Commun. Math. Anal. Appl.
doi: 10.4208/cmaa.2022-0001

Vol. **1**, No. 2, pp. 241-262
May 2022

Self-Similar Solutions of Leray's Type for Compressible Navier-Stokes Equations in Two Dimension

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Received 2 January 2022; Accepted 13 January 2022

Abstract. We study the backward self-similar solution of Leray's type for compressible Navier-Stokes equations in dimension two. The existence of weak solutions is established via a compactness argument with the help of an higher integrability of density. Moreover, if the density belongs to $L^\infty(\mathbb{R}^2)$ and the velocity belongs to $L^2(\mathbb{R}^2)$, the solution is trivial; that is $(\rho,\mathbf{u})=0$.

AMS subject classifications: 35A05, 76A10, 76D03

Key words: Navier-Stokes equations, self-similar solutions, compressible.

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1 Introduction

This work concerns the backward self-similar solution of Leray's type for compressible isentropic Navier-Stokes equations with a positive number M in $\mathbb{R}^n$ as the adiabatic constant $\gamma>\frac{n}{2}$

$$\begin{cases} \partial_t\rho+\operatorname{div}(\rho\mathbf{u})=0, \\ \partial_t(\rho\mathbf{u})+\operatorname{div}(\rho\mathbf{u}\otimes\mathbf{u})-\mu\Delta\mathbf{u}-\xi\nabla\operatorname{div}\mathbf{u}+M\nabla\rho^\gamma=0, \\ (\rho,\mathbf{u})|_{t=0}=(\rho_0,\mathbf{u}_0), \end{cases} \tag{1.1}$$

where $\mu>0$, $\xi\in\mathbb{R}$ with $\mu+\xi>0$, ρ and $\mathbf{u}$ stand for the density and velocity of the flow, respectively. The positive constant M in (1.1) is the squared inverse of

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the Mach number. The weak solution of (1.1) in $\mathbb{R}^n$ had been established first by Lions [15] for $\gamma\geq\frac{3}{2}$ as $n=2$ and $\gamma\geq\frac{9}{5}$ as $n=3$, and later extended by Feireisl *et al.* [6] to allow $\gamma>\frac{n}{2}$ in the Leray sense, see also [21] for $\gamma=1$ in dimension two and [1] for the general pressure in this direction.

Definition 1.1. *A couple* $(\rho,\mathbf{u})$ *with* $\rho\in L^\infty(0,T;L^\gamma(\mathbb{R}^n))$ *and* $\nabla\mathbf{u}\in L^2((0,T)\times\mathbb{R}^n)$ *is said to be a renormalized weak solution to* (1.1) *with finite energy if* $(\rho,\mathbf{u})$ *satisfies the following:*

- *The kinetic energy is bounded, i.e.,* $\rho|\mathbf{u}|^2\in L^\infty(0,T;L^1(\mathbb{R}^n))$*. The density function is nonnegative, i.e.,* $\rho\geq 0$.
- *The momentum equation holds true in the sense of distributions.*
- *The identity*
$$\partial_t b(\rho)+\mathrm{div}\big(\mathbf{u}b(\rho)\big)=\big(b(\rho)-b'(\rho)\rho\big)\mathrm{div}\mathbf{u} \tag{1.2}$$
holds true in the sense of distributions for any function $b(t)\in C([0,\infty)\cap C^1(\mathbb{R}^+)$ *such that* $b(t)-b'(t)t\in C([0,\infty))$ *is bounded on* $[0,\infty)$.
- *The global energy inequality*
$$E\big(\rho(t),\mathbf{u}(t)\big)+\int_0^t\int_{\mathbb{R}^n}\Big(\mu|\nabla\mathbf{u}|^2+\xi|\mathrm{div}\mathbf{u}|^2\Big)dxds\leq E(\rho_0,\mathbf{u}_0) \tag{1.3}$$
holds true with
$$E(\rho,\mathbf{u})=\begin{cases}\dfrac{1}{2}\|\rho|\mathbf{u}|^2\|_{L^1(\mathbb{R}^n)}+\dfrac{M}{\gamma-1}\displaystyle\int_{\mathbb{R}^n}\rho^\gamma dx, & \text{if} \quad \gamma>1,\\ \dfrac{1}{2}\|\rho|\mathbf{u}|^2\|_{L^1(\mathbb{R}^n)}+M\displaystyle\int_{\mathbb{R}^n}\rho\ln(1+\rho)dx & \text{if} \quad \gamma=1.\end{cases}$$

It is worth noticing that a number of efforts have been paid on the existence of weak solutions for both the time-discretized and the steady counterparts of (1.1) as $\gamma\in[1,\frac{n}{2}]$, see for instance [4,12,13,19]. For the time-discretized version of (1.1), the concentration-cancellation occurs in dimension two (see [15, Section 6.6]) and in dimension three as $\gamma>1$ (see [20]); while the concentration set has been verified to be $(\mathcal{H}^1,1)$ rectifiable in dimension three as $\gamma=1$ in [20], where $\mathcal{H}^1$ stands for the one dimensional Hausdorff measure. For the steady version of (1.1) as $\gamma=1$, the concentration cancellation in dimension two has been verified in the work [7] due to a potential type estimate of the density (see [7, Lemma 3.1]), also see [14,17,22,23] for the most recent improvement along this direction in dimension three. For time-dependent version (1.1), in [10] the author considered the cylindric Hausdorff dimension of the concentration for global weak solutions as $\gamma\in[1,\frac{n}{2}]$ in multi-dimensional spaces, while the concentration of the kinetic energy is characterised in terms of parabolic Hausdorff measure in [11] with the help of the decay rate of the gradient of velocity.

A closely related issue for the existence of weak solutions is the regularity of weak solutions. This issue has been studied in [16,24] for Leray-Hopf's weak solutions of incompressible Navier-Stokes equations in dimension three. The onset in [16,24] is based on the original suggestion, due to Leray, to look for a possible singular solution by studying backward self-similar solutions. The backward or forward self-similar solution is motivated in terms of the scaling-invariance property of Navier-Stokes equations. Indeed, the classical dimensional analysis suggests that if $(\rho,\mathbf{u})$ solves (1.1) with a positive number M, then for each $\lambda>0$,

$$\mathbf{u}_\lambda(x,t)=\lambda\mathbf{u}\big(\lambda x,\lambda^2 t\big),\quad \rho_\lambda(x,t)=\rho\big(\lambda x,\lambda^2 t\big),\quad M_\lambda=\lambda^2 M \tag{1.4}$$

also solve (1.1) with a positive number M_λ. In dimension two, as in [16,24], this scaling-invariance property suggests us to study a backward self-similar solution of the form

$$\begin{aligned}
&\mathbf{u}(x,t)=\frac{1}{\sqrt{2(T-t)}}\tilde{\mathbf{u}}\left(\frac{x}{\sqrt{2(T-t)}}\right),\\
&\rho(x,t)=\tilde{\rho}\left(\frac{x}{\sqrt{2(T-t)}}\right),\quad M=\frac{\tilde{M}}{2(T-t)},
\end{aligned}\tag{1.5}$$

where $T\in\mathbb{R}$, $\tilde{\rho}\in\mathbb{R}$ and $\tilde{\mathbf{u}}\in\mathbb{R}^2$ are defined in $\mathbb{R}^2$. Hence $(\rho,\mathbf{u})$ is defined in $\mathbb{R}^2\times(-\infty,T)$ and if $(\tilde{\rho},\tilde{\mathbf{u}})\neq 0$, then $(\rho,\mathbf{u})$ develops a singularity at $t=T$. In terms of the variables $(\tilde{\rho},\tilde{\mathbf{u}})$, the original system (1.1) turns to be a steady problem, after dropping the tilde,

$$\begin{cases}
y\cdot\nabla\rho+\operatorname{div}(\rho\mathbf{u})=0, & \text{(1.6a)}\\
\rho\mathbf{u}+\rho y\cdot\nabla\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}-\mu\Delta\mathbf{u}-\xi\nabla\operatorname{div}\mathbf{u}+M\nabla\rho^\gamma=0, & \text{(1.6b)}
\end{cases}$$

where $\rho=\rho(y)$ and $\mathbf{u}=\mathbf{u}(y)$ for $y\in\mathbb{R}^2$. We remark that the radial forward self-similar smooth solution to the full compressible Navier-Stokes equation had been studied recently in [8,9] in dimension three.

Similar to (1.3), the global energy law of (1.6) takes the form

$$\nu\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2dx+\xi\int_{\mathbb{R}^2}|\operatorname{div}\mathbf{u}|^2dx\leq\frac{2M}{\gamma-1}\int_{\mathbb{R}^2}\rho^\gamma dx. \tag{1.7}$$

In terms of the conservation of mass and the scaling (1.4), the quantity $M\rho$ is invariant in $L^1(\mathbb{R}^2)$ and hence belongs to $L^1(\mathbb{R}^2)$. Moreover, the quantities

$$M\int_{\mathbb{R}^2}\rho^\gamma dy\quad\text{and}\quad\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2dy$$

are invariant with respect to the time variable t in terms of the scaling (1.5) since $L^2(\mathbb{R}^2)$ is invariant under the scaling $\lambda f(\lambda\cdot)$ as in (1.4). Hence, it is reasonable to assume that the quantity

$$\int_{\mathbb{R}^2}\left(M\rho^\gamma+\rho|\mathbf{u}|^2\right)dy \tag{1.8}$$

is bounded for the solution to (1.6). With this bound at hand, the first result of this work concerns the existence of weak solutions $(\rho,\mathbf{u})$ to (1.6) and the result reads as follows.

Theorem 1.1. *As $\gamma>\frac{3}{2}$, if $\rho\in L^1(\mathbb{R}^2)\cap L^\gamma(\mathbb{R}^2)$ and $\rho|\mathbf{u}|^2\in L^1(\mathbb{R}^2)$, then there exists a weak solution $(\rho,\mathbf{u})$ of* (1.6) *which satisfies*

- $\rho\in L^{2\gamma}_{loc}(\mathbb{R}^2)$;
- $\nabla\mathrm{curl}\mathbf{u},\ \nabla\left\{\mathrm{div}\mathbf{u}-\frac{M}{\mu+\xi}\rho^\gamma\right\}\in L_{loc}^{\frac{3(\gamma-1)}{2\gamma-1}}$.

Moreover, the weak solution $(\rho,\mathbf{u})$ satisfies the following local inequality:

$$\begin{aligned}
&\mu\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\phi dy+\xi\int_{\mathbb{R}^2}|\mathrm{div}\mathbf{u}|^2\phi dy\\
\le&\frac{\mu}{2}\int_{\mathbb{R}^2}|\mathbf{u}|^2\Delta\phi dy-\xi\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\phi\mathrm{div}\mathbf{u}dy\\
&+\frac{1}{2}\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2(y+\mathbf{u})\cdot\nabla\phi dy+\frac{2M}{\gamma-1}\int_{\mathbb{R}^2}\rho^\gamma\phi dy\\
&+\frac{M}{\gamma-1}\int_{\mathbb{R}^2}\rho^\gamma(y+\gamma\mathbf{u})\cdot\nabla\phi dy
\end{aligned} \tag{1.9}$$

for any smooth function ϕ.

The main difficulties for the weak stability of (1.6) are the concentration and oscillation of density. Following [15], as $\gamma>\frac{3}{2}$, with the help of the estimate (1.8), the higher integrability of density, $\rho\in L^{2\gamma}_{loc}$, will be justified and this higher integrability implies that the concentration of the energy $\frac{1}{2}\rho|\mathbf{u}|^2+\frac{M}{\gamma-1}\rho^\gamma$ never happens, and the key issue for the construction of weak solutions for (1.6) is the possible oscillation of the pressure term $P(\rho)=\rho^\gamma$. With the aid of the higher integrability, the oscillation issue will be overcome by a convex argument in the spirit of [5, 6, 15, 18] with the help of effective viscous flux in the framework of renormalized solutions.

The second result guarantees that the weak solution established in Theorem 1.1 is indeed a trivial solution if the density belongs to L^∞ and the velocity belongs to $L^2(\mathbb{R}^2)$.

Theorem 1.2. *If* $\rho\in L^\infty(\mathbb{R}^2)$ *and* $\mathbf{u}\in L^2(\mathbb{R}^2)$*, then the weak solution* $(\rho,\mathbf{u})$ *in Theorem* 1.1 *is zero, that is* $(\rho,\mathbf{u})=0$.

Theorem 1.2 coincides with the regularity result of weak solutions for two dimensional compressible Navier-Stokes equation (1.1) in [3] when the density is a priori bounded in L^∞. Keeping in mind, the backward self-similar solutions considers the possible singularity at some positive time $T>0$, the H^1 bound of the initial velocity for the result in [3] is automatically satisfied at some time $T_0\in(0,T)$ in view of the energy (1.3). On the other hand, Theorem 1.2 also provides a similar result in dimension two as that in [8, 9] for three dimensional cases when the density is bounded in L^∞. A combination of Theorems 1.1 and 1.2 guarantees that if the density is bounded in L^∞ and the velocity is bounded in $L^2(\mathbb{R}^2)$, there is no backward self-similar weak solution for (1.1) in dimension two.

The rest of this paper is organized as follows. In Section 2, Theorem 1.1 is established via a three-layer approximation with the help of uniform estimates and compactness arguments. Section 3 is devoted to the regularity of weak solutions and as a byproduct, Theorem 1.2 will be justifies. Throughout this work, the notation $A\lesssim B$ means $A\le CB$ for some harmless nonnegative constant C. Moreover, we denote by $\overline{\phi}$ the weak limit of a sequence ϕ_n in L^1.

2 Preliminary

First of all, we establish the global energy law (1.7).

Lemma 2.1. *For smooth solutions of* (1.6)*, the global energy law* (1.7) *holds true.*

Proof. We multiply Eq. (1.6b) by $\mathbf{u}$ to get

$$\begin{aligned}&\mu\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2dy+\xi\int_{\mathbb{R}^2}|\mathrm{div}\mathbf{u}|^2dy+\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2dy\\ =&-\frac{1}{2}\int_{\mathbb{R}^2}\left(\rho y\cdot\nabla|\mathbf{u}|^2+\rho\mathbf{u}\cdot\nabla|\mathbf{u}|^2\right)dy-M\int_{\mathbb{R}^2}\nabla\rho^\gamma\cdot\mathbf{u}dy.\end{aligned}\tag{2.1}$$

For the first term in the right-hand side of (2.1), using Eq. (1.6a) and applying the integration by parts, we get

$$\begin{aligned}&\int_{\mathbb{R}^2}\left(\rho y\cdot\nabla|\mathbf{u}|^2+\rho\mathbf{u}\cdot\nabla|\mathbf{u}|^2\right)dy\\ =&-\int_{\mathbb{R}^2}\left(y\cdot\nabla\rho+\mathrm{div}(\rho\mathbf{u})\right)|\mathbf{u}|^2dy-2\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2dy\\ =&-2\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2dy.\end{aligned}$$

For the second term in the right-hand side of (2.1), integrating by parts and using Eq. (1.6a), one has

$$\begin{aligned}\int_{\mathbb{R}^2}\nabla\rho^\gamma\cdot\mathbf{u}dy&=\int_{\mathbb{R}^2}\gamma\rho^{\gamma-1}\big(\operatorname{div}(\rho\mathbf{u})-\rho\operatorname{div}\mathbf{u}\big)dy\\&=-\int_{\mathbb{R}^2}\gamma\rho^{\gamma-1}y\cdot\nabla\rho dy-\int_{\mathbb{R}^2}\gamma\rho^\gamma\operatorname{div}\mathbf{u}dy\\&=-\int_{\mathbb{R}^2}y\cdot\nabla\rho^\gamma dy+\gamma\int_{\mathbb{R}^2}\nabla\rho^\gamma\cdot\mathbf{u}dy,\end{aligned}$$

and hence

$$\int_{\mathbb{R}^2}\nabla\rho^\gamma\cdot\mathbf{u}dy=\frac{1}{\gamma-1}\int_{\mathbb{R}^2}y\cdot\nabla\rho^\gamma dy=-\frac{2}{\gamma-1}\int_{\mathbb{R}^2}\rho^\gamma dy.$$

Substituting these two identities into (2.1) yields the global energy inequality (1.6). □

Next we turn to the local energy inequality (1.9).

Lemma 2.2. *For smooth solutions of* (1.6), *the local energy law* (1.9) *also holds true.*

Proof. We multiply Eq. (1.6b) by $\mathbf{u}\phi$ to get

$$\begin{aligned}&\mu\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\phi dy+\xi\int_{\mathbb{R}^2}|\operatorname{div}\mathbf{u}|^2\phi dy+\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2\phi dy\\=&-\frac{1}{2}\int_{\mathbb{R}^2}\Big(\rho y\cdot\nabla|\mathbf{u}|^2+\rho\mathbf{u}\cdot\nabla|\mathbf{u}|^2\Big)\phi dy-M\int_{\mathbb{R}^2}\nabla\rho^\gamma\cdot\mathbf{u}\phi dy\\&+\frac{\mu}{2}\int_{\mathbb{R}^2}|\mathbf{u}|^2\Delta\phi dy-\xi\int_{\mathbb{R}^2}\mathbf{u}\cdot\nabla\phi\operatorname{div}\mathbf{u}dy.\end{aligned}\tag{2.2}$$

For the first term in the right-hand side of (2.2), using Eq. (1.6a) and applying the integration by parts, we get

$$\begin{aligned}&\int_{\mathbb{R}^2}\Big(\rho y\cdot\nabla|\mathbf{u}|^2+\rho\mathbf{u}\cdot\nabla|\mathbf{u}|^2\Big)\phi dy\\=&-\int_{\mathbb{R}^2}\big(y\cdot\nabla\rho+\operatorname{div}(\rho\mathbf{u})\big)|\mathbf{u}|^2\phi dy-2\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2\phi dy\\&-\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2(y+\mathbf{u})\cdot\nabla\phi dy\\=&-2\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2\phi dy-\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2(y+\mathbf{u})\cdot\nabla\phi dy.\end{aligned}$$

For the second term in the right-hand side of (2.2), integrating by parts and using Eq. (1.6a), one has

$$\begin{aligned}\int_{\mathbb{R}^2}\nabla\rho^\gamma\cdot\mathbf{u}\phi dy&=\int_{\mathbb{R}^2}\gamma\rho^{\gamma-1}\big(\mathrm{div}(\rho\mathbf{u})-\rho\mathrm{div}\mathbf{u}\big)\phi dy\\&=-\int_{\mathbb{R}^2}\gamma\rho^{\gamma-1}y\cdot\nabla\rho\phi dy-\int_{\mathbb{R}^2}\gamma\rho^\gamma\mathrm{div}\mathbf{u}\phi dy\\&=-\int_{\mathbb{R}^2}y\cdot\nabla\rho^\gamma\phi dy+\gamma\int_{\mathbb{R}^2}\nabla\rho^\gamma\cdot\mathbf{u}\phi dy+\gamma\int_{\mathbb{R}^2}\rho^\gamma\mathbf{u}\cdot\nabla\phi dy,\end{aligned}$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^2}\nabla\rho^\gamma\cdot\mathbf{u}\phi dy&=\frac{1}{\gamma-1}\left\{\int_{\mathbb{R}^2}y\cdot\nabla\rho^\gamma\phi dy-\gamma\int_{\mathbb{R}^2}\rho^\gamma\mathbf{u}\cdot\nabla\phi dy\right\}\\&=-\frac{2}{\gamma-1}\int_{\mathbb{R}^2}\rho^\gamma\phi dy-\frac{1}{\gamma-1}\int_{\mathbb{R}^2}\rho^\gamma(y+\gamma\mathbf{u})\cdot\nabla\phi dy.\end{aligned}$$

Substituting these two identities into (2.2) yields the global energy inequality (1.9). □

3 Proof of Theorem 1.1

In order to prove Theorem 1.1, one needs to establish the strong convergence of density. The possible concentration and oscillation will be two obstacles for the strong convergence of density. The concentration will be overcome by obtaining an higher integrability of density, while the oscillation can be handled with the help of the convex argument in [15].

To begin with, we first deal with the higher integrability of density.

3.1 Higher integrability of density

For any $\theta>0$, we multiply Eq. (1.6a) by $\phi(-\Delta)^{-1}\nabla(\rho^\theta\phi)$ and integrate the result equation to obtain

$$\begin{aligned}&M\int_{\mathbb{R}^2}\rho^{\gamma+\theta}\phi^2dy\\=&M\int_{\mathbb{R}^2}\rho^\gamma\nabla\phi\cdot\nabla(-\Delta)^{-1}(\rho^\theta\phi)dy-\mu\int_{\mathbb{R}^2}\nabla\mathbf{u}:\Big[\nabla\phi\otimes(-\Delta)^{-1}\nabla(\rho^\theta\phi)\Big]dy\\&-\mu\int_{\mathbb{R}^2}\phi\nabla\mathbf{u}:\nabla^2(-\Delta)^{-1}(\rho^\theta\phi)dy-\xi\int_{\mathbb{R}^2}\mathrm{div}\mathbf{u}\nabla\phi\cdot\nabla(-\Delta)^{-1}(\rho^\theta\phi)dy\end{aligned}$$

$$
\begin{aligned}
&+\xi\int_{\mathbb{R}^2}\mathrm{div}\mathbf{u}\phi^2\rho^\theta dy-\int_{\mathbb{R}^3}\phi\rho\mathbf{u}\cdot\nabla(-\Delta)^{-1}(\rho^\theta\phi)dy\\
&-\int_{\mathbb{R}^2}\phi\left[(\rho y+\rho\mathbf{u})\cdot\nabla\mathbf{u}\right]\cdot\nabla(-\Delta)^{-1}(\rho^\theta\phi)dy=\sum_{i=1}^{7}I_i, \qquad (3.1)
\end{aligned}
$$

where the positive function ϕ is a smooth function with a compact support.

Step one: $\rho\in L^{\frac{11\gamma}{6}}_{loc}(\mathbb{R}^2)$. We set $\theta=\frac{5\gamma}{6}$ in (3.1) and estimate each terms in the right-hand side of (3.1) one by one. Using the Gagliardo-Nirenberg inequality

$$
\|f\|_{L^\infty}\lesssim\|\nabla f\|^{\frac{22}{25}}_{L^{\frac{11}{5}}(\mathbb{R}^2)}\|f\|^{\frac{3}{25}}_{L^3(\mathbb{R}^2)}
$$

for the first term, one has

$$
\begin{aligned}
|I_1|&=M\left|\int_{\mathbb{R}^2}\rho^\gamma\nabla\phi\cdot(-\Delta)^{-1}\nabla\left(\rho^{\frac{5\gamma}{6}}\phi\right)dy\right|\\
&\lesssim M\|\rho^\gamma\nabla\phi\|_{L^1}\left\|(-\Delta)^{-1}\nabla\left(\rho^{\frac{5\gamma}{6}}\phi\right)\right\|_{L^\infty}\\
&\lesssim M\|\rho^\gamma\|_{L^1}\left\|(-\Delta)^{-1}\nabla\left(\rho^{\frac{5\gamma}{6}}\phi\right)\right\|^{\frac{3}{25}}_{L^3}\left\|\nabla(-\Delta)^{-1}\nabla\left(\rho^{\frac{5\gamma}{6}}\phi\right)\right\|^{\frac{22}{25}}_{L^{\frac{11}{5}}}\\
&\lesssim M\|\rho^\gamma\|_{L^1}\left\|\rho^{\frac{5\gamma}{6}}\phi\right\|^{\frac{3}{25}}_{L^{\frac{6}{5}}}\left\|\rho^{\frac{5\gamma}{6}}\phi\right\|^{\frac{22}{25}}_{L^{\frac{11}{5}}}\\
&\lesssim M\|\rho^\gamma\|^{\frac{11}{10}}_{L^1}\left\|\rho^{\frac{11\gamma}{6}}\phi^2\right\|^{\frac{2}{5}}_{L^1}\\
&\lesssim M\eta^{-\frac{2}{3}}\|\rho^\gamma\|^{\frac{11}{6}}_{L^1}+M\eta\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2dy,
\end{aligned}
$$

where η is a positive parameter. Here we also applied the estimate for Riesz's potential

$$
\left\|(-\Delta)^{-1}\partial f\right\|_{L^{\frac{2p}{2-p}}(\mathbb{R}^2)}\lesssim\|f\|_{L^p(\mathbb{R}^2)}\quad\text{as}\quad p\in(1,2).
$$

For the second term, one has

$$
\begin{aligned}
|I_2|&\lesssim\|\nabla\mathbf{u}\|_{L^2}\left\|(-\Delta)^{-1}\nabla\left(\rho^{\frac{5\gamma}{6}}\phi\right)\right\|_{L^3}[\mathrm{meas}(\mathrm{supp}\phi)]^{\frac{1}{6}}\\
&\lesssim\|\nabla\mathbf{u}\|_{L^2}\left\|\rho^{\frac{5\gamma}{6}}\phi\right\|_{L^{\frac{6}{5}}}\lesssim\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\|_{L^1}.
\end{aligned}
$$

Similarly, one has

$$
|I_4|\lesssim\|\mathrm{div}\mathbf{u}\|_{L^2}\|\rho^\gamma\|_{L^1}.
$$

In view of the boundedness of Riesz's transformation in $L^p(\mathbb{R}^2)$ as $p\in(1,\infty)$, the third term can be handled as

$$\begin{aligned}|I_3|&\lesssim\|\nabla\mathbf{u}\|_{L^2}\left\|\nabla^2(-\Delta)^{-1}\left(\rho^{\frac{5\gamma}{6}}\phi\right)\right\|_{L^2}\\&\lesssim\|\nabla\mathbf{u}\|_{L^2}\left\|\rho^{\frac{5\gamma}{6}}\phi\right\|_{L^2}\\&\lesssim\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\|_{L^1}^{\frac{1}{10}}\left(\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2dy\right)^{\frac{2}{5}}\\&\lesssim\eta^{-\frac{2}{3}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{5}{3}}\|\rho^\gamma\|_{L^1}^{\frac{1}{6}}+\eta\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2dy,\end{aligned}$$

and the fifth term is bounded directly as

$$|I_5|\lesssim\|\mathrm{div}\mathbf{u}\|_{L^2}\left\|\rho^{\frac{5\gamma}{6}}\phi\right\|_{L^2}\lesssim\eta^{-\frac{2}{3}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{5}{3}}\|\rho^\gamma\|_{L^1}^{\frac{1}{6}}+\eta\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2dy.$$

As $\gamma>\frac{3}{2}$, the sixth term could be estimated as

$$\begin{aligned}|I_6|&\lesssim\left\|\rho\phi^{\frac{1}{\gamma}}\right\|_{L^\gamma}\left\|\mathbf{u}\phi^{1-\frac{1}{\gamma}}\right\|_{L^{\frac{3\gamma}{2\gamma-3}}}\left\|\nabla(-\Delta)^{-1}\left(\rho^{\frac{5\gamma}{6}}\phi\right)\right\|_{L^3}\\&\lesssim\|\rho^\gamma\|_{L^1}^{\frac{1}{\gamma}}\|\nabla\mathbf{u}\|_{L^2}\left\|\rho^{\frac{5\gamma}{6}}\phi\right\|_{L^{\frac{6}{5}}}\lesssim\|\rho^\gamma\|_{L^1}^{\frac{5\gamma+6}{6\gamma}}\|\nabla\mathbf{u}\|_{L^2}.\end{aligned}$$

Here we applied the embedding of $\|f\|_{L^p_{loc}}\lesssim\|\nabla f\|_{L^2(\mathbb{R}^2)}$ as $p\in[1,\infty)$. For the seventh term, integrating by parts and using Eq. (1.6a), we have

$$\begin{aligned}I_7=&\int_{\mathbb{R}^2}\left[(\rho y+\rho\mathbf{u})\otimes\mathbf{u}\right]:\left[\nabla\phi\otimes\nabla(-\Delta)^{-1}(\rho^\theta\phi)\right]dy\\&+2\int_{\mathbb{R}^2}\phi\rho\mathbf{u}\cdot\nabla(-\Delta)^{-1}(\rho^\theta\phi)dy\\&+\int_{\mathbb{R}^2}\phi\left[(\rho y+\rho\mathbf{u})\otimes\mathbf{u}\right]:\left[\nabla\nabla(-\Delta)^{-1}(\rho^\theta\phi)\right]dy=\sum_{j=1}^3I_{7_j}.\end{aligned}\tag{3.2}$$

The second term I_{7_2} can be estimated as I_6. Since ϕ is compactly supported, the term I_{7_3} can be handled as

$$\begin{aligned}|I_{7_3}|&\lesssim\left\|\rho\phi^{\frac{12}{11\gamma}}\right\|_{L^{\frac{11\gamma}{6}}}\left(1+\left\||\mathbf{u}|^2\phi^{1-\frac{12}{11\gamma}}\right\|_{L^{\frac{99\gamma}{44\gamma-54}}}\right)\left\|\nabla^2(-\Delta)^{-1}\left(\rho^{\frac{5\gamma}{6}}\phi\right)\right\|_{L^{\frac{9}{5}}}\\&\lesssim\left\|\rho\phi^{\frac{12}{11\gamma}}\right\|_{L^{\frac{11\gamma}{6}}}\left(1+\left\||\mathbf{u}|^2\phi^{1-\frac{12}{11\gamma}}\right\|_{L^{\frac{99\gamma}{44\gamma-54}}}\right)\left\|\rho^{\frac{5\gamma}{6}}\phi\right\|_{L^{\frac{9}{5}}}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \left(1+\left\||\mathbf{u}|^2\phi^{1-\frac{12}{11\gamma}}\right\|_{L^{\frac{99\gamma}{44\gamma-54}}}\right)\left(\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2 dy\right)^{\frac{6}{11\gamma}+\frac{1}{3}}\|\rho^\gamma\|_{L^1}^{\frac{2}{9}}\\
&\lesssim \eta\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2 dy+\left(\eta^{-\frac{11\gamma+18}{33\gamma}}\left(1+\left\||\mathbf{u}|^2\phi^{1-\frac{12}{11\gamma}}\right\|_{L^{\frac{99\gamma}{44\gamma-54}}}\right)\|\rho^\gamma\|_{L^1}^{\frac{2}{9}}\right)^{\frac{33\gamma}{22\gamma-18}}.
\end{aligned}
$$

Similarly the term I_{7_1} can be estimated as

$$
\begin{aligned}
|I_{7_1}|&\lesssim \|\rho\|_{L^\gamma}\left(1+\||\mathbf{u}|^2|\nabla\phi|\|_{L^{\frac{18\gamma}{17\gamma-18}}}\right)\left\|\nabla(-\Delta)^{-1}\left(\rho^{\frac{5\gamma}{6}}\phi\right)\right\|_{L^{18}}\\
&\lesssim \|\rho^\gamma\|_{L^1}^{\frac{1}{\gamma}}\left(1+\||\mathbf{u}|^2|\nabla\phi|\|_{L^{\frac{18\gamma}{17\gamma-18}}}\right)\left\|\rho^{\frac{5\gamma}{6}}\phi\right\|_{L^{\frac{9}{5}}}\\
&\lesssim \|\rho^\gamma\|_{L^1}^{\frac{2\gamma+9}{9\gamma}}\left(1+\||\mathbf{u}|^2|\nabla\phi|\|_{L^{\frac{18\gamma}{17\gamma-18}}}\right)\left(\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2 dy\right)^{\frac{1}{3}}\\
&\lesssim \eta\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2 dy+\left[\eta^{-\frac{1}{3}}\|\rho^\gamma\|_{L^1}^{\frac{2\gamma+9}{9\gamma}}\left(1+\||\mathbf{u}|^2|\nabla\phi|\|_{L^{\frac{18\gamma}{17\gamma-18}}}\right)\right]^{\frac{3}{2}}.
\end{aligned}
$$

Since $\nabla\mathbf{u}$ is bounded in $L^2(\mathbb{R}^2)$, $\mathbf{u}$ is bounded in $L^q_{loc}(\mathbb{R}^2)$ for all $q<\infty$. Therefore, we have

$$
\begin{aligned}
|I_7|\lesssim& 3\eta\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2 dy+\left[\eta^{-\frac{1}{3}}\|\rho^\gamma\|_{L^1}^{\frac{2\gamma+9}{9\gamma}}\left(1+\||\mathbf{u}|^2|\nabla\phi|\|_{L^{\frac{18\gamma}{17\gamma-18}}}\right)\right]^{\frac{3}{2}}\\
&+\|\rho^\gamma\|_{L^1}^{\frac{\gamma+1}{\gamma}}\|\nabla\mathbf{u}\|_{L^2}+\left(\eta^{-\frac{11\gamma+18}{33\gamma}}\left(1+\left\||\mathbf{u}|^2\phi^{1-\frac{12}{11\gamma}}\right\|_{L^{\frac{99\gamma}{44\gamma-54}}}\right)\|\rho^\gamma\|_{L^1}^{\frac{2}{9}}\right)^{\frac{33\gamma}{22\gamma-18}}\\
\lesssim& 3\eta\int_{\mathbb{R}^2}\rho^{\frac{11\gamma}{6}}\phi^2 dy+\left[\eta^{-\frac{1}{3}}\|\rho^\gamma\|_{L^1}^{\frac{2\gamma+9}{9\gamma}}\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\right]^{\frac{3}{2}}+\|\rho^\gamma\|_{L^1}^{\frac{\gamma+1}{\gamma}}\|\nabla\mathbf{u}\|_{L^2}\\
&+\left(\eta^{-\frac{11\gamma+18}{33\gamma}}\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\rho^\gamma\|_{L^1}^{\frac{2}{9}}\right)^{\frac{33\gamma}{22\gamma-18}}.
\end{aligned}
$$

The desired estimate $\rho\in L_{loc}^{\frac{11\gamma}{6}}(\mathbb{R}^2)$ follows from the suitable choices of both the smooth function ϕ and the small parameter η.

Step two: $\rho\in L^{2\gamma}_{loc}(\mathbb{R}^2)$. We set $\theta=\lambda$ in (3.1) and then estimate each terms in the right-hand side of (3.1) again one by one. For the first term, one has

$$
|I_1|=M\left|\int_{\mathbb{R}^2}\rho^\gamma\phi(-\Delta)^{-1}\mathrm{div}(\rho^\gamma\nabla\phi)dy\right|
$$

$$
\begin{aligned}
&\lesssim M\|\rho^\gamma\phi\|_{L^2}\left\|(-\Delta)^{-1}\mathrm{div}(\rho^\gamma\nabla\phi)\right\|_{L^3}|\mathrm{meas}(\mathrm{supp}\phi)|^{\frac{1}{6}}\\
&\lesssim M\|\rho^\gamma\phi\|_{L^2}\|\rho^\gamma\nabla\phi\|_{L^{\frac{6}{5}}}|\mathrm{meas}(\mathrm{supp}\phi)|^{\frac{1}{6}}\\
&\lesssim M\|\rho^\gamma\phi\|_{L^2}\|\rho^\gamma\|_{L^1}^{\frac{19}{30}}\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}|\mathrm{meas}(\mathrm{supp}\phi)|^{\frac{1}{6}}\\
&\lesssim M\eta^{-1}\|\rho^\gamma\|_{L^1}^{\frac{19}{15}}\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{15}}|\mathrm{meas}(\mathrm{supp}\phi)|^{\frac{1}{3}}+M\eta\|\rho^\gamma\phi\|_{L^2}^2,
\end{aligned}
$$

where η is again a positive parameter. For the second term, one has

$$
\begin{aligned}
|I_2|&\lesssim\|\nabla\mathbf{u}\|_{L^2}\left\|(-\Delta)^{-1}\mathrm{div}(\rho^\gamma\phi)\right\|_{L^3}|\mathrm{meas}(\mathrm{supp}\phi)|^{\frac{1}{6}}\\
&\lesssim\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\phi\|_{L^{\frac{6}{5}}}|\mathrm{meas}(\mathrm{supp}\phi)|^{\frac{1}{6}}\\
&\lesssim\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\|_{L^1}^{\frac{19}{30}}\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}|\mathrm{meas}(\mathrm{supp}\phi)|^{\frac{1}{6}}.
\end{aligned}
$$

Similarly, one has

$$
|I_4|\lesssim\|\mathrm{div}\mathbf{u}\|_{L^2}\|\rho^\gamma\|_{L^1}^{\frac{19}{30}}\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}|\mathrm{meas}(\mathrm{supp}\phi)|^{\frac{1}{6}}.
$$

The third term can be handled as

$$
\begin{aligned}
|I_3|&\lesssim\|\nabla\mathbf{u}\|_{L^2}\left\|\nabla^2(-\Delta)^{-1}(\rho^\gamma\phi)\right\|_{L^2}\\
&\lesssim\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\phi\|_{L^2}\\
&\lesssim\eta^{-1}\|\nabla\mathbf{u}\|_{L^2}^2+\eta\|\rho^\gamma\phi\|_{L^2}^2,
\end{aligned}
$$

and the fifth term is bounded directly as

$$
|I_5|\lesssim\|\mathrm{div}\mathbf{u}\|_{L^2}\|\rho^\gamma\phi\|_{L^2}\lesssim\eta^{-1}\|\mathrm{div}\mathbf{u}\|_{L^2}^2+\eta\|\rho^\gamma\phi\|_{L^2}^2.
$$

As $\gamma>1$, the sixth term could be estimated as

$$
\begin{aligned}
|I_6|&\lesssim\left\|\rho\phi^{\frac{1}{\gamma}}\right\|_{L^{2\gamma}}\left\|\mathbf{u}\phi^{1-\frac{1}{\gamma}}\right\|_{L^{\frac{6\gamma}{4\gamma-3}}}\left\|\nabla(-\Delta)^{-1}(\rho^\gamma\phi)\right\|_{L^3}\\
&\lesssim\|\rho^\gamma\phi\|_{L^2}^{\frac{1}{\gamma}}\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\phi\|_{L^{\frac{6}{5}}}\\
&\lesssim\|\rho^\gamma\phi\|_{L^2}^{\frac{1}{\gamma}}\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\|_{L^1}^{\frac{19}{30}}\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}\\
&\lesssim\eta\|\rho^\gamma\phi\|_{L^2}^2+\eta^{-\frac{1}{2\gamma-1}}\left(\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\|_{L^1}^{\frac{19}{30}}\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}\right)^{\frac{2\gamma}{2\gamma-1}}.
\end{aligned}
$$

For the seventh term, we use the identity (3.2). Note that the second term I_{7_2} can be estimated as I_6. Since ϕ is compactly supported, the term I_{7_3} can be handled as

$$
\begin{aligned}
|I_{7_3}| &\lesssim \left\|\rho\phi^{\frac{1}{\gamma}}\right\|_{L^{2\gamma}}\left(1+\left\||\mathbf{u}|^2\phi^{1-\frac{1}{\gamma}}\right\|_{L^{\frac{2\gamma}{\gamma-1}}}\right)\left\|\nabla^2(-\Delta)^{-1}(\rho^\gamma\phi)\right\|_{L^2}\\
&\lesssim \left(1+\left\||\mathbf{u}|^2\phi^{1-\frac{1}{\gamma}}\right\|_{L^{\frac{2\gamma}{\gamma-1}}}\right)\|\rho^\gamma\phi\|_{L^2}^{\frac{\gamma+1}{\gamma}}\\
&\lesssim \eta\|\rho^\gamma\phi\|_{L^2}^2+\left(\eta^{-\frac{\gamma+1}{2\gamma}}\left(1+\left\||\mathbf{u}|^2\phi^{1-\frac{1}{\gamma}}\right\|_{L^{\frac{2\gamma}{\gamma-1}}}\right)\right)^{\frac{2\gamma}{\gamma-1}}.
\end{aligned}
$$

Similarly the first term can be estimated as

$$
\begin{aligned}
|I_{7_1}| &\lesssim \|\rho\|_{L^\gamma}\left(1+\left\||\mathbf{u}|^2|\nabla\phi|\right\|_{L^{\frac{3\gamma}{2\gamma-3}}}\right)\left\|\nabla(-\Delta)^{-1}(\rho^\gamma\phi)\right\|_{L^3}\\
&\lesssim \|\rho^\gamma\|_{L^1}^{\frac{1}{\gamma}}\left(1+\left\||\mathbf{u}|^2|\nabla\phi|\right\|_{L^{\frac{3\gamma}{2\gamma-3}}}\right)\|\rho^\gamma\phi\|_{L^{\frac{6}{5}}}\\
&\lesssim \|\rho^\gamma\|_{L^1}^{\frac{1}{\gamma}+\frac{19}{30}}\left(1+\left\||\mathbf{u}|^2|\nabla\phi|\right\|_{L^{\frac{3\gamma}{2\gamma-3}}}\right)\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}.
\end{aligned}
$$

Since $\nabla\mathbf{u}$ is bounded in $L^2(\mathbb{R}^2)$, $\mathbf{u}$ is bounded in $L^q_{loc}(\mathbb{R}^2)$ for all $q<\infty$. Therefore, we have

$$
\begin{aligned}
|I_7| \lesssim\; & \|\rho^\gamma\|_{L^1}^{\frac{1}{\gamma}+\frac{19}{30}}\left(1+\left\||\mathbf{u}|^2|\nabla\phi|\right\|_{L^{\frac{3\gamma}{2\gamma-3}}}\right)\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}\\
&+\left(\eta^{-\frac{\gamma+1}{2\gamma}}\left(1+\left\||\mathbf{u}|^2\phi^{1-\frac{1}{\gamma}}\right\|_{L^{\frac{2\gamma}{\gamma-1}}}\right)\right)^{\frac{2\gamma}{\gamma-1}}\\
&+\eta^{-\frac{1}{2\gamma-1}}\left(\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\|_{L^1}^{\frac{19}{30}}\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}\right)^{\frac{2\gamma}{2\gamma-1}}+2\eta\|\rho^\gamma\phi\|_{L^2}^2\\
\lesssim\; & \|\rho^\gamma\|_{L^1}^{\frac{1}{\gamma}+\frac{19}{30}}\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}+\left(\eta^{-\frac{\gamma+1}{2\gamma}}\left(1+\|\nabla\mathbf{u}\|_{L^2}^2\right)\right)^{\frac{2\gamma}{\gamma-1}}\\
&+\eta^{-\frac{1}{2\gamma-1}}\left(\|\nabla\mathbf{u}\|_{L^2}\|\rho^\gamma\|_{L^1}^{\frac{19}{30}}\|\rho^\gamma\|_{L^{\frac{11}{6}}_{loc}}^{\frac{11}{30}}\right)^{\frac{2\gamma}{2\gamma-1}}+2\eta\|\rho^\gamma\phi\|_{L^2}^2.
\end{aligned}
$$

The desired estimate $\rho\in L^{2\gamma}_{loc}$ follows from the suitable choices of both the smooth function ϕ and the small parameter η.

3.2 Compactness

In this subsection we are going to verify the compactness result which will play a crucial role in the existence proofs allowing us to pass to the limit in the approximation solutions.

We consider a sequence of functions $\{(\rho_n,\mathbf{u}_n)\}_{n=1}^{\infty}$ which satisfies

$$\rho_n\geq 0,\quad \rho_n\in L^{\gamma}\cap L^{2\gamma}_{loc},\quad \nabla\mathbf{u}_n\in L^2(\mathbb{R}^2),\quad \rho_n|\mathbf{u}_n|^2\in L^1(\mathbb{R}^2).$$

Without loss of generality, we may assume, extracting subsequences if necessary, that ρ_n converges weakly to some $\rho\geq 0$ in $L^{\gamma}\cap L^{2\gamma}_{loc}$, and that $\nabla\mathbf{u}_n$ converges weakly in L^2 to some function $\nabla\mathbf{u}\in L^2$. Moreover, we claim that $\rho|\mathbf{u}|^2\in L^1$. Indeed, by Egorov theorem, we deduce that for all $\varepsilon>0$, there exists $\Omega_\varepsilon\subset B_{\frac{1}{\varepsilon}}$ such that $|\Omega_\varepsilon|<\varepsilon$ such that $|\mathbf{u}_n|^2$ converges uniformly to $|\mathbf{u}|^2$ on $\Omega_\varepsilon^c\cap B_{\frac{1}{\varepsilon}}$. Hence, we have

$$C\geq\int_{\mathbb{R}^2}\rho_n|\mathbf{u}_n|^2dx\geq\int_{\Omega_\varepsilon^c\cap B_{\frac{1}{\varepsilon}}}\rho_n|\mathbf{u}_n|^2dx\to\int_{\Omega_\varepsilon^c\cap B_{\frac{1}{\varepsilon}}}\rho|\mathbf{u}|^2dx$$

as $n\to\infty$, and our claim follows upon letting ε go to 0.

Next we assume that

$$\operatorname{div}(\rho_n\mathbf{u}_n)+y\cdot\nabla\rho_n-\varepsilon_n\Delta\rho_n=0 \tag{3.3}$$

with $\lim_{n\to\infty}\varepsilon_n=0$ and for $b>0$ and $\gamma>1$ the sequence $\operatorname{div}\mathbf{u}_n-b\rho_n^{\gamma}$ converges a.e in $\mathbb{R}^2$. We set $\varepsilon>0$ and $\theta>0$ which has to be taken small enough in the argument below and will be determined later on. Then the function $(\varepsilon+\rho_n)^{\theta}$ satisfies

$$\begin{aligned}&\operatorname{div}\big(\mathbf{u}_n(\varepsilon+\rho_n)^{\theta}\big)+y\cdot\nabla(\varepsilon+\rho_n)^{\theta}-\varepsilon_n\Delta(\varepsilon+\rho_n)^{\theta}\\ \geq\ &\theta\varepsilon\operatorname{div}\mathbf{u}_n(\varepsilon+\rho_n)^{\theta-1}+(1-\theta)\operatorname{div}\mathbf{u}_n(\varepsilon+\rho_n)^{\theta}.\end{aligned} \tag{3.4}$$

In order to justify this computation which is straightforward if $(\rho_n,\mathbf{u}_n)$ is smooth, one can apply the standard regularisation argument as the exponent θ is sufficiently small with a simply observation that when ϕ is smooth and positive, then

$$-\Delta\phi^{\theta}=-\theta\Delta\phi\phi^{\theta-1}+\theta(1-\theta)\phi^{\theta-2}|\nabla\phi|^2\geq-\theta\Delta\phi\phi^{\theta-1}.$$

It is easy to wee that $\operatorname{div}\mathbf{u}_n-b\rho_n^{\gamma}$ converges to $\operatorname{div}\mathbf{u}-b\overline{\rho^{\gamma}}$ in L^r_{loc} for all $r<2$. With this information at hand, we may pass to the limit in (3.4), extracting subsequence if necessary, and deduce easily

$$\begin{aligned}&\operatorname{div}\Big(\mathbf{u}\overline{(\varepsilon+\rho)^{\theta}}\Big)+y\cdot\nabla\overline{(\varepsilon+\rho)^{\theta}}\\ \geq\ &\theta\varepsilon\overline{\operatorname{div}\mathbf{u}(\varepsilon+\rho)^{\theta-1}}+(1-\theta)\operatorname{div}\mathbf{u}\overline{(\varepsilon+\rho)^{\theta}}\\ &\quad+(1-\theta)b\Big\{\overline{\rho^{\gamma}(\varepsilon+\rho)^{\theta}}-\overline{\rho^{\gamma}}\,\overline{(\varepsilon+\rho)^{\theta}}\Big\},\end{aligned}$$

provided $\theta<\gamma$. Choosing $\theta\in(0,1)$, we find at least formally that the quantity $(\overline{(\varepsilon+\rho)^\theta})^{\frac{1}{\theta}}$ satisfies

$$\begin{aligned}
&\operatorname{div}\left(\mathbf{u}\left(\overline{(\varepsilon+\rho)^\theta}\right)^{\frac{1}{\theta}}\right)+y\cdot\nabla\left(\overline{(\varepsilon+\rho)^\theta}\right)^{\frac{1}{\theta}}\\
\geq&\,\varepsilon\overline{\operatorname{div}\mathbf{u}(\varepsilon+\rho)^{\theta-1}}\left(\overline{(\varepsilon+\rho)^\theta}\right)^{\frac{1}{\theta}-1}\\
&+\frac{(1-\theta)}{\theta}b\left\{\overline{\rho^\gamma(\varepsilon+\rho)^\theta}-\overline{\rho^\gamma}\,\overline{(\varepsilon+\rho)^\theta}\right\}\left(\overline{(\varepsilon+\rho)^\theta}\right)^{\frac{1}{\theta}-1}
\end{aligned} \tag{3.5}$$

in the sense of distributions. We remark that the rigorous justification of this inequality is a bit tedious, but can be made rigorously by verifying (3.5) for $h_R(t)=Rh(t/R)$ as $R\geq 1$ and then letting $R\to\infty$, where $h\in C^\infty([0,\infty))$, $h'\in C_0^\infty([0,\infty))$, $0\leq h'\leq 1$ and $h(t)=t$ if $t\in[0,1]$. We left this detailed verification to the interested reader.

We next claim that (3.5) implies

$$\operatorname{div}\left(\mathbf{u}\overline{\rho^\theta}^{\frac{1}{\theta}}\right)+y\cdot\nabla\overline{\rho^\theta}^{\frac{1}{\theta}}\geq b\frac{1-\theta}{\theta}\left(\overline{\rho^{\gamma+\theta}}-\overline{\rho^\gamma}\,\overline{\rho^\theta}\right)\overline{\rho^\theta}^{\frac{1}{\theta}-1}. \tag{3.6}$$

Indeed, this inequality follows from the limit of (3.5) as $\varepsilon\to 0$ once we notice that

$$\varepsilon^{1-\theta}\overline{\operatorname{div}\mathbf{u}(\varepsilon+\rho)^{\theta-1}}\left(\overline{(\varepsilon+\rho)^\theta}\right)^{\frac{1}{\theta}-1}$$

is uniformly bounded with respect to ε in L^1_{loc}.

On the other hand, passing to the limit in (3.3) as $n\to\infty$, we obtain

$$y\cdot\nabla\rho+\operatorname{div}(\rho\mathbf{u})=0 \tag{3.7}$$

in the sense of distributions. Since $f(t)=t^\theta$ as $t\geq 0$ is a concave function, there holds $\overline{\rho^\theta}^{\frac{1}{\theta}}\leq\rho$ a.e. Then subtracting (3.7) from (3.6), we obtain

$$b\frac{1-\theta}{\theta}\left(\overline{\rho^{\gamma+\theta}}-\overline{\rho^\gamma}\,\overline{\rho^\theta}\right)\overline{\rho^\theta}^{\frac{1}{\theta}-1}\leq\operatorname{div}(\mathbf{u}r)+y\cdot\nabla r,$$

where $0\leq r\leq\rho$ a.e. In particular, $r\in L^\gamma\cap L^{2\gamma}_{loc}$ and $r|\mathbf{u}|^2\in L^1$. Moreover, since $y\cdot\nabla r=\operatorname{div}(yr)-2r\leq\operatorname{div}(yr)$, there follows

$$b\frac{1-\theta}{\theta}\left(\overline{\rho^{\gamma+\theta}}-\overline{\rho^\gamma}\,\overline{\rho^\theta}\right)\overline{\rho^\theta}^{\frac{1}{\theta}-1}\leq\operatorname{div}((\mathbf{u}+y)r).$$

Since $\rho \in L^1(\mathbb{R}^2)$, integrating this inequality over $\mathbb{R}^2$, we obtain

$$\int_{\mathbb{R}^2} \left(\overline{\rho^{\gamma+\theta}} - \overline{\rho^\gamma}\,\overline{\rho^\theta}\right) \overline{\rho^\theta}^{\frac{1}{\theta}-1} dx \le 0. \tag{3.8}$$

Since

$$\overline{\rho^{\gamma+\theta}} \ge \left(\overline{\rho^\gamma}\right)^{\frac{\gamma+\theta}{\gamma}} \quad \text{and} \quad \overline{\rho^{\gamma+\theta}} \ge \left(\overline{\rho^\theta}\right)^{\frac{\gamma+\theta}{\theta}} \quad \text{a.e.,} \tag{3.9}$$

we have $\overline{\rho^{\gamma+\theta}} - \overline{\rho^\gamma}\,\overline{\rho^\theta} \ge 0$ a.e. Therefore it follows from (3.8) that

$$\left(\overline{\rho^{\gamma+\theta}} - \overline{\rho^\gamma}\,\overline{\rho^\theta}\right) \overline{\rho^\theta}^{\frac{1}{\theta}-1} = 0 \quad \text{a.e. in } \mathbb{R}^2.$$

This identity, combining with (3.9), yields

$$\overline{\rho^{\gamma+\theta}} = \left(\overline{\rho^\gamma}\right)^{\frac{\gamma+\theta}{\gamma}}, \quad \overline{\rho^{\gamma+\theta}} = \left(\overline{\rho^\theta}\right)^{\frac{\gamma+\theta}{\theta}} \quad \text{a.e. on } \{\overline{\rho^\theta} > 0\}.$$

Therefore ρ_n converges strongly to ρ in $L^{\gamma+\theta}(\{\overline{\rho^\theta}>0\})$. Finally, on the set $\{\overline{\rho^\theta}=0\}$, clearly ρ_n^θ converges strongly to 0 in $L^1(B_R)$ for all $R<\infty$. Hence , we deduce that ρ_n^θ converges to ρ^θ in $L^1(B_R)$ for all $R<\infty$, or equivalently, ρ_n converges to ρ in $L^1(B_R)$ for all $R<\infty$.

The strong convergence of ρ_n in $L^1(B_R)$, combined with the uniform bound in $L^{2\gamma}_{loc}$, yields the strong convergence of ρ_n in $L^p(B_R)$ for any $1 \le p < 2\gamma$. Therefore $\rho_n \mathbf{u}_n$ converges weakly to $\rho\mathbf{u}$ in $L^{\frac{2\gamma}{\gamma+1}}_{loc}$ and $\rho_n \mathbf{u}_n \otimes \mathbf{u}_n$ converges weakly to $\rho\mathbf{u}\otimes\mathbf{u}$ in L^p_{loc} for any $p \in (1,2\gamma)$.

3.3 Approximation

In order to establish the existence of weak solutions, following [6,15], we would consider the following approximation:

$$\begin{cases} y\cdot\nabla\rho + \operatorname{div}(\rho\mathbf{u}) - \varepsilon\Delta\rho = 0, \\ \rho\mathbf{u} + \rho y\cdot\nabla\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} - \mu\Delta\mathbf{u} - \xi\nabla\operatorname{div}\mathbf{u} + M\nabla\rho^\gamma + \varepsilon\nabla\mathbf{u}\nabla\rho = 0, \end{cases} \tag{3.10}$$

where the parameter ε is a positive number. For the approximated system, similar to (1.3), a direct computation justifies the global energy law

$$\nu\int_{\mathbb{R}^2} |\nabla\mathbf{u}|^2 dx + \xi\int_{\mathbb{R}^2} |\operatorname{div}\mathbf{u}|^2 dx + M\varepsilon\gamma\int_{\mathbb{R}^2} \rho^{\gamma-2}|\nabla\rho|^2 dx \le \frac{2M}{\gamma-1}\int_{\mathbb{R}^2} \rho^\gamma dx. \tag{3.11}$$

With this energy law in hand, the existence of weak solutions for the approximation system follows from the standard theory for elliptic equations.

For the convergence of the sequence of approximated solutions, the compactness framework in the previous subsection guarantees the convergence of all terms in (3.10), except the terms $\varepsilon\Delta\rho$ and $\varepsilon\nabla\mathbf{u}\nabla\rho$, in the sense of distributions. Moreover, the local energy estimate (1.9) holds true by using the lower semicontinuity of weak convergence in L^2.

Case 1: $\gamma>2$. In order to handle the convergence of $\varepsilon\Delta\rho$, we need the following bound to pass to the limit as ε goes to 0^+:

$$\varepsilon\nabla\rho_\varepsilon\to 0 \quad \text{in } L^2 \quad \text{as} \quad \varepsilon\to 0^+.$$

Once this is verified, we conclude that $\varepsilon\Delta\rho_\varepsilon\to 0$ in the sense of distributions. Indeed, we first observe that the energy law (3.11) yields for all $\delta>0$

$$\big\|\varepsilon|\nabla\rho_\varepsilon|1_{\{\rho_\varepsilon\ge\delta\}}\big\|_{L^2}\lesssim\varepsilon^{\frac12}\delta^{1-\frac{\gamma}{2}}. \tag{3.12}$$

On the other hand, multiplying the equation satisfied by ρ_ε by $\rho_\varepsilon\wedge\delta=\min\{\rho,\delta\}$, it follows

$$\begin{aligned}&\varepsilon\int_{B_R}|\nabla\rho_\varepsilon|^2 1_{\{\rho_\varepsilon<\delta\}}dx\\ \lesssim&\left|\int_{B_R}\rho_\varepsilon(y+\mathbf{u}_\varepsilon)\cdot\nabla\rho_\varepsilon 1_{\{\rho_\varepsilon<\delta\}}dx\right|\lesssim\delta\big\||\nabla\rho_\varepsilon|1_{\{\rho_\varepsilon<\delta\}}\big\|_{L^2(B_R)},\end{aligned}$$

and hence

$$\big\|\varepsilon|\nabla\rho_\varepsilon|1_{\{\rho_\varepsilon<\delta\}}\big\|_{L^2(B_R)}\lesssim(\varepsilon\delta)^{\frac12}.$$

Combining this inequality with (3.12), we know

$$\big\|\varepsilon|\nabla\rho_\varepsilon|\big\|_{L^2(B_R)}\lesssim(\varepsilon\delta)^{\frac12}+\varepsilon^{\frac12}\delta^{1-\frac{\gamma}{2}},$$

and we complete the proof upon choosing $\delta=\varepsilon^s$ with $s\in(0,\frac{1}{\gamma-2})$.

Case 2: $\gamma\in(\frac32,2]$. In this case, we modified the approximate system (3.10) by adding an artificial pressure term, parameterized by $\beta>0$

$$\begin{cases}y\cdot\nabla\rho+\operatorname{div}(\rho\mathbf{u})-\varepsilon\Delta\rho=0,\\ \rho\mathbf{u}+\rho y\cdot\nabla\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}-\mu\Delta\mathbf{u}-\xi\nabla\operatorname{div}\mathbf{u}+M\nabla(\rho^\gamma+\beta\rho^3)+\varepsilon\nabla\mathbf{u}\nabla\rho=0.\end{cases} \tag{3.13}$$

Then for any fixed $\beta>0$, following the argument in Case 1, we could take the limit as $\varepsilon\to 0$ to get

$$\begin{cases} y\cdot\nabla\rho+\text{div}(\rho\mathbf{u})=0, \\ \rho\mathbf{u}+\rho y\cdot\nabla\mathbf{u}+\rho\mathbf{u}\cdot\nabla\mathbf{u}-\mu\Delta\mathbf{u}-\xi\nabla\text{div}\mathbf{u}+M\nabla\left(\rho^\gamma+\beta\rho^{\frac{5}{2}}\right)=0 \end{cases} \tag{3.14}$$

in the sense of distributions. Finally for the convergence as $\beta\to 0$, a similar argument as in the Subsection 3.1 yields a uniform bound of $\rho^{2\gamma}$ in L^1_{loc}, and hence $\beta\rho^{\frac{5}{2}}$ converges to zero as $\beta\to 0$.

Finally we turn to the estimates of $\nabla\text{curl}\mathbf{u}$ and $\nabla\{\text{div}\mathbf{u}-\frac{M}{\mu+\xi}\rho^\gamma\}$. These estimates follow directly from the estimate for elliptic equations. In fact, we write the momentum equation in the form of

$$-\nabla\left[(\mu+\xi)\text{div}\mathbf{u}-M\rho^\gamma\right]+\mu\text{curlcurl}\mathbf{u}=-\rho\mathbf{u}\cdot\nabla\mathbf{u}-\rho y\cdot\nabla\mathbf{u}-\rho\mathbf{u},$$

where curl(curl$\mathbf{u}$) means $[-\nabla^\perp(\text{curl}\mathbf{u})]$ in dimension two. Since $\rho\in L^{2\gamma}_{loc}$, $\nabla\mathbf{u}\in L^2(\mathbb{R}^2)$, and hence $\mathbf{u}\in L^p_{loc}$ for any $p\in(1,\infty)$, we easily knows

$$-\rho\mathbf{u}\cdot\nabla\mathbf{u}-\rho y\cdot\nabla\mathbf{u}-\rho\mathbf{u}\in L_{loc}^{\frac{3(\gamma-1)}{2\gamma-1}},$$

due to the inequality

$$\frac{1}{2}+\frac{1}{2\gamma}<\frac{2\gamma-1}{3(\gamma-1)}.$$

4 Proof of Theorem 1.2

This section is devoted to proving Theorem 1.2 which justifies that the weak solutions obtained in Theorem 1.1 are trivial if the density belong to L^∞. The strategy of the argument is motivated by the argument for the Liouville theorem for steady Navier-Stokes equations in [2].

Let $\phi(s)\in C_0^\infty(\mathbb{R}^+)$ be a smooth cut-off function such that

$$\phi(s)=\begin{cases} 1, & \text{if} \quad s<1, \\ 0, & \text{if} \quad s>2. \end{cases}$$

We set $\phi_R(y)=\phi(|y|/R)$ for each $R>0$. We multiply the first equation in (1.6) by $\phi_R(x)$, integrate over $\mathbb{R}^2$, and integrate by parts to obtain

$$2\int_{\mathbb{R}^2}\rho(y)\phi_R(y)dy=-\int_{\mathbb{R}^2}y\cdot\nabla\phi_R(y)\rho(y)dy-\int_{\mathbb{R}^2}\rho\mathbf{u}(y)\cdot\nabla\phi_R(y)dy. \tag{4.1}$$

On one hand, the definition of ϕ_R gives

$$y\cdot\nabla\phi_R(y)=\frac{|y|}{R}\phi'\left(\frac{|y|}{R}\right).$$

Therefore $\mathrm{supp}(y\cdot\nabla\phi)\subset B_{2R}\setminus B_R$ and $|y\cdot\nabla\phi_R(y)|\lesssim 1$. Hence

$$\left|\int_{\mathbb{R}^2} y\cdot\nabla\phi_R(y)\rho(y)dy\right|\lesssim\int_{B_{2R}\setminus B_R}\rho(y)dy\to 0\quad\text{as}\quad R\to\infty,$$

since $\rho\in L^1(\mathbb{R}^2)$. On the other hand, since $|\nabla\phi_R(y)|\lesssim R^{-1}$ and $\mathrm{supp}(\nabla\phi_R)\subset B_{2R}\setminus B_R$, we have

$$\begin{aligned}&\left|\int_{\mathbb{R}^2}\rho\mathbf{u}(y)\cdot\nabla\phi_R(y)dy\right|\lesssim R^{-1}\int_{B_{2R}\setminus B_R}|\rho\mathbf{u}|dy\\ \lesssim& R^{-1}\left(\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2dy\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\rho^\gamma\right)^{\frac{1}{2\gamma}}R^{1-\frac{1}{\gamma}}\lesssim R^{-\frac{1}{\gamma}}\to 0\end{aligned}$$

as $R\to\infty$. Therefore we deduce from (4.1) that

$$\int_{\mathbb{R}^2}\rho(y)\phi_R(y)dy\to 0\quad\text{as}\quad R\to\infty. \tag{4.2}$$

In particular, passing $R\to 0$, we get

$$\int_{\mathbb{R}^2}\rho(y)dy=0.$$

Since $\rho\geq 0$, one knows $\rho(y)=0$ in $\mathbb{R}^2$.

Next we multiple Eq. (1.6b) by $\mathbf{u}\phi_R$, and integrate over $\mathbb{R}^2$, and integrate by part to obtain,

$$\begin{aligned}&\mu\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2\phi_R(y)dy+(\mu+\lambda)\int_{\mathbb{R}^2}|\mathrm{div}\mathbf{u}|^2\phi_Rdy\\ =&-\mu\int_{\mathbb{R}^2}\mathbf{u}\cdot(\nabla\phi_R\cdot\nabla)\mathbf{u}dy-(\mu+\lambda)\int_{\mathbb{R}^2}\mathrm{div}\mathbf{u}(\mathbf{u}\cdot\nabla)\phi_Rdy\\ &-M\int_{\mathbb{R}^2}\phi_R\mathbf{u}\cdot\nabla\rho^\gamma dy-\int_{\mathbb{R}^2}\phi_R\rho|\mathbf{u}|^2dy\\ &-\int_{\mathbb{R}^2}\phi_R\rho\mathbf{u}\cdot\nabla\mathbf{u}\cdot\mathbf{u}dy-\int_{\mathbb{R}^2}\phi_R\rho y\cdot\nabla\mathbf{u}\cdot\mathbf{u}dy\quad=\sum_{k=1}^6 K_k.\end{aligned} \tag{4.3}$$

We estimate K_k $(k=1,\dots,6)$ term by term. First of all, we have

$$|K_1| \lesssim \frac{1}{R}\int_{B_{2R}\backslash B_R}\left|\phi'\left(\frac{|y|}{R}\right)\right||\nabla\mathbf{u}||\mathbf{u}|dy$$
$$\lesssim \frac{1}{R}\|\mathbf{u}\|_{L^2(\mathbb{R}^2)}\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^2)} \to 0$$

as $R\to\infty$. The estimate of K_2 is similar to I_1 and we have

$$|K_2| \lesssim \frac{1}{R}\int_{B_{2R}\backslash B_R}\left|\phi'\left(\frac{|y|}{R}\right)\right||\nabla\mathbf{u}||\mathbf{u}|dy$$
$$\lesssim \frac{1}{R}\|\mathbf{u}\|_{L^2(\mathbb{R}^2)}\|\nabla\mathbf{u}\|_{L^2(\mathbb{R}^2)} \to 0$$

as $R\to\infty$. For the third term, integration by parts gives

$$K_3 = M\int_{\mathbb{R}^2}\rho^\gamma\mathbf{u}\cdot\nabla\phi_R dy + M\int_{\mathbb{R}^2}\phi_R \mathrm{div}\mathbf{u}\rho^\gamma dy.$$

Since

$$\left|\int_{\mathbb{R}^2}\rho^\gamma\mathbf{u}\cdot\nabla\phi_R dy\right| \lesssim R^{-1}\int_{B_{2R}}\rho^\gamma|\mathbf{u}|dy,$$
$$\left|\int_{\mathbb{R}^2}\phi_R \mathrm{div}\mathbf{u}\rho^\gamma dy\right| \lesssim \int_{B_{2R}}\rho^\gamma|\mathrm{div}\mathbf{u}|dy,$$

we can apply the estimate (4.2) and the assumption that $\rho\in L^\infty(\mathbb{R}^2)$ to conclude that

$$|K_3| \lesssim R^{-1}\int_{B_{2R}}\rho^\gamma|\mathbf{u}|dy + \int_{B_{2R}}\rho^\gamma|\mathrm{div}\mathbf{u}|dy$$
$$\lesssim R^{-1}\int_{B_{2R}}\rho^{\frac{1}{2}}|\mathbf{u}|dy + \int_{B_{2R}}\rho^{\frac{1}{2}}|\mathrm{div}\mathbf{u}|dy$$
$$\lesssim R^{-1}\|\rho\|^{\frac{1}{2}}_{L^\gamma(\mathbb{R}^2)}\|\mathbf{u}\|_{L^2(\mathbb{R}^2)}R^{1-\frac{1}{\gamma}} + \left(\int_{B_{2R}}\rho dy\right)^{\frac{1}{2}}\|\mathrm{div}\mathbf{u}\|_{L^2(\mathbb{R}^2)} \to 0$$

as $R\to\infty$. Here in the last step we used the estimate (4.2).

For the last three terms, using Eq. (1.6a), an integration by parts yields

$$K_4+K_5+K_6 = -\frac{1}{2}\int_{\mathbb{R}^2}\phi_R\rho\mathbf{u}\cdot\nabla|\mathbf{u}|^2dy - \frac{1}{2}\int_{\mathbb{R}^2}\phi_R\rho y\cdot\nabla|\mathbf{u}|^2dy - \int_{\mathbb{R}^2}\phi_R\rho|\mathbf{u}|^2dy$$
$$= \frac{1}{2}\int_{\mathbb{R}^2}|\mathbf{u}|^2\rho\mathbf{u}\cdot\nabla\phi_R dy + \frac{1}{2}\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2 y\cdot\nabla\phi_R dy.$$

Note that

$$\left|\int_{\mathbb{R}^2}\rho|\mathbf{u}|^2 y\cdot\nabla\phi_R dy\right|\lesssim\int_{B_{2R}\setminus B_R}\rho|\mathbf{u}|^2 dy.$$

Moreover, the Ladyzenskaya's inequality

$$\|\mathbf{u}\|_{L^4}\lesssim\|\mathbf{u}\|_{L^2}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2}^{\frac{1}{2}},$$

combined with $\rho\in L^\infty(\mathbb{R}^2)$, yields

$$\begin{aligned}\left|\int_{\mathbb{R}^2}|\mathbf{u}|^2\rho\mathbf{u}\cdot\nabla\phi_R dy\right|&\lesssim R^{-1}\int_{B_{2R}\setminus B_R}|\mathbf{u}|^3 dy\\ &\lesssim R^{-1}\left(\int_{B_{2R}}dy\right)^{\frac{1}{4}}\|\mathbf{u}\|_{L^4(\mathbb{R}^2)}^3\lesssim R^{-\frac{1}{2}}\|\mathbf{u}\|_{L^4(\mathbb{R}^2)}^{\frac{3}{4}}.\end{aligned}$$

Therefore the fact that $\rho|\mathbf{u}|^2\in L^1(\mathbb{R}^2)$ implies

$$|K_4+K_5+K_6|\lesssim R^{-\frac{1}{2}}\|\mathbf{u}\|_{L^4(\mathbb{R}^2)}^{\frac{3}{4}}+\int_{B_{2R}\setminus B_R}\rho|\mathbf{u}|^2 dy\to 0$$

as $R\to\infty$.

Passing $R\to\infty$ in (4.3), we conclude that

$$\mu\int_{\mathbb{R}^2}|\nabla\mathbf{u}|^2 dy+(\mu+\lambda)\int_{\mathbb{R}^2}|\mathrm{div}\mathbf{u}|^2 dy=0,$$

and $\mathbf{u}$ is a constant vector in $\mathbb{R}^2$, which, combined with the integrability conditions $\mathbf{u}\in L^2(\mathbb{R}^2)$, yields $\mathbf{u}=0$. Moreover, from the momentum equation once again, we get $\nabla\rho^\gamma=0$, and thus the density is again a constant. This fact, combined with the fact that $\rho\in L^1(\mathbb{R}^2)$, yields that $\rho=0$ in $\mathbb{R}^2$.

Acknowledgements

X. Hu's work was partially supported by grants from the Research Grants Council of the Hong Kong (Project No. CityU 11300417, 11301919, 11300420).

References

[1] D. Bresch, P. Jabin, *Global existence of weak solutions for compressible Navier-Stokes equations: Thermodynamically unstable pressure and anisotropic viscous stress tensor*, Ann. of Math. 188 (2018), 577–684.

[2] D. Chae, *Remarks on the Liouville type results for the compressible Navier-Stokes equations in* $\mathbb{R}^N$, Nonlinearity 25 (2012), 1345–1349.

[3] B. Desjardins, *Regularity of weak solutions of the compressible isentropic Navier-Stokes equations*, Comm. Partial Differential Equations 22 (1997), 977–1008.

[4] R. Erban, *On the existence of solutions to the Navier-Stokes equations of a two-dimensional compressible flow*, Math. Methods Appl. Sci. 26 (2003), 489–517.

[5] E. Feireisl, *Dynamics of Viscous Compressible Fluids*, Oxford Lecture Series in Mathematics and its Applications, 26, Oxford University Press, 2004.

[6] E. Feireisl, A. Novotný, H. Petzeltová, *On the existence of globally defined weak solutions to the Navier-Stokes equations*, J. Math. Fluid Mech. 3 (2001), 358–392.

[7] J. Frehse, M. Steinhauer, W. Weigant, *The Dirichlet problem for viscous compressible isothermal Navier-Stokes equations in two dimensions*, Arch. Ration. Mech. Anal. 198 (2010), 1–12.

[8] P. Germain, T. Iwabuchi, *Self-similar solutions of the compressible Navier-Stokes equations*, Arch. Ration. Mech. Anal. 240 (2021), 1645–1673.

[9] P. Germain, T. Iwabuchi, T. Léger, *On self-similar solutions to degenerate compressible Navier-Stokes equations*, Comm. Math. Phys. 381 (2021), 1001–1030.

[10] X. Hu, *Hausdorff dimensions of concentrations for isentropic compressible Navier-Stokes equations*, Arch. Ration. Mech. Anal. 234 (2019), 375–416.

[11] X. Hu, *Weak solutions for compressible isentropic Navier-Stokes equations in dimensions three*, Arch. Ration. Mech. Anal. 242 (2021), 1907–1945.

[12] S. Jiang, P. Zhang, *On spherically symmetric solutions of the compressible isentropic Navier-Stokes equations*, Comm. Math. Phys. 215 (2001), 559–581.

[13] S. Jiang, P. Zhang, *Remarks on weak solutions to the Navier-Stokes equations for 2-D compressible isothermal fluids with spherically symmetric initial data*, Indiana Univ. Math. J. 51 (2002), 345–355.

[14] S. Jiang, C. Zhou, *Existence of weak solutions to the three-dimensional steady compressible Navier-Stokes equations*, Ann. Inst. H. Poincare Anal. Non Lineaire 28 (2011), 485-498.

[15] P. Lions, *Mathematical Topics in Fluid Mechanics. Vol. 2. Compressible Models*, Oxford Lecture Series in Mathematics and its Applications, 10, Oxford Science Publications, Clarendon Press, 1998.

[16] J. Necas, M. Ruzicka, V. Sverak, *On Leray's self-similar solutions of the Navier-Stokes equations*, Acta Math. 176 (1996), 283–294.

[17] A. Novotný, M. Pokorný, *Weak and variational solutions to steady equations for compressible heat conducting fluids*, SIAM J. Math. Anal. 43 (2011), 1158–1188.

[18] A. Novotný, I. Straskraba, *Introduction to the Mathematical Theory of Compressible Flow*, Oxford Lecture Series in Mathematics and its Applications, 27, Oxford University Press, 2004.

[19] M. Padula, *Existence of global solutions for 2-dimensional viscous compressible flows*, J. Funct. Anal. 69 (1986), 1–20.

[20] P. Plotnikov, J. Sokolowski, *Concentrations of solutions to time-discretizied compressible Navier-Stokes equations*, Comm. Math. Phys. 258 (2005), 567–608.

[21] P. Plotnikov, W. Weigant, *Isothermal Navier-Stokes equations and Radon transform*, SIAM J. Math. Anal. 47 (2015), 626–653.

[22] P. Plotnikov, W. Weigant, *Steady 3D viscous compressible flows with adiabatic exponent* $\gamma \in (1,\infty)$, J. Math. Pures Appl. (9) 104 (2015), 58–82.

[23] P. Plotnikov, W. Weigant, *Weak Solutions to 2D and 3D Compressible Navier-Stokes Equations in Critical Cases*, Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, 1601–1671, Springer, 2018.

[24] T. Tsai, *On Leray's self-similar solutions of the Navier-Stokes equations satisfying local energy estimates*, Arch. Rational Mech. Anal. 143 (1998), 29–51.