(|(\Phi,\Psi,W)|^2(0,\tau)+\|\Gamma(\tau)\|_1^2\Big)d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}|u_x^s|\left(\Psi^2+W^2\right)dxd\tau\\
\lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\mathrm{e}^{-\delta\beta}+\int_0^t\|\omega(\tau)\|^2d\tau\\
&\quad+\int_0^t\int_{\mathbb{R}_+}|\Psi G_1+WG_2+\Gamma G_3|dxd\tau+\int_0^t\int_{\mathbb{R}_+}|\theta_x^s|^2dxd\tau.
\end{aligned}\tag{3.23}$$

Finally, the last two terms on the right-hand side of (3.24) can be estimated as follows:

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}|\Psi G_1+WG_2+\Gamma G_3|dxd\tau\\
\lesssim&\left(\delta+N(t)\right)\int_0^t\int_{\mathbb{R}_+}|u_x^s||(\Psi,W)|^2dxd\tau\\
&\quad+N(t)\int_0^t\int_{\mathbb{R}_+}|(\psi,\psi,\zeta)(\tau)|^2dxd\tau+\int_0^t\int_{\mathbb{R}_+}|\theta_x^s|^2dxd\tau
\end{aligned}\tag{3.24}$$

and

$$\int_0^t\int_{\mathbb{R}_+}|\theta_x^s|^2dxd\tau\lesssim\delta^2\mathrm{e}^{-\delta\beta}\int_0^t\int_{\mathbb{R}_+}\mathrm{e}^{-c\delta\xi}\mathrm{e}^{-c\delta\tau}d\xi d\tau\lesssim\mathrm{e}^{-\delta\beta}.\tag{3.25}$$

Therefore, substituting (3.24), (3.25) into (3.23) and using (3.2), we can get (3.11). This completes the proof of Lemma 3.2. □

Remark 3.1. We see that from the last equation of (2.39)

$$w_x=-q^s+a\Gamma+b\zeta(\theta+\theta^s)\left(\theta^2+\theta\theta^s+(\theta^s)^2\right).\tag{3.26}$$

By applying Lemmas 3.1 and 3.2, we can obtain the following estimate of w_x on the boundary

$$\begin{aligned}&\int_0^t w_x^2(0,\tau)d\tau \lesssim \int_0^t (q_x^s)^2(0,\tau)d\tau + \int_0^t \left(\Gamma^2+\zeta^2\right)(0,\tau)d\tau\\ &\lesssim \|(\Phi_0,\Psi_0,W_0)\|^2 + \|(\phi_0,\psi_0,\zeta_0)\|^2 + \mathrm{e}^{-\delta\beta}\\ &\quad + \left(\delta+N(t)\right)\int_0^t \|(\phi,\psi,\zeta,\phi_x,\psi_x,\zeta_x,\zeta_{xx})(\tau)\|^2 d\tau. \end{aligned} \tag{3.27}$$

4 High-order energy estimates on fluid dynamic perturbations

In this section, we will focus on establishing the high-order energy estimates in two steps. Firstly, the first-order energy estimates are established in Subsection 4.1. Then the second-order energy estimates are established in Subsection 4.2.

4.1 First-order energy estimates

To establish the first-order energy estimate, we will first establish the following estimate on the time derivatives.

Lemma 4.1. *Under the assumption of Proposition* 2.2*, if* $N(t)$ *and* δ *are suitably small, it holds that*

$$\begin{aligned}&\|(\phi_t,\psi_t,\zeta_t)(t)\|^2 + \int_0^t \left(\phi_t^2+\zeta_t^2\right)(0,\tau) + \|(\omega_t,\omega_{tx})(\tau)\|^2 d\tau\\ &\lesssim \|(\phi_0,\psi_0,\zeta_0)\|^2 + \|(\phi_t,\psi_t,\zeta_t)(0)\|^2 + \mathrm{e}^{-\delta\beta}\\ &\quad + \left(\delta+N(t)\right)\int_0^t \left(\|(\phi,\psi,\zeta)(\tau)\|_2^2 + \|\omega_{txx}(\tau)\|^2\right)d\tau. \end{aligned} \tag{4.1}$$

Proof. Differentiating each equation in (2.14) with respect to t, then multiplying them by $\frac{R\theta}{\rho}\phi_t$, $\rho\psi_t$, $\frac{\rho}{\theta}\zeta_t$ and $\frac{\rho}{4b\theta^4}w_t$ respectively, and then adding all the resulting equations together, we have

$$\left(\frac{R\theta}{2\rho}\phi_t^2+\frac{\rho}{2}\psi_t^2+\frac{C_v\rho}{2\theta}\zeta_t^2\right)_t + \frac{\rho w_{xt}^2}{4b\theta^4} + \left(\frac{\rho}{4b\theta^4}\right)_x w_t w_{xt} + \frac{a\rho w_t^2}{4b\theta^4} + I_{4x} = I_5, \tag{4.2}$$

where

$$I_4 := \frac{R\theta u}{2\rho}\phi_t^2 + \frac{\rho u}{2}\psi_t^2 + \frac{C_v\rho u}{2\theta}\zeta_t^2 + R\theta\phi_t\psi_t + R\rho\zeta_t\psi_t + \frac{\rho}{\theta}\zeta_t\omega_t - \frac{\rho w_t}{4b\theta^4}w_{xt} \tag{4.3}$$

$$\begin{aligned}I_5:=&\left\{\left(\frac{R\theta}{2\rho}\right)_t+\left(\frac{R\theta u}{2\rho}\right)_x\right\}\phi_t^2+\left\{\left(\frac{C_v\rho}{2\theta}\right)_t+\left(\frac{C_v\rho u}{2\theta}\right)_x\right\}\zeta_t^2+R\theta_x\phi_t\psi_t\\&+R\rho_x\zeta_t\psi_t+(Q_{1t}-\rho_t\psi_x-u_t\phi_x)\frac{R\theta}{\rho}\phi_t+\left[Q_{2t}-u_t\psi_x-\left(\frac{R\theta}{\rho}\right)_t\phi_x\right]\rho\psi_t\\&+(Q_{3t}-C_vu_t\zeta_x-R\theta_t\psi_x)\frac{\rho}{\theta}\zeta_t+\left(\frac{\rho}{\theta}\right)_x\zeta_t\omega_t\\&+\left\{q^s_{txx}-12b\theta^2\theta_t\zeta_x-\left[4b\theta^s_x\zeta\left(\theta^2+\theta\theta^s+(\theta^s)^2\right)\right]_t\right\}\frac{w_t}{4b\theta^4}.\end{aligned}\tag{4.4}$$

For the boundary term I_4, we note that

$$\begin{aligned}-I_4(0,t)=&-\left(\frac{R\theta u}{2\rho}\phi_t^2+\frac{\rho u}{2}\psi_t^2+\frac{C_v\rho u}{2\theta}\zeta_t^2+R\theta\phi_t\psi_t+R\rho\zeta_t\psi_t+\frac{\rho}{\theta}\zeta_t\omega_t-\frac{\rho w_t}{4b\theta^4}w_{xt}\right)(0,t)\\&\geq c\left(\phi_t^2+\zeta_t^2\right)(0,t)-C\left(\psi_t^2+\omega_t^2+\omega_t\omega_{tx}\right)(0,t)\\&\geq c\left(\phi_t^2+\zeta_t^2\right)(0,t)-C\delta\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta t}.\end{aligned}\tag{4.5}$$

Then integrating (4.2) over $\mathbb{R}_+\times[0,t]$ and using (4.5), we obtain

$$\begin{aligned}&\|(\phi_t,\psi_t,\zeta_t)(t)\|^2+\int_0^t\left(\left(\phi_t^2+\zeta_t^2\right)(0,\tau)+\|(\omega_t,\omega_{tx})(\tau)\|^2\right)d\tau\\&\lesssim\|(\phi_t,\psi_t,\zeta_t)(0)\|^2+\mathrm{e}^{-c\delta\beta}+\int_0^t\left(\psi_t^2+\omega_t^2+|\omega_{tx}\omega_t|\right)(0,\tau)d\tau\\&\quad+\left(\delta+N(t)\right)\int_0^t\int_{\mathbb{R}^+}|(\phi,\psi,\zeta,\phi_x,\psi_x,\zeta_x,\zeta_{xx})|^2(x,\tau)dxd\tau\\&\lesssim\|(\phi_t,\psi_t,\zeta_t)(0)\|^2+\mathrm{e}^{-c\delta\beta}+\left(\delta+N(t)\right)\int_0^t\left(\|(\phi,\psi,\zeta)(\tau)\|_2^2+\|\omega_{txx}(\tau)\|^2\right)d\tau,\end{aligned}\tag{4.6}$$

where for the last inequality, we employed the following boundary estimate:

$$\begin{aligned}&\int_0^t\left(\psi_t^2+\omega_t^2+|\omega_{tx}\omega_t|\right)(0,\tau)d\tau\\&\lesssim\int_0^t\delta\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta\tau}\left(1+|\omega_{tx}|(0,\tau)\right)d\tau\\&\lesssim\int_0^t\delta\mathrm{e}^{-c\delta\beta}\mathrm{e}^{-c\delta\tau}\left(1+\|\omega_{tx}\|^{\frac12}\|\omega_{txx}\|^{\frac12}\right)d\tau\\&\lesssim\mathrm{e}^{-c\delta\beta}+\delta\int_0^t\|(\omega_{tx},\omega_{txx})(\tau)\|^2d\tau.\end{aligned}\tag{4.7}$$

By Lemmas 2.1, 3.1 and 3.2, we can get (4.1) from (4.6). □

Based on the estimates on the time derivative, in order to derive the first order energy estimates, we need to control boundary integral terms with respect to the spatial derivatives first. Let us rewrite system (2.14) as follows:

$$\begin{cases} u\phi_x+\rho\psi_x=Q_1-\phi_t, \\ u\psi_x+R\zeta_x+\dfrac{R\theta}{\rho}\phi_x=Q_2-\psi_t, \\ C_v u\zeta_x+R\theta\psi_x=Q_3-C_v\zeta_t-w_x. \end{cases} \tag{4.8}$$

So

$$\begin{aligned} \psi_x=&\frac{u(Q_2-\psi_t)}{u^2-R\gamma\theta}-\frac{R\theta(Q_1-\phi_t)}{\rho(u^2-R\gamma\theta)}-\frac{\gamma-1}{u^2-R\gamma\theta}(Q_3-C_v\zeta_t-w_x) \\ =&O(1)|(Q_1,Q_2,Q_3,\phi_t,\psi_t,\zeta_t,w_x)|. \end{aligned} \tag{4.9}$$

Therefore, based on the expressions in (2.15), estimate (3.27), Lemmas 2.1, 3.1, 3.2 and 4.1, we obtain

$$\begin{aligned} &\int_0^t \psi_x^2(0,\tau)d\tau\lesssim\int_0^t |(Q_1,Q_2,Q_3,\phi_t,\psi_t,\zeta_t,w_x)|^2(0,\tau)d\tau \\ \lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|^2+\mathrm{e}^{-\delta\beta} \\ &+\big(\delta+N(t)\big)\int_0^t\Big(\|(\phi,\psi,\zeta)(\tau)\|_2^2+\|\omega_{txx}(\tau)\|^2\Big)d\tau, \end{aligned} \tag{4.10}$$

$$\begin{aligned} &\int_0^t |(\phi_x,\zeta_x)|^2(0,\tau)d\tau \\ \lesssim&\int_0^t |(Q_1,Q_3)|^2(0,\tau)d\tau+\int_0^t |(\phi_t,\zeta_t,w_x,\psi_x)|^2(0,\tau)d\tau \\ \lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|^2+\mathrm{e}^{-\delta\beta} \\ &+\big(\delta+N(t)\big)\int_0^t\Big(\|(\phi,\psi,\zeta)(\tau)\|_2^2+\|\omega_{txx}(\tau)\|^2\Big)d\tau, \end{aligned} \tag{4.11}$$

$$\begin{aligned} &\int_0^t \omega_{xx}^2(0,\tau)d\tau\lesssim\int_0^t\left[\omega^2+\zeta_x^2+|\theta_x^s\zeta|^2+(q_{xx}^s)^2\right](0,\tau)d\tau \\ \lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|^2 \\ &+\mathrm{e}^{-\delta\beta}+\big(\delta+N(t)\big)\int_0^t\Big(\|(\phi,\psi,\zeta)(\tau)\|_2^2+\|\omega_{txx}(\tau)\|^2\Big)d\tau. \end{aligned} \tag{4.12}$$

Now we are ready to establish the first-order energy estimate.

Lemma 4.2. *Under the assumption of Proposition* 2.2*, if* $N(t)$ *and* δ *are suitably small, it holds that*

$$\begin{aligned}&\|(\phi_x,\psi_x,\zeta_x)(t)\|^2+\int_0^t\Big(|(\phi_x,,\psi_x,\zeta_x,\omega_{xx})|^2(0,\tau)+\|w_x(\tau)\|_1^2\Big)d\tau\\ \lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_1^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|^2+\mathrm{e}^{-\delta\beta}\\ &+(\delta+N(t))\int_0^t\Big(\|(\phi,\psi,\zeta)(\tau)\|_2^2+\|\omega_{txx}(\tau)\|^2\Big)d\tau. \end{aligned} \tag{4.13}$$

Proof. Multiplying the equations in (2.14) by $-\frac{R\theta}{\rho}\phi_{xx}$, $-\rho\psi_{xx}$, $-\frac{\rho\zeta_{xx}}{\theta}$ and $-\frac{\rho\omega_{xx}}{4b\theta^4}$ one by one, respectively, we have

$$\begin{aligned}&\left(\frac{R\theta}{\rho}\frac{\phi_x^2}{2}\right)_t-\left(\frac{R\theta}{\rho}\phi_x\phi_t+\frac{R\theta u}{\rho}\frac{\phi_x^2}{2}\right)_x\\ =&\left\{\left(\frac{R\theta}{2\rho}\right)_t-\left(\frac{R\theta u}{2\rho}\right)_x\right\}\phi_x^2-\frac{R\theta}{\rho}\phi_{xx}Q_1-\left(\frac{R\theta}{\rho}\right)_x\phi_t\phi_x, \end{aligned} \tag{4.14}$$

$$\begin{aligned}&\left(\frac{\rho\psi_x^2}{2}\right)_t-\left(\rho u\frac{\psi_x^2}{2}+\rho\psi_t\psi_x\right)_x-R\theta\phi_x\psi_{xx}-R\rho\zeta_x\psi_{xx}\\ =&-\rho\psi_{xx}Q_2-(\rho u)_x\psi_x^2, \end{aligned} \tag{4.15}$$

$$\begin{aligned}&\left(\frac{C_v\rho}{\theta}\frac{\zeta_x^2}{2}\right)_t-\left(\frac{C_v\rho u}{\theta}\frac{\zeta_x^2}{2}+\frac{C_v\rho}{\theta}\zeta_t\zeta_x\right)_x-R\rho\zeta_{xx}\psi_x-\frac{\rho\zeta_{xx}}{\theta}\omega_x\\ =&\left\{\left(\frac{C_v\rho}{\theta}\right)_t-\left(\frac{C_v\rho u}{\theta}\right)_x\right\}\frac{\zeta_x^2}{2}-\frac{\rho\zeta_{xx}}{\theta}Q_3-\left(\frac{C_v\rho}{\theta}\right)_x\zeta_t\zeta_x, \end{aligned} \tag{4.16}$$

$$\begin{aligned}&\frac{\rho\omega_{xx}^2}{4b\theta^4}-\left(\frac{a\rho\omega\omega_x}{4b\theta^4}\right)_x+\frac{a\rho\omega_x^2}{4b\theta^4}+\left(\frac{a\rho}{4b\theta^4}\right)_x\omega\omega_x-\frac{\rho\zeta_x}{\theta}\omega_{xx}\\ =&\left\{4b\theta_x^s\zeta\left[\theta^2+\theta\theta^s+(\theta^s)^2\right]-q_{xx}^s\right\}\frac{\rho\omega_{xx}}{4b\theta^4}. \end{aligned} \tag{4.17}$$

Adding them together, we get

$$\left(\frac{R\theta}{\rho}\frac{\phi_x^2}{2}+\frac{\rho\psi_x^2}{2}+\frac{C_v\rho}{\theta}\frac{\zeta_x^2}{2}\right)_t+\frac{\rho\omega_{xx}^2}{4b\theta^4}+\frac{a\rho\omega_x^2}{4b\theta^4}+\left(\frac{a\rho}{4b\theta^4}\right)_x\omega\omega_x-I_{6x}=I_7, \tag{4.18}$$

where

$$I_6:=\frac{R\theta}{\rho}\phi_t\phi_x+\frac{R\theta u}{2\rho}\phi_x^2+\rho u\frac{\psi_x^2}{2}+\rho\psi_t\psi_x+R\theta\phi_x\psi_x$$

$$
\begin{aligned}
&+R\rho\zeta_x\psi_x+\frac{C_v\rho}{\theta}\zeta_t\zeta_x+\frac{C_v\rho u}{\theta}\frac{\zeta_x^2}{2}-\frac{a\rho\omega\omega_x}{4b\theta^4}-\frac{\rho\zeta_x}{\theta}\omega_x\\
\lesssim\, & O(1)|(\phi_x,\psi,\zeta_x,\phi_t,\psi_t,\zeta_t,\omega,\omega_x)|^2,\\
I_7:=&\left\{\left(\frac{R\theta}{2\rho}\right)_t-\left(\frac{R\theta u}{2\rho}\right)_x\right\}\phi_x^2-R\theta_x\psi_x\phi_x-R\rho_x\zeta_x\psi_x-(\rho u)_x\psi_x^2\\
&+\left\{\left(\frac{C_v\rho}{\theta}\right)_t-\left(\frac{C_v\rho u}{\theta}\right)_x\right\}\frac{\zeta_x^2}{2}-\frac{R\theta}{\rho}\phi_{xx}Q_1-\left(\frac{R\theta}{\rho}\right)_x\phi_t\phi_x\\
&-\rho\psi_{xx}Q_2-\rho_x\psi_t\psi_x-\frac{\rho\zeta_{xx}}{\theta}Q_3-\left(\frac{C_v\rho}{\theta}\right)_x\zeta_t\zeta_x-\left(\frac{\rho}{\theta}\right)_x\zeta_x\omega_x\\
&+\left\{4b\theta_x^s\zeta\left[\theta^2+\theta\theta^s+(\theta^s)^2\right]-q_{xx}^s\right\}\frac{\rho\omega_{xx}}{4b\theta^4}\\
\lesssim\, & \big(\delta+N(t)\big)|(\phi_{xx},\psi_{xx},\zeta_{xx},\omega_{xx},\phi_x,\psi_x,\zeta_x,\omega_x,\phi,\psi,\zeta,\omega)|^2.
\end{aligned}
$$

Therefore, integrating (4.18) over $\mathbb{R}_+\times[0,t]$ and using (4.10)-(4.12), we can get (4.13). □

4.2 Second-order energy estimates

Now, we are going to derive the second-order energy estimates. First for the time derivatives, differentiating (2.14) with respect to t, we get the system that

$$
\begin{cases}
\phi_{tt}+u\phi_{tx}+\rho\psi_{tx}=\widetilde{Q}_1, & (4.19a)\\
\psi_{tt}+u\psi_{tx}+R\zeta_{tx}+\dfrac{R\theta}{\rho}\phi_{tx}=\widetilde{Q}_2, & (4.19b)\\
C_v\zeta_{tt}+C_vu\zeta_{tx}+R\theta\psi_{tx}+\omega_{tx}=\widetilde{Q}_3, & (4.19c)\\
-\omega_{txx}+a\omega_t+4b\theta^3\zeta_{tx}+4b\theta_x^s\zeta_t\left(\theta^2+\theta\theta^s+\left(\theta^s\right)^2\right)=\widetilde{Q}_4, & (4.19d)
\end{cases}
$$

where

$$
\begin{aligned}
&\widetilde{Q}_1=Q_{1t}-(u_t\phi_x+\rho_t\psi_x),\\
&\widetilde{Q}_2=Q_{2t}-u_t\psi_x-\left(\frac{R\theta}{\rho}\right)_t\phi_x,\\
&\widetilde{Q}_3=Q_{3t}-C_vu_t\zeta_x-R\theta_t\psi_x,\\
&\widetilde{Q}_4=q_{xx}^s-4b\left(\theta^3\right)_t\zeta_x-4b\left\{\theta_x^s\left[\theta^2+\theta\theta^s+(\theta^s)^2\right]\right\}_t\zeta.
\end{aligned}
\tag{4.20}
$$

Then we have the following lemma on the estimates of the time derivatives.

Lemma 4.3. *Under the assumptions of Proposition* 2.2*, if* $N(t)$ *and* δ *are suitably small, it holds that*

$$\begin{aligned}&\|(\phi_{tt},\psi_{tt},\zeta_{tt})(t)\|^2+\int_0^t\Big[(\phi_{tt}^2+\zeta_{tt}^2)(0,\tau)+\|(\omega_{tt},\omega_{ttx})(\tau)\|^2\Big]d\tau\\ \lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2\\&+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\mathrm{e}^{-\delta\beta}+\delta\|(\omega_{tx},\omega_{txx})(t)\|^2\\&+(\delta+N(t))\int_0^t\Big(\|(\phi,\psi,\zeta)(\tau)\|_2^2+\|\omega_{tx}(\tau)\|_1^2\Big)d\tau. \qquad (4.21)\end{aligned}$$

Proof. Differentiating the equations in (4.19) with respect to t, then multiplying the resulted equations one by one by $\frac{R\theta}{\rho}\phi_{tt}$, $\rho\psi_{tt}$, $\frac{\rho}{\theta}\zeta_{tt}$ and $\frac{\rho}{4b\theta^4}w_{tt}$ respectively, and adding them together, we get

$$\left(\frac{R\theta}{2\rho}\phi_{tt}^2+\frac{\rho}{2}\psi_{tt}^2+\frac{C_v\rho}{2\theta}\zeta_{tt}^2\right)_t+\frac{\rho w_{xtt}^2}{4b\theta^4}+\left(\frac{\rho}{4b\theta^4}\right)_x w_{tt}w_{xtt}+\frac{a\rho w_{tt}^2}{4b\theta^4}+J_{1x}=J_2, \qquad (4.22)$$

where

$$\begin{aligned}J_1:=&\frac{R\theta u}{2\rho}\phi_{tt}^2+\frac{\rho u}{2}\psi_{tt}^2+\frac{C_v\rho u}{2\theta}\zeta_{tt}^2+R\theta\phi_{tt}\psi_{tt}+R\rho\psi_{tt}\zeta_{tt}\\&+\frac{\rho}{\theta}\zeta_{tt}w_{tt}-\frac{\rho}{4b\theta^4}w_{tt}w_{xtt}, \qquad (4.23)\\ J_2:=&\left\{\left(\frac{R\theta}{2\rho}\right)_t+\left(\frac{R\theta u}{2\rho}\right)_x\right\}\phi_{tt}^2+(\widetilde{Q}_{1t}-\rho_t\psi_{tx}-u_t\phi_{tx})\frac{R\theta}{\rho}\phi_{tt}\\&+\left[\widetilde{Q}_{2t}-u_t\psi_{tx}-\left(\frac{R\theta}{\rho}\right)_t\phi_{tx}\right]\rho\psi_{tt}+R\theta_x\phi_{tt}\psi_{tt}-R\rho_x\psi_{tt}\zeta_{tt}\\&+\left\{\left(\frac{C_v\rho}{2\theta}\right)_t+\left(\frac{C_v\rho u}{2\theta}\right)_x\right\}\zeta_{tt}^2+(Q_{3t}-C_vu_t\zeta_{tx}-R\theta_t\psi_{tx})\frac{\rho}{\theta}\zeta_{tt}\\&+\left\{\widetilde{Q}_4-12b\theta^2\theta_t\zeta_{tx}-\Big[4b\theta_x^s\zeta_t(\theta^2+\theta\theta^s+(\theta^s)^2)\Big]_t\right\}\frac{w_{tt}}{4b\theta^4}. \qquad (4.24)\end{aligned}$$

In particular, one has

$$J_1(0,\tau)\geq c(\phi_{tt}^2+\zeta_{tt}^2)(0,\tau)-C(\psi_{tt}^2+w_{tt}^2)(0,\tau)-\frac{\rho_-}{4b\theta_-^4}\int_0^t w_{tt}w_{xtt}(0,\tau)d\tau, \qquad (4.25)$$

$$|J_2|\lesssim O(1)(\delta+N(t))|(\phi_{tt},\psi_{tt},\zeta_{tt},\phi_{tx},\psi_{tx},\zeta_{tx},\phi_x,\psi_x,\zeta_x,\phi,\psi,\zeta)|^2. \qquad (4.26)$$

Integrating (4.22) over $\mathbb{R}_+$, we obtain

$$\frac{d}{dt}\|(\phi_{tt},\psi_{tt},\zeta_{tt})(t)\|^2+(\phi_{tt}^2+\zeta_{tt}^2)(0,t)+\|(w_{tt},w_{ttx})(t)\|^2$$

$$\begin{aligned}&\lesssim \left(\psi_{tt}^2+w_{tt}^2\right)(0,t)+\frac{\rho_-}{4b\theta_-^4}w_{tt}w_{xtt}(0,t)\\&\quad +O(1)\left(\delta+N(t)\right)\|(\phi_{tt},\psi_{tt},\zeta_{tt},\phi_{tx},\psi_{tx},\zeta_{tx},\phi_x,\psi_x,\zeta_x,\phi,\psi,\zeta)(t)\|^2. \end{aligned}\tag{4.27}$$

Note that

$$\begin{aligned}&\int_0^t \frac{\rho}{4b\theta^4}w_{tt}w_{xtt}(0,\tau)d\tau=\frac{\rho_-}{4b\theta_-^4}\int_0^t w_{tt}w_{xtt}(0,\tau)d\tau\\&=\frac{\rho_-}{4b\theta_-^4}\int_0^t\left\{(w_{tt}w_{xt})_t-w_{ttt}w_{xt}\right\}(0,\tau)d\tau\\&\lesssim |w_{tt}w_{xt}|(0,t)+|w_{tt}w_{xt}|(0,0)-\frac{\rho_-}{4b\theta_-^4}\int_0^t w_{ttt}w_{xt}(0,\tau)d\tau\\&\lesssim\delta\|(w_{tx},w_{txx})(t)\|^2+\delta\int_0^t w_{tx}^2(0,\tau)d\tau+\delta\\&\lesssim\delta\|(w_{tx},w_{txx})(t)\|^2+\delta\int_0^t\|(w_{tx},w_{txx})(\tau)\|^2d\tau+\delta. \end{aligned}\tag{4.28}$$

Integrating (4.27) on $[0,t]$ and using (4.28), we can get (4.21). □

To control the derivatives with respect to ∂_{tx} on the boundary, we rewrite system (4.19) as follows:

$$\begin{cases}u\phi_{tx}+\rho\psi_{tx}=\widetilde{Q}_1-\phi_{tt},\\ u\psi_{tx}+R\zeta_{tx}+\dfrac{R\theta}{\rho}\phi_{tx}=\widetilde{Q}_2-\psi_{tt},\\ C_vu\zeta_{tx}+R\theta\psi_{tx}=\widetilde{Q}_3-C_v\zeta_{tt}-w_{tx}.\end{cases}\tag{4.29}$$

Then

$$\begin{aligned}&\int_0^t w_{tx}^2(0,\tau)d\tau\lesssim\int_0^t\|w_{tx}(\tau)\|\|w_{txx}(\tau)\|d\tau\\&\lesssim\nu\int_0^t\|w_{txx}(\tau)\|^2d\tau+C_\nu\int_0^t\|w_{tx}(\tau)\|^2d\tau, \end{aligned}\tag{4.30}$$

$$\begin{aligned}&\int_0^t|(\phi_{tx},\psi_{tx},\zeta_{tx})|^2(0,\tau)d\tau\\&\lesssim\int_0^t|(\phi_{tt},\psi_{tt},\zeta_{tt},w_{tx})|^2(0,\tau)d\tau+\int_0^t\left|(\widetilde{Q}_1,\widetilde{Q}_2,\widetilde{Q}_3)\right|^2(0,\tau)d\tau\\&\lesssim\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2+\mathrm{e}^{-\delta\beta}\\&\quad+\left(\nu+\delta+N(t)\right)\int_0^t\left(\|(\phi,\psi,\zeta)(\tau)\|_2^2+\|\omega_{txx}(\tau)\|^2\right)d\tau, \end{aligned}\tag{4.31}$$

where $\nu>0$ is a constant small enough. Therefore, one also has

$$\begin{aligned}&\int_0^t |w_{txx}|^2(0,\tau)d\tau \lesssim \int_0^t \left|(\omega_t,\zeta_{tx},\theta_x^s\zeta_t,\zeta,\zeta_t,\widetilde{Q}_4)\right|^2(0,\tau)d\tau\\ \lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2\\ &+e^{-\delta\beta}+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2\\ &+\big(\nu+\delta+N(t)\big)\int_0^t \Big(\|(\phi,\psi,\zeta)(\tau)\|_2^2+\|\omega_{txx}(\tau)\|^2\Big)d\tau. \end{aligned} \tag{4.32}$$

Now we are ready to introduce the following lemma.

Lemma 4.4. *Under the assumption of Proposition* 2.2, *if $N(t)$ and δ are suitably small, it holds that*

$$\begin{aligned}&\|(\phi_{tx},\psi_{tx},\zeta_{tx})(t)\|^2+\int_0^t \Big(|(\phi_{tx},\psi_{tx},\zeta_{tx},w_{tx},w_{txx})|^2(0,\tau)+\|w_{tx}(\tau)\|_1^2\Big)d\tau\\ \lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2+\mathrm{e}^{-\delta\beta}\\ &+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\big(\delta+N(t)\big)\int_0^t\|(\phi,\psi,\zeta)(\tau)\|_2^2d\tau. \end{aligned} \tag{4.33}$$

Proof. Differentiating the equations in (4.19) with respect to x, then multiplying each of the resulting equations by $\frac{R\theta}{\rho}\phi_{tx}$, $\rho\psi_{tx}$, $\frac{\rho}{\theta}\zeta_{tx}$ and $\frac{\rho}{4b\theta^4}w_{tx}$ respectively, and adding all resulting equations together, we get

$$\left(\frac{R\theta}{2\rho}\phi_{tx}^2+\frac{\rho}{2}\psi_{tx}^2+\frac{C_v\rho}{2\theta}\zeta_{tx}^2\right)_t+\frac{\rho w_{txx}^2}{4b\theta^4}+\left(\frac{\rho}{4b\theta^4}\right)_x w_{tx}w_{txx}+\frac{a\rho w_{tx}^2}{4b\theta^4}+J_{3x}=J_4, \tag{4.34}$$

where

$$\begin{aligned}J_3:=&\frac{R\theta u}{2\rho}\phi_{tx}^2+\frac{\rho u}{2}\psi_{tx}^2+\frac{C_v\rho u}{2\theta}\zeta_{tx}^2+R\theta\phi_{tx}\psi_{tx}+R\rho\psi_{tx}\zeta_{tx}\\ &+\frac{\rho}{\theta}\zeta_{tx}w_{tx}-\frac{\rho}{4b\theta^4}w_{tx}w_{txx}\\ =&O(1)|(\phi_{tx},\psi_{tx},\zeta_{tx},w_{tx},w_{txx})(x,t)|^2,\end{aligned} \tag{4.35}$$

$$\begin{aligned}J_4:=&\left\{\left(\frac{R\theta}{2\rho}\right)_t+\left(\frac{R\theta u}{2\rho}\right)_x\right\}\phi_{tx}^2+\big(\widetilde{Q}_{1x}-\rho_x\psi_{tx}-u_x\phi_{tx}\big)\frac{R\theta}{\rho}\phi_{tx}\\ &+\left[\widetilde{Q}_{2x}-u_x\psi_{tx}-\left(\frac{R\theta}{\rho}\right)_x\phi_{tx}\right]\rho\psi_{tx}+R\theta_x\phi_{tx}\psi_{tx}-R\rho_x\psi_{tx}\zeta_{tx}\\ &+\left\{\left(\frac{C_v\rho}{2\theta}\right)_t+\left(\frac{C_v\rho u}{2\theta}\right)_x\right\}\zeta_{tx}^2+\big(\widetilde{Q}_{3x}-C_vu_x\zeta_{tx}-R\theta_x\psi_{tx}\big)\frac{\rho}{\theta}\zeta_{tx}\end{aligned} \tag{4.36}$$

$$
\begin{aligned}
&+\left\{\widetilde{Q}_{4x}-12b\theta^2\theta_x\zeta_{tx}-\Big[4b\theta^s_x\zeta_t\big(\theta^2+\theta\theta^s+(\theta^s)^2\big)\Big]_x\right\}\frac{w_{tx}}{4b\theta^4}\\
=&O(1)|(\phi_{tx},\psi_{tx},\zeta_{tx},w_{tx},\phi_{xx},\psi_{xx},\zeta_{xx},w_{xx},\phi_x,\psi_x,\zeta_x,w_x,\phi,\psi,\zeta,w)(x,t)|^2.
\end{aligned}
$$

Integrating (4.34) over $\mathbb{R}_+\times[0,t]$ and using the boundary estimates (4.30)-(4.32), we can get (4.33). □

To control the other derivatives of second order, differentiating (3.11) with respect to x, we get the system that

$$
\begin{cases}
\phi_{tx}+u\phi_{xx}+\rho\psi_{xx}=Q_5, & (4.37a)\\
\psi_{tx}+u\psi_{xx}+R\zeta_{xx}+\dfrac{R\theta}{\rho}\phi_{xx}=Q_6, & (4.37b)\\
C_v\zeta_{tx}+C_vu\zeta_{xx}+R\theta\psi_{xx}+\omega_{xx}=Q_7, & (4.37c)\\
-\omega_{xxx}+a\omega_x+4b\theta^3\zeta_{xx}+4b\theta^s_x\zeta_x\left(\theta^2+\theta\theta^s+\left(\theta^s\right)^2\right)=Q_8, & (4.37d)
\end{cases}
$$

where

$$
\begin{aligned}
&Q_5=Q_{1x}-(u_x\phi_x+\rho_x\psi_x),\\
&Q_6=Q_{2x}-u_x\psi_x-\left(\frac{R\theta}{\rho}\right)_x\phi_x,\\
&Q_7=Q_{3t}-C_vu_t\zeta_x-R\theta_t\psi_x,\\
&Q_8=q^s_{xxx}-4b\left(\theta^3\right)_x\zeta_x-4b\left\{\theta^s_x\left[\theta^2+\theta\theta^s+(\theta^s)^2\right]\right\}_x\zeta.
\end{aligned}
\tag{4.38}
$$

It can be rewritten as

$$
\begin{cases}
u\phi_{xx}+\rho\psi_{xx}=Q_5-\phi_{tx},\\
u\psi_{xx}+R\zeta_{xx}+\dfrac{R\theta}{\rho}\phi_{xx}=Q_6-\psi_{tx},\\
C_vu\zeta_{xx}+R\theta\psi_{xx}=Q_7-C_v\zeta_{tx}-w_{xx}.
\end{cases}
\tag{4.39}
$$

Therefore,

$$
\begin{aligned}
&\int_0^t|(\phi_{xx},\psi_{xx},\zeta_{xx})|^2(0,\tau)d\tau\\
\lesssim&\int_0^t|(\phi_{tx},\psi_{tx},\zeta_{tx},w_{xx})|^2(0,\tau)d\tau+\int_0^t|(Q_5,Q_6,Q_7)|^2(0,\tau)d\tau\\
\lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2\\
&+\mathrm{e}^{-\delta\beta}+\big(\delta+N(t)\big)\int_0^t\|(\phi,\psi,\zeta)(\tau)\|_2^2d\tau,
\end{aligned}
\tag{4.40}
$$

$$
\begin{aligned}
&\int_0^t |\omega_{xxx}|^2(0,\tau)d\tau\\
\lesssim &\int_0^t |(\omega_x,\zeta_{xx},\theta_x^s\zeta_x,\zeta,\zeta_x,Q_8)|^2(0,\tau)d\tau\\
\lesssim &\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2\\
&\quad +\mathrm{e}^{-\delta\beta}+\big(\delta+N(t)\big)\int_0^t \|(\phi,\psi,\zeta)(\tau)\|_2^2 d\tau. \qquad (4.41)
\end{aligned}
$$

Then we have the following estimates.

Lemma 4.5. *Under the assumption of Proposition* 2.2, *if* $N(t)$ *and* δ *are suitably small, it holds that*

$$
\begin{aligned}
&\|(\phi_{xx},\psi_{xx},\zeta_{xx})(t)\|^2+\int_0^t \Big(|(\phi_{xx},\psi_{xx},\zeta_{xx},w_{xxx})|^2(0,\tau)+\|w_{xx}(\tau)\|_1^2\Big)d\tau\\
\lesssim &\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2+\mathrm{e}^{-\delta\beta}\\
&+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\big(\delta+N(t)\big)\int_0^t \|(\phi,\psi,\zeta)(\tau)\|_2^2 d\tau. \qquad (4.42)
\end{aligned}
$$

Proof. Differentiating the equations in (2.14) with respect to x twice, then multiplying each of them by $\frac{R\theta}{\rho}\phi_{xx}$, $\rho\psi_{xx}$, $-\frac{\rho}{\theta}\zeta_{xxx}$ and $-\frac{\rho}{4b\theta^4}w_{xxx}$ respectively, and adding the resulting equations together, we get

$$
\begin{aligned}
&\left(\frac{R\theta}{\rho}\frac{\phi_{xx}^2}{2}+\rho\frac{\psi_{xx}^2}{2}+\frac{C_v\rho}{2\theta}\zeta_{xx}^2\right)_t+\frac{a\rho\omega_{xx}^2}{4b\theta^4}+\rho\frac{\omega_{xxx}^2}{4b\theta^4}+\Big(\frac{a\rho}{4b\theta^4}\Big)_x\omega_{xx}\omega_{xxx}+J_{9x}\\
=&O(1)\big(\delta+N(t)\big)|(\phi_{xx},\psi_{xx},\zeta_{xx},\omega_{xxx})||(\phi_{xx},\psi_{xx},\zeta_{xx},\zeta_{tx},\phi_x,\psi_x,\zeta_x,\phi,\psi,\zeta)|, \quad (4.43)
\end{aligned}
$$

where

$$
\begin{aligned}
J_9:=&\frac{R\theta u}{\rho}\frac{\phi_{xx}^2}{2}+\rho u\frac{\psi_{xx}^2}{2}+R\theta\psi_{xx}\phi_{xx}-\frac{C_v\rho u}{2\theta}\zeta_{xx}^2\\
&-\frac{C_v\rho}{\theta}\zeta_{tx}\zeta_{xx}-\frac{\rho}{\theta}\omega_{xx}\zeta_{xx}-\frac{a\rho w_x w_{xx}}{4b\theta^4}\\
=&O(1)|(\phi_{xx},\psi_{xx},\zeta_{xx},\zeta_{tx},\omega_x,\omega_{xx})|^2. \qquad (4.44)
\end{aligned}
$$

Integrating (4.43) over $\mathbb{R}_+\times[0,t]$ and using (4.40) with (4.41), we can get (4.42). This completes the proof. □

Combining Lemmas 3.1-3.2 and 4.1-4.5 together, we can get

$$
\|(\Phi,\Psi,W)(t)\|^2+\|(\phi,\psi,\zeta)(t)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(t)\|_1^2+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(t)\|^2
$$

$$
\begin{aligned}
&+\int_0^t |(\Phi,\Psi,W)|^2(0,\tau)d\tau+\int_0^t\int_{\mathbb{R}_+}\left(|u_x^s||(\Psi,W)|^2+\Gamma^2+\Gamma_x^2\right)dxd\tau\\
&+\int_0^t |(\phi,\phi_x,\psi_x,\zeta_x,w_x,\phi_{xx},\psi_{xx},\zeta_{xx},w_{xx},w_{xxx},\phi_{tx},\psi_{tx},\zeta_{tx},w_{tx},w_{txx})|^2(0,\tau)d\tau\\
&+\int_0^t\left(\|w(\tau)\|_3^2+\|w_t(\tau)\|_2^2\right)d\tau\\
\lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_1^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2\\
&+\mathrm{e}^{-\delta\beta}+\left(\delta+N(t)\right)\int_0^t\|(\phi,\psi,\zeta)(\tau)\|_2^2d\tau. \qquad (4.45)
\end{aligned}
$$

Finally in this section, we will deal with the term $\int_0^t\|(\phi,\psi,\zeta)(\tau)\|_2^2d\tau$, which appears in the right hand side of estimate (4.45).

Lemma 4.6. *Under the assumptions of Proposition* 2.2*, if* $N(t)$ *and* δ *are suitably small, it holds that*

$$
\begin{aligned}
\int_0^t\|(\phi,\psi,\zeta)(\tau)\|_2^2d\tau\lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2 \qquad (4.46)\\
&+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\mathrm{e}^{-\delta\beta}.
\end{aligned}
$$

Proof. Multiplying the second and the third equations in (2.39) by $R\theta^s\Phi_x$ and $2\Psi_x$, respectively, we get

$$
\begin{aligned}
&(R\theta^s\Psi\Phi_x)_t-R\theta_t^s\Psi\Phi_x-(R\theta^s\Psi\Phi_t)_x+R\theta_x^s\Psi\Phi_t\\
&\quad-R\theta^s\Psi_x^2-R\theta^s u_x^s\Phi\Psi_x+(R\theta^s\Phi_x)^2\\
=&\left(G_1-RW_x-(\gamma-1)u_x^s\Psi+\frac{R\theta^s}{\rho^s}\rho_x^s\Phi\right)R\theta^s\Phi_x, \qquad (4.47)\\
&2C_v\left[(W\Psi_x)_t-(W\Psi_t)_x\right]+2C_vW_x(\Psi_t+u^s\Psi_x)+2R\theta^s\Psi_x^2\\
&\quad-2\Psi_x\left(\frac{q_x^s}{\rho^s}\Phi+\frac{p_x^s}{\rho^s}\Psi-\Gamma_x\right)=2G_2\Psi_x. \qquad (4.48)
\end{aligned}
$$

Then

$$
\begin{aligned}
&(R\theta^s\Psi\Phi_x+2C_vW\Psi_x)_t-(R\theta^s\Psi\Phi_t+2C_vW\Psi_t)_x\\
&\quad+R\theta^s\Psi_x^2+(R\theta^s\Phi_x)^2-R\theta_t^s\Psi\Phi_x+R\theta_x^s\Psi\Phi_t\\
&\quad-R\theta^s u_x^s\Phi\Psi_x+2C_vW_x(\Psi_t+u^s\Psi_x)\\
=&\left[G_1-RW_x-(\gamma-1)u_x^s\Psi+\frac{R\theta^s}{\rho^s}\rho_x^s\Phi\right]R\theta^s\Phi_x\\
&\quad+2\Psi_x\left(\frac{q_x^s}{\rho^s}\Phi+\frac{p_x^s}{\rho^s}\Psi-\Gamma_x\right)+2G_2\Psi_x. \qquad (4.49)
\end{aligned}
$$

Multiplying the fourth equation in (2.39) by W_x, we get

$$\frac{4b(\theta^s)^3}{\rho}W_x^2 \lesssim \Gamma_{xx}^2+\Gamma^2+(\theta_x^s\Phi)^2+(u_x^s\Psi)^2+G_3^2. \tag{4.50}$$

Hence, integrating (4.49) over $\mathbb{R}_+\times[0,t]$ and by Lemma 2.1 and estimate (4.50), we obtain that

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}_+}\Big(R\theta^s\Psi_x^2+(R\theta^s\Phi_x)^2+W_x^2\Big)dxd\tau \\
\lesssim&\|(\Psi,\Phi_x,W,\Psi_x)(t)\|^2+\|(\Psi,\Phi_x,W,\Psi_x)(0)\|^2+\mathrm{e}^{-c\delta\beta} \\
&+\int_0^t|(\Phi,\Psi,W)|^2(0,\tau)d\tau+\int_0^t\int_{\mathbb{R}_+}\Big(|u_x^s||(\Psi,W)|^2+\Gamma_{xx}^2+\Gamma^2+(u_x^s)^2\Big)dxd\tau \\
&+\int_0^t|(\Phi_x,\Psi_x,W_x)|^2(0,\tau)d\tau+\big(\delta+N(t)\big)\int_0^t\|(\phi_x,\psi_x,\zeta_x)(\tau)\|^2d\tau.
\end{aligned}\tag{4.51}$$

Recalling the relation between (ϕ,ψ,ζ) and (Φ,Ψ,W) in (2.18), we know

$$\phi^2=\Phi_x^2,\quad \psi^2\lesssim\Psi_x^2+(u_x^s\Phi)^2,\quad \zeta^2\lesssim W_x^2+(\theta_x^s\Phi)^2+(u_x^s\Psi)^2+\rho^2\psi^4.$$

Therefore

$$\begin{aligned}
&\int_0^t\|(\phi,\psi,\zeta)(\tau)\|^2d\tau \\
\lesssim&\|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2 \\
&+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\mathrm{e}^{-\delta\beta}+\big(\delta+N(t)\big)\int_0^t\|(\phi_x,\psi_x,\zeta_x)(\tau)\|_1^2d\tau.
\end{aligned}\tag{4.52}$$

Furthermore, it follows from $(3.11)_4$ that

$$|\zeta_x(x,t)|^2\lesssim|(\omega_x,\omega,\theta_x^s\zeta,q_{xx})(x,t)|^2. \tag{4.53}$$

In addition, multiplying $(3.11)_2$ and $(3.11)_3$ by $\frac{R\theta}{\rho}\phi_x$ and $2\psi_x$, respectively, we obtain

$$\begin{aligned}
&\left(\frac{R\theta}{\rho}\psi\phi_x+2C_v\zeta\psi_x\right)_t-\left(\frac{R\theta}{\rho}\psi\phi_t+2C_v\zeta\psi_t\right)_x+\left(\frac{R\theta}{\rho}\phi_x\right)^2+R\theta\psi_x^2 \\
=&\left(\frac{R\theta}{\rho}\right)_t\psi\phi_x+\left(\frac{R\theta}{\rho}\right)_x\psi(u\phi_x+\rho\psi_x-Q_1)+C_vR\zeta_x^2 \\
&+O(1)\big(|\zeta_x\phi_x|+|\psi_x\omega_x|+|(\phi_x,\zeta_x)Q_2|+|\psi_x(Q_1,Q_3)|\big).
\end{aligned}\tag{4.54}$$

Integrating (4.54) over $\mathbb{R}_+ \times [0,t]$ and using (4.45) and (4.52) with (4.53), we have

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}_+} \left(\psi_x^2+\psi_x^2+\zeta_x^2\right) dx d\tau \\
&\lesssim \|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2 \\
&\quad +\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\mathrm{e}^{-\delta\beta}+\left(\delta+N(t)\right)\int_0^t \|(\phi,\psi,\zeta)(\tau)\|_2^2 d\tau. \qquad (4.55)
\end{aligned}$$

Next, from (4.37d), it holds that

$$|\zeta_{xx}(x,t)|^2 \lesssim |(\omega_{xxx},\omega_x,\theta_x^s\zeta_x,Q_8)(x,t)|^2. \qquad (4.56)$$

Finally, multiplying (4.37b) and (4.37c) by $\frac{R\theta}{\rho}\phi_{xx}$ and $2\psi_{xx}$, respectively, we obtain

$$\begin{aligned}
&\int_0^t \int_{\mathbb{R}_+} \left(\psi_{xx}^2+\psi_{xx}^2+\zeta_{xx}^2\right) dx d\tau \\
&\lesssim \|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2 \\
&\quad +\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\mathrm{e}^{-\delta\beta}+\left(\delta+N(t)\right)\int_0^t \|(\phi,\psi,\zeta)(\tau)\|_2^2 d\tau. \qquad (4.57)
\end{aligned}$$

Combining the estimates (4.52), (4.55) and (4.57) together, we can get (4.46). □

5 Estimates on radiative perturbation

In this section, we focus on the proof of the estimates related to the rediative heat flux.

Lemma 5.1. *Under the assumptions in Proposition* 2.2, *if $N(t)$ and δ are suitably small, it holds that*

$$\begin{aligned}
&\|\Gamma(t)\|^2+\|w(t)\|_3^2 \\
&\lesssim \|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2 \\
&\quad +\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2+\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\mathrm{e}^{-\delta\beta}. \qquad (5.1)
\end{aligned}$$

Proof. By the last equation of (2.39), we see that

$$-\Gamma_{xx}+a\Gamma+b\zeta(\theta+\theta^s)\left[\theta^2+\theta\theta^s+(\theta^s)^2\right]=q^s. \qquad (5.2)$$

Multiplying Eq. (5.2) by Γ, we have

$$-(\Gamma_x\Gamma)_x+\Gamma_x^2+a\Gamma^2+b\Gamma\zeta(\theta+\theta^s)\left[\theta^2+\theta\theta^s+(\theta^s)^2\right]=q_x^s\Gamma. \qquad (5.3)$$

Integrating (5.2) over $[0,+\infty)$, and choosing δ and $N(t)$ suitably small, we have

$$\|(\Gamma,\Gamma_x)(t)\|^2 \lesssim \int_0^\infty \left[\zeta^2+(q_x^s)^2\right]dx+|\Gamma_x\Gamma(0,t)|. \tag{5.4}$$

Multiplying $(2.14)_4$ by $-w_{xx}$, we have

$$\begin{aligned}&-(aww_x)_x+w_{xx}^2+aw_x^2\\&=\left\{4b\theta^3\zeta_x+4b\theta_x^s\zeta-\left[\theta^2+\theta\theta^s+(\theta^s)^2\right]-q_{xx}^s\right\}w_{xx}.\end{aligned} \tag{5.5}$$

Integrating (5.5) over $[0,+\infty)$, and choosing δ and $N(t)$ suitably small, we have

$$\|(w_x,w_{xx})(t)\|^2 \lesssim \int_0^\infty \left[\zeta_x^2+\zeta^2+(q_{xx}^s)^2\right]dx+|aw_xw(0,t)|. \tag{5.6}$$

Multiplying (2.14d) by $-w_{xxx}$, we have

$$\begin{aligned}&-(aw_xw_{xx})_x+w_{xxx}^2+aw_{xx}^2\\&=\left\{4b\theta^3\zeta_x+4b\theta_x^s\zeta\left[\theta^2+\theta\theta^s+(\theta^s)^2\right]-q_{xx}^s\right\}_x w_{xxx}.\end{aligned} \tag{5.7}$$

Integrating (5.7) over $[0,+\infty)$, and choosing δ and $N(t)$ suitably small, we have

$$\|(w_{xx},w_{xxx})(t)\|^2 \lesssim \int_0^\infty \left[\zeta_{xx}^2+\zeta_x^2+\zeta^2+(q_{xxx}^s)^2\right]dx+|w_{xx}w_x(0,t)|. \tag{5.8}$$

Combining the estimates (5.4), (5.6) and (5.8) together, one has

$$\begin{aligned}&\|\Gamma(t)\|^2+\|w(t)\|_3^2\\&\lesssim \|\zeta(t)\|_2^2+\int_0^\infty \left[(q_x^s)^2+(q_{xx}^s)^2+(q_{xxx}^s)^2\right]dx\\&\quad+|\Gamma_x\Gamma(0,t)|+|aw_xw(0,t)|+|aw_{xx}w_x(0,t)|.\end{aligned} \tag{5.9}$$

The last three terms on the right-hand side of (5.9) are estimated as follows one by one:

$$\begin{aligned}|\Gamma_x\Gamma(0,t)|&=|w(0,t)||\Gamma(0,t)|\lesssim |q^s(0,t)|\|\Gamma(t)\|_\infty\\&\lesssim \delta\|\Gamma(t)\|^{\frac12}\|\Gamma_x(t)\|^{\frac12}\lesssim \delta\|\Gamma(t)\|^2+\delta\|\Gamma_x(t)\|^{\frac23}\\&\lesssim \delta\|(\Gamma,\Gamma_x)(t)\|^2+\delta,\end{aligned} \tag{5.10}$$

$$\begin{aligned}|w_xw|(0,t)&\lesssim |q^s(0,t)|\|w_x(t)\|_\infty\\&\lesssim \delta\|w_x(t)\|^{\frac12}\|w_{xx}(t)\|^{\frac12}\lesssim \delta\|w_x(t)\|^2+\delta\|w_{xx}(t)\|^{\frac23}\\&\lesssim \delta\|(w_x,w_{xx})(t)\|^2+\delta,\end{aligned} \tag{5.11}$$

$$
\begin{aligned}
|w_{xx}w_x(0,t)| &= \left| w_x \left\{ aw + 4b\theta^3\zeta_x + 4b\theta_x^s\zeta\left[\theta^2 + \theta\theta^s + (\theta^s)^2\right] - q_{xx}^s \right\} \right| (0,t) \\
&\lesssim |w_x(q_x^s, q_{xx}^s)|(0,t) + |w_x(\zeta_x,\zeta)|(0,t) \\
&\lesssim \delta\|(w_x,w_{xx})(t)\|^2 + \delta + \|w_x(t)\|_\infty \|(\zeta_x,\zeta)(t)\|_\infty \\
&\lesssim \delta\|(w_x,w_{xx})(t)\|^2 + \delta + \|w_x(t)\|^{\frac{1}{2}} \|w_{xx}(t)\|^{\frac{1}{2}} \\
&\quad \times \left(\|(\zeta_x,\zeta)(t)\|^{\frac{1}{2}} + \|(\zeta_x,\zeta_{xx})(t)\|^{\frac{1}{2}} \right) \\
&\lesssim \left(\delta + \frac{1}{4} \right) \|(w_x,w_{xx})(t)\|^2 + \delta + \|\zeta(t)\|_2^2. \qquad (5.12)
\end{aligned}
$$

Inserting (5.10)-(5.12) into (5.9), we can get (5.1). This completes the proof of Lemma 5.1. $\square$

Lemma 5.2. *Under the assumptions of Proposition* 2.2*, if* $N(t)$ *and* δ *are suitably small, it holds that*

$$
\begin{aligned}
&\|w_t(t)\|_2^2 + \|w_{tt}(t)\|_1^2 \\
\lesssim &\|(\Phi_0,\Psi_0,W_0)\|^2 + \|(\phi_0,\psi_0,\zeta_0)\|_2^2 \\
&\quad + \|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2 + \|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2 + \mathrm{e}^{-\delta\beta}. \qquad (5.13)
\end{aligned}
$$

Proof. Multiplying (4.19d) by w_t, we get

$$
\begin{aligned}
&-(w_{tx}w_t)_x + w_{tx}^2 + aw_t^2 \\
=&-\left\{ 4b\theta^3\zeta_{tx} + 4b\theta_x^s\zeta_t\left[\theta^2 + \theta\theta^s + (\theta^s)^2\right] - \widetilde{Q}_4 \right\} w_t. \qquad (5.14)
\end{aligned}
$$

Multiplying (4.19d) by $-w_{txx}$, we get

$$
\begin{aligned}
&-(aw_{tx}w_t)_x + w_{txx}^2 + aw_{tx}^2 \\
=&\left\{ 4b\theta^3\zeta_{tx} + 4b\theta_x^s\zeta_t\left[\theta^2 + \theta\theta^s + (\theta^s)^2\right] - \widetilde{Q}_4 \right\} w_{txx}. \qquad (5.15)
\end{aligned}
$$

Integrating (5.14) and (5.15) over $[0,+\infty)$, and choosing δ and $N(t)$ suitably small, it holds

$$
\|w_t(t)\|_2^2 \lesssim \int_0^\infty \left(\zeta_{tx}^2 + \zeta_t^2 + \zeta^2 + \widetilde{Q}_4^2 \right) dx + |w_{tx}w_t(0,t)|, \qquad (5.16)
$$

where the last terms on the right-hand side of (5.16) are estimated as follows:

$$
\begin{aligned}
&|w_{tx}w_t(0,t)| \lesssim |q_t^s(0,t)| \|w_{tx}\|_\infty \\
\lesssim &|q_t^s(0,t)| \|w_{tx}\|^{\frac{1}{2}} \|w_{txx}\|^{\frac{1}{2}} \lesssim \delta \left(\|w_{txx}\|^2 + \|w_{tx}\|^2 \right) + \delta. \qquad (5.17)
\end{aligned}
$$

Plugging (5.17) into (5.16), we get

$$\begin{aligned}\|w_t(t)\|_2^2 \lesssim & \|(\Phi_0,\Psi_0,W_0)\|^2+\|(\phi_0,\psi_0,\zeta_0)\|_2^2+\|(\phi_t,\psi_t,\zeta_t)(0)\|_1^2 \\ & +\|(\phi_{tt},\psi_{tt},\zeta_{tt})(0)\|^2+\mathrm{e}^{-\delta\beta}. \end{aligned} \tag{5.18}$$

On the other hand, multiplying (4.19d) by $-w_{tt}$, we get

$$\begin{aligned} & -(aw_{ttx}w_{tt})_x+w_{ttx}^2+aw_{tt}^2 \\ = & 12b\theta^2\theta_t\zeta_{tx}w_{tt}+\left(4b\theta^3\zeta_{tt}w_{tt}\right)_x-12b\theta^2\theta_x\zeta_{tt}w_{tt}-4b\theta^3\zeta_{tt}w_{ttx} \\ & +\left\{4b\theta_x^s\zeta_t\left[\theta^2+\theta\theta^s+(\theta^s)^2\right]-\widetilde{Q}_4\right\}_t w_{tt}. \end{aligned} \tag{5.19}$$

Integrating (5.19) over $[0,+\infty)$, and choosing δ and $N(t)$ suitably small, it holds

$$\begin{aligned}\|w_{tt}(t)\|_1^2 \lesssim & \|(\zeta,\phi,\psi)(t)\|_2^2+|w_{tt}\zeta_{tt}(0,t)| \\ \lesssim & \|(\zeta,\phi,\psi)(t)\|_2^2+\delta. \end{aligned} \tag{5.20}$$

Combining (5.16) and (5.18) together, we can get (5.13). This completes the proof of Lemma 5.2. □

Now we can prove Proposition 2.2 to conclude the proof of Theorem 2.1 by Propositions 2.1 and 2.2.

Proof of Proposition 2.2. Using the results (4.45), (4.46), (5.1), and (5.13), we can obtain (2.48). This completes the proof of Proposition 2.2. □

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