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Asymptotic Behavior of Solutions to One-Dimensional Compressible Navier-Stokes-Poisson Equations with Large Initial Data

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Abstract. In this paper, we are concerned with the large time behavior of global solutions to the Cauchy problem of one-dimensional compressible Navier-Stokes-Poisson equations with density and/or temperature dependent transport coefficients and large initial data. The initial data are assumed to be without vacuum and mass concentrations, and the same is shown to be hold for the global solution constructed. The proof is based on some detail analysis on uniform positive lower and upper bounds of the specific volume and the absolute temperature.

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Key words: Navier-Stokes-Poisson equations, global solutions with large data, density and/or temperature dependent transport coefficients.

1 Introduction and main results

1.1 The problem and our main results

The compressible Navier-Stokes-Poisson (denote as NSP in the sequel) equations

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which take form of compressible Navier-Stokes equations coupled with self-consistent Poisson equation are often used to simulate the motion of viscous fluid under the influence of self-consistent electrostatic potential force. In this paper, we will consider one-dimensional non-isentropic compressible NSP equations, which can be written in the Lagrange coordinates as

$$v_t - u_x = 0, \tag{1.1a}$$

$$u_t + p(v,\theta)_x = \left(\frac{\mu u_x}{v}\right)_x + \frac{\phi_x}{v}, \tag{1.1b}$$

$$\left(E + \frac{u^2}{2}\right)_t + \big(up(v,\theta)\big)_x = \left(\frac{\mu u u_x}{v}\right)_x + \left(\frac{\kappa\theta_x}{v}\right)_x + \frac{u\phi_x}{v}, \tag{1.1c}$$

$$\left(\frac{\phi_x}{v}\right)_x = 1 - ve^{-\phi}, \quad \lim_{|x|\to\infty}\phi(t,x) = 0. \tag{1.1d}$$

Here $t \geq 0$ and $x \in \mathbb{R}$ are the time and Lagrangian spatial variables. The unknown functions $v(t,x)$, $u(t,x)$, $\theta(t,x)$ and $\phi(t,x)$ stand for the specific volume, the velocity, the absolute temperature, and the self-consistent potential, respectively. p and E are the pressure and internal energy. $\mu > 0$ and $\kappa > 0$ are transport coefficients which are assumed to be smooth functions of the specific volume v and/or the absolute temperature θ. Throughout this paper, we focus on the case that the background doping profile is a positive constant which, without loss of generality, can be normalized to be 1. Moreover, we consider the ideal, polytropic gas

$$p = \frac{R\theta}{v}, \quad E = c_v\theta. \tag{1.2}$$

Here c_v and R are the specific heat at constant volume and the specific gas constant, respectively.

This paper is concerned with the problem on the construction of global smooth nonvacuum solutions $(v(t,x),u(t,x),\theta(t,x),\ \phi(t,x))$ together with the precise description of their large time behaviors to the Cauchy problem of the NSP system (1.1), (1.2) with prescribed large initial data

$$\big(v(0,x),u(0,x),\theta(0,x)\big) = \big(v_0(x),u_0(x),\theta_0(x)\big), \tag{1.3}$$

which is further assumed to satisfy the following far field conditions:

$$\lim_{|x|\to\infty}\big(v_0(x),u_0(x),\theta_0(x)\big) = \big(v_\pm,u_\pm,\theta_\pm\big). \tag{1.4}$$

This paper is concentrated on the case when the far fields $(v_\pm,u_\pm,\theta_\pm)$ of the initial data $(v_0(x),u_0(x),\theta_0(x))$ are the same, i.e. $(v_-,u_-,\theta_-) = (v_+,u_+,\theta_+)$, and without loss of generality, one can assume $v_- = v_+ = 1, u_- = u_+ = 0, \theta_- = \theta_+ = 1$ in the rest of this paper.

Such a problem has been studied by many mathematicians and many results have been obtained. Before stating our main results, we first recall some former results closely related.

For the one-dimensional compressible Navier-Stokes equations, Kanel' [13] first studied the isentropic case with $p=Rv^{-\gamma}(\gamma>1)$ and positive constant viscosity and obtained the existence and large time behavior of global solutions with large initial data. For the non-isentropic case with positive constant transport coefficients, the existence of global solutions in bounded domain with large initial data was first obtained by Kazhikhov and Shelukhin in [17]. After that, many efforts have been made to study the asymptotic behavior of global solutions constructed by Kazhikhov and Shelukhin in [17] as $t\to\infty$. With some additional smallness conditions imposed on the initial data, the corresponding results are confirmed in [4, 9, 14, 15, 23, 24] and the references therein. To remove the smallness assumptions, Jiang [10, 11] firstly introduced a localized representation of the specific volume to deduce its pointwise upper and lower bounds independent of the time and space variables, with which the global solutions constructed by Kazhikhov and Shelukhin in [17] are shown to be convergent locally in space as time goes to infinity for large initial data and the temperature is bounded from below and above locally in space for all $t\geq 0$. For the problem on whether such global solutions converge to a constant steady state uniformly with respect to the spatial variable as time goes to infinity for large initial data or not, Li and Liang [19] gave a positive answer by deriving lower and upper bounds for the temperature independent of both time and space. Then, Wang and Zhao [28] extended the corresponding results to the case when transport coefficients depend on the density and temperature, which are assumed to be proportional to $h(v)\theta^{\alpha}$ for certain non-degenerate smooth function $h(v)$. From then on, there are a lot of results on the stability of basic wave patterns to the one-dimensional compressible Navier-Stokes equations with large initial perturbation, see [6–8, 26] and the references therein.

When the potential force is taken into account, the corresponding results with small initial perturbations were well-established: For multidimensional Cauchy problem, the global existence of smooth solutions to the compressible NSP system away from vacuum with optimal temporal decay estimates was established in [18] for the isentropic case, and in [5, 31] for non-isentropic case. Then, the pointwise decay estimates of global solutions for bipolar compressible NSP system were obtained by Wu and Wang in [29, 30] for isentropic flow and non-isentropic flow, respectively. While for the one-dimensional case, the nonlinear stability of some basic wave patterns to NSP system are studied in [1–3].

The story for the case with large initial perturbation is quite different and fewer results are available up to now on the global existence and asymptotic behavior of solutions to the compressible NSP equations with large initial data. For one-dimensional Cauchy problem, Tan *et al.* [25] studied the global solutions for a class of degenerate density and/or temperature dependent transport coefficients with large initial data away from vacuum. Moreover, they obtained the large time behavior of the global solutions when the adiabatic exponent $\gamma > 1$ is assumed to satisfy that $\gamma - 1$ is sufficiently small. After that, Liu *et al.* [21] considered the initial boundary value problems for compressible NSP system on the exterior domain, and under the radial symmetry assumption, the global existence of solutions with large initial data on a domain exterior to a ball in $\mathbb{R}^n (n \geq 1)$ is proved in [21]. For three-dimensional case, Liu *et al.* [22] considered an initial value problem of compressible NSP equations subject to large and non-flat doping profile in whole space $\mathbb{R}^3$ and established the global well-posedness of strong solutions with large oscillations and vacuum provided the initial data are of small energy and the steady state is strictly away from vacuum. For the nonlinear stability of some basic wave patterns under large initial perturbation, the only result available up to now is on the nonlinear stability of rarefaction waves for isentropic compressible NSP obtained in [32].

Even so, for one-dimensional compressible non-isentropic NSP system, to the best of our knowledge, the only result on the precise description of the large time behaviors of global solutions to its Cauchy problem obtained up to now is on the Nishida-Smoller type global large solution constructed in [25] and the main purpose of this paper is to show that the Cauchy problem (1.1)-(1.4) admits a unique global smooth nonvacuum solution $(v(t,x),u(t,x),\theta(t,x),\phi(t,x))$ and the nonvacuum constant equilibrium state $(1,0,1,0)$ is time-asymptotically nonlinear stable for large initial data provided that the transport coefficients μ and κ satisfy some assumptions listed below.

In this paper, we assume that the heat conductivity κ takes the following form:

$$\kappa(v,\theta) = \kappa_1 + \kappa_2 v \theta^b \tag{1.5}$$

for some positive constants κ_1, κ_2, and b. As for the viscosity coefficient μ, we assume that $\mu = \mu(v)$ is a smooth function of v for $v > 0$ which satisfies

$$\mu(v) \simeq \begin{cases} v^{-\ell_1}, & v \to 0^+, \\ v^{\ell_2}, & v \to \infty \end{cases} \tag{1.6}$$

for some nonnegative constants $\ell_1 \geq 0$, $\ell_2 \geq 0$ and

$$v \left| \frac{d\mu(v)}{dv} \right|^2 \lesssim |\mu(v)|^3 \tag{1.7}$$

holds for $v>0$. Here and in the rest of this paper, $f(x)\lesssim g(x)$ holds for all $x\in A$ means that there exists a generic positive constant $C>0$ such that $f(x)\leq Cg(x)$ holds for all $x\in A$. $f(x)\simeq g(x)(x\in A)$ means both $f(x)\lesssim g(x)$ and $g(x)\lesssim f(x)$ hold for $x\in A$.

Under the above assumptions imposed on the transport coefficients μ and κ, we now turn to state our main result. To do so, for each given constant $0<w<1$, we first introduce the notation

$$H(w):=\sup_{w\leq\sigma\leq w^{-1}}\left|\left(\mu(\sigma),\mu'(\sigma),\mu''(\sigma),\mu'''(\sigma)\right)\right| \tag{1.8}$$

and then our result can be stated as follows.

Theorem 1.1. *Suppose that*

(i) *The viscosity coefficient $\mu(v)$ is assumed to satisfy* (1.6) *and* (1.7).

(ii) *The initial data $(v_0(x),u_0(x),\theta_0(x))$ satisfy*

$$\begin{aligned}
&\left(v_0(x)-1,u_0(x),\theta_0(x)-1\right)\in H^3(\mathbb{R}),\\
&\|(v_0-1,u_0,\theta_0-1)\|_{H^3(\mathbb{R})}\leq\Pi_0,\\
&V_0\leq v_0(x)\leq V_0^{-1},\quad \theta_0(x)\geq V_0,\quad \forall x\in\mathbb{R},
\end{aligned}$$

where Π_0 and $V_0\leq 1$ are given positive constants.

(iii) *The parameters b,ℓ_1,ℓ_2 and κ_2 are assumed to satisfy*

$$b>\frac{29}{4},\quad \ell_1>1,\quad \ell_2>1,\quad \kappa_2>1, \tag{1.9}$$

$$\kappa_2\geq C_0\left(\frac{\lambda_1^{\lambda_1}}{(\lambda_1-1)^{\lambda_1-1}}\right)^{\frac{2b+3}{14\varsigma_1}}, \tag{1.10}$$

where

$$\lambda_1=\frac{7}{2b+3}\left(b+\frac{3}{2}+\frac{1}{2\ell_1}\right)>1$$

and C_0 is some constant depending only on Π_0,V_0 and κ_1.

Then the Cauchy problem (1.1)-(1.6) *admits a unique solution $(v(t,x),u(t,x),\theta(t,x),\phi(t,x))$ satisfying*

$$\begin{aligned}
&\left(v(t,x)-1,u(t,x),\theta(t,x)-1\right)\in C\left([0,\infty);H^3(\mathbb{R})\right),\\
&v_x(t,x)\in L^2\left([0,\infty);H^2(\mathbb{R})\right),\\
&\left(u_x(t,x),\theta_x(t,x)\right)\in L^2\left([0,\infty);H^3(\mathbb{R})\right),\\
&\phi(t,x)\in C^0\left([0,\infty);H^5(\mathbb{R})\right),
\end{aligned}$$

and

$$\inf_{(t,x)\in[0,+\infty)\times\mathbb{R}}\{v(t,x),\theta(t,x)\}>0,\qquad \sup_{(t,x)\in[0,+\infty)\times\mathbb{R}}\{v(t,x),\theta(t,x)\}<+\infty.$$

In addition, we have the following large time behavior:

$$\lim_{t\to+\infty}\sup_{x\in\mathbb{R}}|(v(t,x)-1,u(t,x),\theta(t,x)-1,\phi(t,x))|=0.$$

Remark 1.1. Several remarks concerning Theorem 1.1 are listed below:

(1) The restrictions (1.9) and (1.10) imposed on the parameters b,ℓ_1,ℓ_2, and κ_2 mean that both the viscosity μ and the heat conductivity κ can not be constants. We are convinced that such assumptions are due to the limitations of our argument and can be relaxed to cover the case when the transport coefficients are positive constants. Such a problem is under our current research.

(2) Only the case when $(v_-,u_-,\theta_-)=(v_+,u_+,\theta_+)=(1,0,1)$ is considered in this paper. When $(v_-,u_-,\theta_-)\neq(v_+,u_+,\theta_+)$, the large time behaviors of the corresponding global solutions will be well-described by some basic wave patterns consisting of rarefaction waves, viscous shock waves, viscous contact waves, and/or their linear superpositions and the nonlinear stability of these wave patterns with large initial perturbation is under our current study.

1.2 Main ideas to prove Theorem 1.1

Now we outline our main ideas used to prove Theorem 1.1. As is well-known, the key point to guarantee the global solvability of the Cauchy problem (1.1)-(1.6) is to deduce the desired positive lower and upper bounds, which can depend on the time variable, on the specific volume and the absolute temperature, while to derive its large behavior, generally speaking, one had to show further that the above bounds on the specific volume and the absolute temperature are independent of the time variable also.

To the best of our knowledge, the effective methods available up to now to deduce the uniform positive lower and upper bounds on the specific volume and the absolute temperature rely heavily on the assumptions imposed on the state equations between the five thermodynamic variables and the transport coefficients.

When the transport coefficients are positive constants, an effective method to yield the desired uniform positive lower and upper bounds on the specific volume for one-dimensional Navier-Stokes equations satisfying the state equations for ideal polytropic gas is the method developed by Jiang in [10, 11], which is based on a localized representation of the specific volume, it seems, however, that such a method does not work for the compressible NSP equations (1.1), (1.2).

When $\mu=\mu(v)$ and $\kappa=\kappa(v,\theta)$, the argument used in [19,28] tells us that one can utilize Kanel's method introduced in [13] to derive an estimate on the lower and upper bounds on the specific volume in terms of the $\|\theta\|_{L^\infty([0,T]\times\mathbb{R})}$ and then use method introduced by Li and Liang in [19] to deduce the desired estimates on the absolute temperature, which in turn yields also the desired estimates on the specific volume. But unfortunately, unlike the case for one-dimensional compressible Navier-Stokes equation for ideal polytropic gas, cf. [19,28] for example, if we apply this method to our problem, we can only get that

$$\|\theta(t)-1\|^2_{L^\infty(\mathbb{R})}\lesssim 1+\|\theta-1\|^{\alpha}_{L^\infty}$$

holds for some positive constant $\alpha>2$, from which one can not deduce the desired estimate on $\|\theta\|_{L^\infty([0,T]\times\mathbb{R})}$.

Even so, motivated by the work of Liao and Zhao [20] on a viscous radiative and reactive gas, where the constitutive relations for the pressure and the internal energy consist of a linear term in θ corresponding to the perfect polytropic contribution and a fourth-order radiative part due to the Stefan-Boltzmann radiative law [27]

$$p=\frac{R\theta}{v}+\frac{a\theta^4}{3},\quad E=c_v\theta+av\theta^4$$

and the following four auxiliary functions:

$$\begin{aligned}
\widetilde{X}(t)&=\int_0^t\int_{\mathbb{R}}\left(\frac{1}{1+\|\theta\|^{\varsigma_2}_{L^\infty_{t,x}}}+\frac{\theta^{b+3}}{1+\|\theta\|^{\varsigma_1}_{L^\infty_{t,x}}}\right)\theta_t^2(\tau,x)\mathrm{d}x\mathrm{d}\tau,\\
\widetilde{Y}(t)&=\max_{0\le\tau\le t}\left\{\int_{\mathbb{R}}\left(\frac{1}{1+\|\theta\|^{2\varsigma_2}_{L^\infty_{t,x}}}+\theta^{2b}\right)\theta_x^2(\tau,x)\mathrm{d}x\right\},\\
\widetilde{Z}(t)&=\max_{0\le\tau\le t}\left\{\int_{\mathbb{R}}u_{xx}^2(\tau,x)\mathrm{d}x\right\},\\
\widetilde{W}(t)&=\int_0^t\int_{\mathbb{R}}\frac{\mu(v,\theta)u_{xt}^2}{v}(\tau,x)\mathrm{d}x\mathrm{d}\tau
\end{aligned}$$

are introduced in [20] to deal with the fourth-order radiative part $\frac{a}{3}\theta^4,av\theta^4$ appeared in both $p(v,\theta)$ and $E(v,\theta)$.

By making full use of the positive contribution of the fourth-order radiative part $\frac{a}{3}\theta^4, av\theta^4$ appeared in both $p(v,\theta)$ and $E(v,\theta)$, they can indeed prove that

$$\begin{aligned}
&\widetilde{X}(T)+\widetilde{Y}(T)\lesssim 1+\frac{\varepsilon}{1+\|\theta\|^{\varsigma_1}_{L^\infty_{t,x}}+\|\theta\|^{\varsigma_2}_{L^\infty_{t,x}}}\widetilde{W}(T)+\widetilde{Z}(T)^{\widetilde{\lambda}_1},\\
&\widetilde{W}(T)\lesssim 1+\left(1+\|\theta\|^{\varsigma_1}_{L^\infty_{t,x}}+\|\theta\|^{\varsigma_2}_{L^\infty_{t,x}}\right)\widetilde{X}(T)+\widetilde{Y}(T)+\widetilde{Z}(T)^{\widetilde{\lambda}_2},\\
&\widetilde{Z}(T)\lesssim 1+\left(1+\|\theta\|^{\varsigma_1}_{L^\infty_{t,x}}+\|\theta\|^{\varsigma_2}_{L^\infty_{t,x}}\right)\left(\widetilde{X}(T)+\widetilde{Y}(T)\right)+\widetilde{Z}(T)^{\widetilde{\lambda}_3}
\end{aligned}$$

hold for some constants $0<\widetilde{\lambda}_i<1$ for $i=1,2,3$, from which one can deduce that $\widetilde{X}(T),\widetilde{Y}(T),\widetilde{Z}(T)$, and $\widetilde{W}(T)$ are bounded from above, and, as a consequence, one can deduce the desired upper bound estimate on the absolute temperature $\theta(t,x)$.

We borrowed this idea and introduced three auxiliary functions $X(t),Y(t)$ and $Z(t)$ in (3.1). However, when we tried to control $X(T)+Y(T)$, the term $\int_0^t\int_{\mathbb{R}}\frac{\mu\kappa\theta_t u_x^2}{v^2}dxd\tau$ appeared, which can not be bounded by $Z(T)$. Besides that, the lack of the estimates on the derivatives of electric potential ϕ such as $\int_0^t\int_{\mathbb{R}}\phi_t^2dxd\tau$ and $\int_0^t\int_{\mathbb{R}}\phi_{xt}^2dxd\tau$ lead to some difficulties when we want to bound $Z(T)$. Thus some new strategies should be developed and our main ideas can be summarized as follows:

(i) We first apply Kanel's method [13] to yield an estimate of the lower and upper bound of $v(t,x)$ in terms of $\|\theta\|_{L^\infty([0,T]\times\mathbb{R})}$ simultaneously in Lemma 2.3.

(ii) Since we had to deal with the higher power of $Y(T)$ in the control of $X(T)+Y(T)$, we estimate the corresponding difficult terms carefully and pay particular attention to the relation of their estimates with respect to the parameter κ_2. Then such a difficulty can indeed be overcome by choosing the parameter κ_2 sufficiently large. Moreover, in order to derive the desired bound of $Z(T)$, we made efforts on the estimates of $\int_0^t\int_{\mathbb{R}}\frac{\phi_{xt}^2}{v}dxd\tau$, $\int_0^t\int_{\mathbb{R}}v\phi_t^2dxd\tau$ and $\int_{\mathbb{R}}u_t^2dx$, based on which we can then deduce the bound of $X(T),Y(T)$, and $Z(T)$.

(iii) By using the delicate energy method, we can derive the higher-order derivatives estimates in Section 4. Then using the continuation argument designed by Wang and Zhao [28], we can prove our main Theorem 1.1.

1.3 Outline of this paper and notations

The rest of this paper is organized as follows: Section 2 is devoted to derive pointwise lower and upper bounds for the specific volume $v(t,x)$. Section 3 focuses on the uniform-in-time upper bound and a local-in-time lower bound of the absolute temperature $\theta(t,x)$. In Section 4, we derive some estimates on higher order derivatives of the solution $(v(t,x),u(t,x),\theta(t,x),\phi(t,x))$.

Notations. For simplicity, we use $\|\cdot\|_\infty$ to denote the norm in $L^\infty([0,T]\times\mathbb{R})$ with $T>0$ being some given positive constant, $\|\cdot\|$ and $\|\cdot\|_q$ are used to denote the norm $\|\cdot\|_{L^2(\mathbb{R})}$ and the norm $\|\cdot\|_{H^q(\mathbb{R})}$, respectively.

We introduce $A\lesssim B$ (or $B\gtrsim A$) if $A\le CB$ holds uniformly for some constant C depending solely on $\Pi_0,V_0,H(V_0)$ and κ_1, and $A\simeq B$ if $A\lesssim B$ and $B\gtrsim A$.

2 Pointwise bounds for the specific volume

We define, for some positive constants N,m_1,m_2,s, and t, the set of functions $X(s,t;m_1,m_2,N)$ as follows:

$$
\begin{aligned}
&X(s,t;m_1,m_2,N)\\
:=&\left\{(v(\tau,x),u(\tau,x),\theta(\tau,x),\phi(\tau,x)):\left|\begin{array}{l}\big(v(\tau,x)-1,u(\tau,x),\theta(\tau,x)-1\big)\in C\big([s,t];H^3(\mathbb{R})\big),\\ \phi(t,x)\in C\big([s,t];H^5(\mathbb{R})\big),\\ v_x(\tau,x)\in L^2\big(s,t;H^2(\mathbb{R})\big),\\ \big(u_x(\tau,x),\theta_x(\tau,x)\big)\in L^2\big(s,t;H^3(\mathbb{R})\big),\\ \mathcal{E}(s,t)\le N^2,\\ v(\tau,x)\ge m_1,\ \theta(\tau,x)\ge m_2,\ \forall(\tau,x)\in[s,t]\times R\end{array}\right.\right\},
\end{aligned}
$$

where

$$
\mathcal{E}(s,t):=\sup_{s\le\tau\le t}\left\{\|(v-1,u,\theta-1)(\tau)\|_3^2\right\}+\int_s^t\left(\|v_x(\tau)\|_2^2+\|(u_x,\theta_x)(\tau)\|_3^2\right)\mathrm{d}\tau.
$$

Theorem 1.1 will be proved by combining local solvability result, certain a priori estimates, and the continuation argument. First, from the well-established local existence result for hyperbolic-parabolic system, cf. [12], one can deduce that there exists a sufficiently small positive constant $t_1>0$ and certain positive constants m_1,m_2, and N such that the Cauchy problem (1.1)-(1.6) admits a unique local solution $(v(t,x),u(t,x),\theta(t,x),\phi(t,x))\in X(0,t_1;m_1,m_2,N)$. Suppose that such a local solution $(v(t,x),u(t,x),\theta(t,x),\ \phi(t,x))$ has been extended to the time interval $[0,T]$ for some $T\ge t_1$ and satisfies $(v(t,x),u(t,x),\theta(t,x),\phi(t,x))\in X(0,T;m_1,m_2,N)$, to prove Theorem 1.1, we only need to derive certain a priori estimates on the solution $(v(t,x),u(t,x),\ \theta(t,x),\phi(t,x))$ in terms of the initial data $(v_0(x),u_0(x),\theta_0(x))$ but independent of the constants m_1,m_2, and N.

Now we turn to deduce the desired a priori estimates. Our goal in this section is to get the desired lower and upper bound of the specific volume $v(t,x)$ in terms of $\|\theta\|_\infty$ and for this purpose, we need to obtain the basic energy estimates.

Lemma 2.1. *Under the assumption of Theorem* 1.1, *we have for* $0\le t\le T$ *that*

$$\begin{aligned}&\int_{\mathbb{R}}\left(\eta+\frac{\phi_x^2}{2v}+v\left(1-\phi e^{-\phi}-e^{-\phi}\right)\right)(t,x)\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu u_x^2}{v\theta}+\frac{\kappa\theta_x^2}{v\theta^2}\right)(\tau,x)\mathrm{d}x\mathrm{d}\tau\\&\le\int_{\mathbb{R}}\left(\eta+\frac{\phi_x^2}{2v}+v\left(1-\phi e^{-\phi}-e^{-\phi}\right)\right)(0,x)\mathrm{d}x:=e_0,\end{aligned}\tag{2.1}$$

where

$$\eta=\eta(v,u,\theta)=R\Phi(v)+\frac{u^2}{2}+c_v\Phi(\theta)\quad \textit{and}\quad \Phi(x)=x-\ln x-1.$$

Proof. Multiplying (1.1a), (1.1b), and (1.1c) by $R(1-v^{-1}),u$, and $(1-\theta^{-1})$, respectively, and adding the resulting identity together, we find

$$\eta_t+\left\{(p-R)u-\frac{\mu uu_x}{v}-\frac{\kappa(\theta-1)\theta_x}{v\theta}\right\}_x+\left(\frac{\mu u_x^2}{v\theta}+\frac{\kappa\theta_x^2}{v\theta^2}\right)=\frac{u\phi_x}{v}.\tag{2.2}$$

The right-hand side of the above identity can be rewritten as

$$\begin{aligned}\int_{\mathbb{R}}\frac{u\phi_x}{v}\mathrm{d}x&=\int_{\mathbb{R}}\frac{u\phi_x}{v}\left(\left(\frac{\phi_x}{v}\right)_x+ve^{-\phi}\right)\mathrm{d}x\\&=\int_{\mathbb{R}}u\left(\frac{1}{2}\left(\frac{\phi_x}{v}\right)^2\right)_x\mathrm{d}x+\int_{\mathbb{R}}u\phi_xe^{-\phi}\mathrm{d}x\\&=-\frac{1}{2}\int_{\mathbb{R}}\frac{\phi_x^2}{v^2}v_t\mathrm{d}x+\int_{\mathbb{R}}e^{-\phi}v_t\mathrm{d}x.\end{aligned}$$

Due to

$$\begin{aligned}\int_{\mathbb{R}}e^{-\phi}v_t\mathrm{d}x&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}ve^{-\phi}\mathrm{d}x+\int_{\mathbb{R}}e^{-\phi}v\phi_t\mathrm{d}x\\&=\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\left(ve^{-\phi}+v\phi e^{-\phi}\right)\mathrm{d}x-\int_{\mathbb{R}}\phi\left(ve^{-\phi}\right)_t\mathrm{d}x\\&=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}v\left(1-e^{-\phi}-\phi e^{-\phi}\right)\mathrm{d}x+\int_{\mathbb{R}}\phi\left(\frac{\phi_x}{v}\right)_{xt}\mathrm{d}x\\&=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}v\left(1-e^{-\phi}-\phi e^{-\phi}\right)\mathrm{d}x-\int_{\mathbb{R}}v\cdot\frac{\phi_x}{v}\cdot\left(\frac{\phi_x}{v}\right)_t\mathrm{d}x\end{aligned}$$

$$=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} v\big(1-e^{-\phi}-\phi e^{-\phi}\big)\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{\phi_x^2}{2v}\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}}\frac{\phi_x^2}{v^2}v_t\mathrm{d}x,$$

we can get that

$$\int_{\mathbb{R}}\frac{u\phi_x}{v}\mathrm{d}x=-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}} v\big(1-e^{-\phi}-\phi e^{-\phi}\big)\mathrm{d}x-\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}}\frac{\phi_x^2}{2v}\mathrm{d}x. \tag{2.3}$$

Inserting the above identity (2.3) into (2.2), we finish the proof of Lemma 2.1. □

The basic energy estimate obtained in Lemma 2.1 actually implies the following two corollaries, which will be used later.

Corollary 2.1. *Let $\alpha_1<\alpha_2$ be the two positive roots of the equation*

$$\Phi(x)=\frac{e_0}{\min\{R,c_v\}},$$

then for any $k\in\mathbb{Z}$, one can get that

$$\alpha_1\le\int_k^{k+1}v(t,x)\mathrm{d}x,\quad \int_k^{k+1}\theta(t,x)\mathrm{d}x\le\alpha_2,\quad 0\le t\le T \tag{2.4}$$

and there exist $a_k(t),b_k(t)\in[k,k+1]$ such that

$$\alpha_1\le v\big(t,a_k(t)\big),\quad \theta\big(t,b_k(t)\big)\le\alpha_2,\quad 0\le t\le T. \tag{2.5}$$

The main purpose of our next corollary is to show that the electric potential $\phi(t,x)$ can be bounded from above and below.

Corollary 2.2. *For all $(t,x)\in[0,T]\times\mathbb{R}$, we have*

$$\phi(t,x)\simeq 1. \tag{2.6}$$

Proof. For each $0\le t\le T$, set $\Lambda(t)=\{x\in\mathbb{R}:|\phi(t,x)|\ge 1\}$, it suffices to prove that there exists a positive constant C depending only on e_0 such that

$$|\phi(t,x_0)|\le C$$

holds for any $x_0\in\Lambda(t)$ and $0\le t\le T$.

Without loss of generality, we assume that $\phi(t,x_0)\geq 1$ and define the following function:

$$\widetilde{\Phi}(t,x)=\int_1^{\phi(t,x)}\sqrt{1-ze^{-z}-e^{-z}}\mathrm{d}z,$$

which satisfies for any $(t,x)\in[0,T]\times\mathbb{R}$ that

$$\begin{aligned}\left|\widetilde{\Phi}(t,x)\right|&\lesssim\left|\int_0^{\phi(t,x)}\sqrt{1-ze^{-z}-e^{-z}}\mathrm{d}z\right|+1\\&=\left|\int_{-\infty}^{x}\sqrt{1-\phi(t,y)e^{-\phi(t,y)}-e^{-\phi(t,y)}}\phi_y(t,y)\mathrm{d}y\right|+1\\&\lesssim\left(\int_{\mathbb{R}}v(1-\phi e^{-\phi}-e^{-\phi})\mathrm{d}y\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\frac{\phi_y^2}{v}\mathrm{d}y\right)^{\frac{1}{2}}+1\lesssim 1.\end{aligned}$$

Meanwhile, it follows from

$$\sqrt{1-ze^{-z}-e^{-z}}\gtrsim 1,\quad z\in[1,+\infty),$$

that

$$\left|\widetilde{\Phi}(t,x_0)\right|\gtrsim|\phi(t,x_0)|-1.$$

As a consequence, we can deduce that there exists a positive constant C, which depends only on e_0, such that

$$|\phi(t,x_0)|\leq C$$

holds for $0\leq t\leq T$. This completes the proof of Corollary 2.2. $\square$

The derivation of pointwise bounds for $v(t,x)$ relies on the following lemma.

Lemma 2.2. *Suppose that the conditions of Theorem* 1.1 *hold, then we can obtain for* $0\leq t\leq T$ *that*

$$\begin{aligned}&\int_{\mathbb{R}}\left(\frac{\mu v_x}{v}\right)^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\theta v_x^2}{v^3}+\frac{\phi_x^2}{v}\right)\mathrm{d}x\mathrm{d}\tau\\&\lesssim 1+\|\theta\|_\infty+\left\|v^{-1}\right\|_\infty^{\ell_1+1}+\|v\|_\infty^{\ell_2}.\end{aligned}\tag{2.7}$$

Proof. According to the chain rule, we have

$$\left(\frac{\mu u_x}{v}\right)_x=\left(\frac{\mu v_t}{v}\right)_x=\left(\frac{\mu v_x}{v}\right)_t,\tag{2.8}$$

which combined with (1.1b) implies

$$\left(\frac{\mu v_x}{v}\right)_t + R\frac{\theta v_x}{v^2} = u_t + R\frac{\theta_x}{v} - \frac{\phi_x}{v}.$$

Multiplying the above identity by $\frac{\mu v_x}{v}$ and integrating the resultant over $[0,t]\times\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}\left(\frac{\mu v_x}{v}\right)^2(t,x)\mathrm{d}x + R\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}(\tau,x)\mathrm{d}x\mathrm{d}\tau \\
=&\frac{1}{2}\int_{\mathbb{R}}\left(\frac{\mu v_x}{v}\right)^2(0,x)\mathrm{d}x + \underbrace{\int_0^t\int_{\mathbb{R}} u_t\cdot\frac{\mu v_x}{v}\mathrm{d}x\mathrm{d}\tau}_{\mathcal{I}_1} \\
&\quad \underbrace{+R\int_0^t\int_{\mathbb{R}}\frac{\mu v_x\theta_x}{v^2}\mathrm{d}x\mathrm{d}\tau}_{\mathcal{I}_2} \underbrace{-\int_0^t\int_{\mathbb{R}}\frac{\phi_x}{v}\cdot\frac{\mu v_x}{v}\mathrm{d}x\mathrm{d}\tau}_{\mathcal{I}_3}.
\end{aligned}$$

It follows from (2.1) and (2.8) that

$$\begin{aligned}
\mathcal{I}_1 =& \int_{\mathbb{R}}\frac{\mu u v_x}{v}(t,x)\mathrm{d}x - \int_{\mathbb{R}}\frac{\mu u v_x}{v}(0,x)\mathrm{d}x + \int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}\mathrm{d}x\mathrm{d}\tau \\
\le& C + \frac{1}{4}\int_{\mathbb{R}}\left(\frac{\mu v_x}{v}\right)^2\mathrm{d}x + \int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}\mathrm{d}x\mathrm{d}\tau, \\
\mathcal{I}_2 \le& \frac{R}{2}\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}\mathrm{d}x\mathrm{d}\tau + C\int_0^t\int_{\mathbb{R}}\frac{\mu\theta_x^2}{v\theta}\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

For the last term $\mathcal{I}_3$, we introduce the function

$$g(v) = \int_1^v \frac{\mu(z)}{z}\mathrm{d}z,$$

then $\lim_{|x|\to\infty} g(v(t,x)) = 0$ and

$$\begin{aligned}
&\int_0^t\int_{\mathbb{R}}\frac{\phi_x}{v}\cdot\frac{\mu v_x}{v}\mathrm{d}x\mathrm{d}\tau - \int_0^t\int_{\mathbb{R}}\frac{\mu(e^\phi)}{v}\phi_x^2\mathrm{d}x\mathrm{d}\tau \\
=&\int_0^t\int_{\mathbb{R}}\frac{\phi_x}{v}\cdot\left(g(v)-g(e^\phi)\right)_x\mathrm{d}x\mathrm{d}\tau \\
=&-\int_0^t\int_{\mathbb{R}}(1-ve^{-\phi})\left(g(v)-g(e^\phi)\right)\mathrm{d}x\mathrm{d}\tau \ge 0.
\end{aligned}$$

Here we have used the monotonicity of $g(v)$ with respect to v.

As a consequence, combining Corollary 2.2, we have

$$\begin{aligned}&\int_{\mathbb{R}}\left(\frac{\mu v_x}{v}\right)^2(t,x)\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu\theta v_x^2}{v^3}+\frac{\phi_x^2}{v}\right)(\tau,x)\mathrm{d}x\mathrm{d}\tau\\ \lesssim&1+\int_0^t\int_{\mathbb{R}}\left(\frac{\mu u_x^2}{v}+\frac{\mu\theta_x^2}{v\theta}\right)\mathrm{d}x\mathrm{d}\tau\lesssim 1+\|\theta\|_\infty+\left\|\frac{\mu\theta}{\kappa}\right\|_\infty.\end{aligned}$$

Since $b\geq\frac{29}{4}$, we have

$$\frac{\mu\theta}{\kappa}=\frac{\mu(v)\theta}{\kappa_1+\kappa_2 v\theta^b}\lesssim\begin{cases}\mu(v), & \theta\leq 1,\\ \dfrac{\mu(v)}{v}, & \theta\geq 1,\end{cases}$$

which combined with (1.6) implies (2.7). Thus the proof of Lemma 2.2 is complete. □

By applying Kanel's technique, we can obtain the pointwise bounds for the specific volume $v(t,x)$ in the following lemma.

Lemma 2.3. *Assume that the conditions of the Theorem* 1.1 *hold, then we can deduce that*

$$\|v^{-1}\|_\infty\lesssim 1+\|\theta\|_\infty^{\varsigma_1},\quad \|v\|_\infty\lesssim 1+\|\theta\|_\infty^{\varsigma_2},\tag{2.9}$$

where

$$\varsigma_1:=\frac{1}{2\ell_1},\quad \varsigma_2:=\frac{1}{2\ell_2+1}.\tag{2.10}$$

Proof. Let

$$\Psi(v):=\int_1^v\frac{\mu(z)}{z}\sqrt{\Phi(z)}\mathrm{d}z.$$

In light of (1.6), we have

$$\|v\|_\infty^{\frac{1}{2}+\ell_2}+\|v^{-1}\|_\infty^{\ell_1}\lesssim 1+\|\Psi(v)\|_\infty.$$

Estimates (2.1) and (2.7) imply for any $(t,x)\in[0,T]\times\mathbb{R}$,

$$\begin{aligned}|\Psi(v(t,x))|&\leq\left|\int_{-\infty}^x\Psi_v\big(v(t,y)\big)\cdot v_y(t,y)\mathrm{d}y\right|\\ &\leq\left\|\left(\frac{\mu v_x}{v}\right)(t)\right\|\cdot\left\|\sqrt{\Phi(v(t))}\right\|\\ &\lesssim 1+\|\theta\|_\infty^{\frac{1}{2}}+\left\|v^{-1}\right\|_\infty^{\frac{\ell_1+1}{2}}+\|v\|_\infty^{\frac{\ell_2}{2}}.\end{aligned}$$

In view of $\ell_1>1$, $\ell_2>1$, and the Young's inequality, we finish the proof of Lemma 2.3. □

Lemmas 2.2 and 2.3 yield the following result.

Corollary 2.3. *Suppose that the conditions of Theorem* 1.1 *hold, we can get for* $0\le t\le T$ *that*

$$\int_{\mathbb{R}}\Big(\frac{\mu v_x}{v}\Big)^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\bigg(\frac{\mu\theta v_x^2}{v^3}+\frac{\phi_x^2}{v}\bigg)\mathrm{d}x\mathrm{d}\tau\lesssim 1+\|\theta\|_\infty. \tag{2.11}$$

3 Pointwise bounds for the absolute temperature

In this section we will derive a uniform-in-time upper bound and a local-in-time lower bound for the absolute temperature θ. For this purpose, motivated by [16, 20], we introduce the following auxiliary functions:

$$\begin{aligned}
X(t)&:=\int_0^t\int_{\mathbb{R}}\frac{\kappa(v(\tau,x),\theta(\tau,x))}{v(\tau,x)}\theta_t^2(\tau,x)\mathrm{d}x\mathrm{d}\tau,\\
Y(t)&:=\sup_{0\le\tau\le t}\int_{\mathbb{R}}\frac{|\kappa(v(\tau,x),\theta(\tau,x))|^2}{v(\tau,x)^2}\theta_x^2(\tau,x)\mathrm{d}x,\\
Z(t)&:=\sup_{0\le\tau\le t}\int_{\mathbb{R}}u_{xx}^2(\tau,x)\mathrm{d}x.
\end{aligned}\tag{3.1}$$

The functions $X(t)$, $Y(t)$, and $Z(t)$ are introduced to control $\|\theta\|_\infty$, $\|u_x(t)\|$, and $\|u_x\|_\infty$, respectively. Actually, we have

Lemma 3.1. *Suppose that the conditions of Theorem* 1.1 *hold, we can get that*

$$\|\theta\|_\infty\lesssim 1+\widetilde{Y}(T)^{\frac{1}{2b+3}}, \tag{3.2}$$

where $\widetilde{Y}(T)=\kappa_2^{-2}Y(T)\le Y(T)$.

Proof. For any $0\le t\le T$ and $x\in[k,k+1]$, we utilize (2.4) and (2.5) to obtain

$$\begin{aligned}
&\big|\theta(t,x)^\alpha-\theta(t,b_k(t))^\alpha\big|\\
\le&\alpha\left|\int_k^{k+1}\theta^{\alpha-1}(t,y)\theta_y(t,y)\mathrm{d}y\right|\\
\le&\alpha\left|\int_k^{k+1}\frac{\kappa^2\theta_y^2}{v^2}(t,y)\mathrm{d}y\right|^{\frac12}\cdot\left|\int_k^{k+1}\frac{v^2}{\kappa^2}\theta^{2\alpha-2}(t,y)\mathrm{d}y\right|^{\frac12}\\
\le&\alpha Y(T)^{\frac12}\cdot\kappa_2^{-1}\|\theta\|_\infty^{\alpha-b-\frac32}\cdot\left|\int_k^{k+1}\theta(t,y)\mathrm{d}y\right|^{\frac12}\\
\lesssim&\kappa_2^{-1}Y(T)^{\frac12}\|\theta\|_\infty^{\alpha-b-\frac32}.
\end{aligned}$$

Taking $\alpha = b+\frac{3}{2}$, we conclude the estimate (3.2) and consequently complete the proof of Lemma 3.1. $\square$

The next lemma follows immediately from the Gagliardo-Nirenberg inequality and the Sobolev inequality.

Lemma 3.2. *For each* $0\le t\le T$, *one has*

$$\begin{aligned} &\sup_{0\le\tau\le t}\|u_x(\tau)\|\lesssim 1+Z(t)^{\frac{1}{4}},\\ &\|u_x\|_{L^\infty([0,t]\times\mathbb{R})}=\sup_{[\tau,x)\in[0,t]\times\mathbb{R}}\left\{|u_x(\tau,x)|\right\}\lesssim 1+Z(t)^{\frac{3}{8}}. \end{aligned}\tag{3.3}$$

Now we start to show that $X(T)$ and $Y(T)$ can be controlled by $Y(T)$ and $Z(T)$.

Lemma 3.3. *Under the assumptions of Theorem* 1.1 *we have*

$$X(T)+Y(T)\lesssim 1+\kappa_2^{-\frac{14\varsigma_1}{2b+3}}Y(T)^{\lambda_1}+Z(T)^{\frac{7}{8}}. \tag{3.4}$$

Here

$$\lambda_1=\frac{7(b+3/2+\varsigma_1)}{2b+3}>1.$$

Proof. In the same manner as in [16,27], if we set

$$K(v,\theta)=\int_0^\theta\frac{\kappa(v,z)}{v}\mathrm{d}z,$$

then by using

$$\left(\frac{\kappa}{v}\theta_t\right)_x-\left(\frac{\kappa}{v}\theta_x\right)_t=\left(\frac{\kappa}{v}\right)_v(v_x\theta_t-v_t\theta_x),$$

one can easily verify that

$$K_t=\frac{\kappa}{v}\theta_t+K_v u_x, \tag{3.5}$$

$$K_{xt}=\left(\frac{\kappa}{v}\theta_x\right)_t+\left(\frac{\kappa}{v}\right)_v v_x\theta_t+K_v u_{xx}+K_{vv}v_x u_x, \tag{3.6}$$

$$|K|\lesssim\frac{\theta}{v}+\kappa_2\theta^{b+1},\quad \left|\left(\frac{\kappa}{v}\right)_v\right|\lesssim\frac{1}{v^2},\quad |K_v|\lesssim\frac{\theta}{v^2}. \tag{3.7}$$

Now multiplying (1.1c) by K_t and integrating the resulting identity over $[0,t]\times\mathbb{R}$, we arrive at

$$\begin{aligned}c_vX(t)+\frac{1}{2}Y(t)=&\frac{1}{2}Y(0)-c_v\int_0^t\int_{\mathbb{R}}K_v\theta_tu_x\mathrm{d}x\mathrm{d}\tau-R\int_0^t\int_{\mathbb{R}}\frac{\kappa u_x\theta\theta_t}{v^2}\mathrm{d}x\mathrm{d}\tau\\&-R\int_0^t\int_{\mathbb{R}}\frac{K_v\theta u_x^2}{v}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}\cdot\Big(\frac{\kappa}{v}\theta_t+K_vu_x\Big)\mathrm{d}x\mathrm{d}\tau\\&-\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x}{v}\cdot\Big(\frac{\kappa}{v}\Big)_vv_x\theta_t\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x}{v}(K_vu_{xx}+K_{vv}v_xu_x)\mathrm{d}x\mathrm{d}\tau\\=:&\frac{1}{2}Y(0)+\sum_{i=1}^6\mathcal{J}_i.\end{aligned}$$

We now turn to control $\mathcal{J}_i$, $i=1,2,\dots,6$ term by term. Using Hölder's inequality, Young's inequality, (2.1), (2.9), and (3.2), we have

$$\begin{aligned}|\mathcal{J}_1|\le& C\int_0^t\int_{\mathbb{R}}\left|\frac{\sqrt{\kappa}}{\sqrt{v}}\theta_t\right|\cdot\left|\frac{\sqrt{\mu}u_x}{\sqrt{v}\sqrt{\theta}}\right|\cdot\frac{\sqrt{\theta^3}}{v\sqrt{\mu\kappa}}\mathrm{d}x\mathrm{d}\tau\\\le&\frac{c_v}{8}X(T)+C\left\|\frac{\theta^3}{v^2\mu\kappa}\right\|_\infty\\\le&\frac{c_v}{8}X(T)+C+CY(T)^{\frac{3+\varsigma_1}{2b+3}},\\|\mathcal{J}_2|\le& C\int_0^t\int_{\mathbb{R}}\left|\frac{\sqrt{\kappa}}{\sqrt{v}}\theta_t\right|\cdot\left|\frac{\sqrt{\mu}u_x}{\sqrt{v\theta}}\right|\cdot\frac{\sqrt{\kappa\theta^3}}{v\sqrt{\mu}}\mathrm{d}x\mathrm{d}\tau\\\le&\frac{c_v}{8}X(T)+\left\|\frac{\kappa\theta^3}{\mu v^2}\right\|_\infty\\\le&\frac{c_v}{8}X(T)+C+CY(T)^{\frac{3+\varsigma_1}{2b+3}}+C\kappa_2^{-\frac{3}{2b+3}}Y(T)^{\frac{b+3}{2b+3}},\\|\mathcal{J}_3|\le& C\int_0^t\int_{\mathbb{R}}\frac{\theta^2u_x^2}{v^3}\mathrm{d}x\mathrm{d}\tau\le C+CY(T)^{\frac{3+\varsigma_1}{2b+3}},\end{aligned}$$

where we have used the facts that

$$\left\|\frac{1}{\mu v}\right\|_\infty+\left\|\frac{1}{\mu}\right\|_\infty+\left\|\frac{v}{\mu}\right\|_\infty\lesssim1.$$

As for the term $\mathcal{J}_4$, we have to treat it carefully. Since

$$|\mathcal{J}_4|\le\left|\int_0^t\int_{\mathbb{R}}\frac{\mu\kappa\theta_tu_x^2}{v^2}\mathrm{d}x\mathrm{d}\tau\right|+\left|\int_0^t\int_{\mathbb{R}}\frac{K_v\mu u_x^3}{v}\mathrm{d}x\mathrm{d}\tau\right|=:\mathcal{J}_{41}+\mathcal{J}_{42}$$

it follows from Hölder's inequality, Young's inequality, (2.1), (2.9), (3.2), and (3.3) that

$$
\begin{aligned}
|\mathcal{J}_{41}| &\leq \int_0^t \int_{\mathbb{R}} \left|\frac{\sqrt{\kappa}}{\sqrt{v}}\theta_t\right| \cdot \left|\frac{\sqrt{\mu}u_x}{\sqrt{v\theta}}\right| \cdot \left|\frac{\sqrt{\mu\kappa\theta}}{v}u_x\right| \mathrm{d}x\mathrm{d}\tau \\
&\leq \frac{c_v}{4}X(T)+C\left\|\frac{\mu\kappa\theta}{v^2}\right\|_\infty \cdot \left(1+Z(T)^{\frac{3}{4}}\right) \\
&\leq \frac{c_v}{4}X(T)+C+CZ(T)^{\frac{7}{8}}+C\left\|\frac{\mu\theta}{v^2}\right\|_\infty^7+C\left\|\frac{\kappa_2\mu\theta^{b+1}}{v}\right\|^7 \\
&\leq \frac{c_v}{4}X(T)+C+CZ(T)^{\frac{7}{8}}+Y(T)^{\frac{7[(\ell_1+1)\varsigma_1+\varsigma_1+1]}{2b+3}} \\
&\quad +\left(\kappa_2 \cdot \kappa_2^{-\frac{2}{2b+3}((\ell_1+1)\varsigma_1+b+1)}Y(T)^{\frac{(\ell_1+1)\varsigma_1+b+1}{2b+3}}\right)^7 \\
&\leq \frac{c_v}{4}+C+Y(T)^{\frac{\frac{21}{2}+14\varsigma_1}{2b+3}}+\kappa_2^{-\frac{14\varsigma_1}{2b+3}}Y(T)^{\lambda_1}+CZ(T)^{\frac{7}{8}}, \\
|\mathcal{J}_{42}| &\leq C\int_0^t \int_{\mathbb{R}} \frac{\mu\theta u_x^3}{v^3}\mathrm{d}x\mathrm{d}\tau \leq C\int_0^t \int_{\mathbb{R}} \left|\frac{\mu u_x^2}{v\theta}\right| \cdot \left|\frac{\theta^2}{v^2}\right| \cdot |u_x| \mathrm{d}x\mathrm{d}\tau \\
&\leq C\left(1+Y(T)^{\frac{2+2\varsigma_1}{2b+3}}\right)\left(1+Z(T)^{\frac{3}{8}}\right) \\
&\lesssim 1+Y(T)^{\frac{4+4\varsigma_1}{2b+3}}+Z(T)^{\frac{3}{4}},
\end{aligned}
$$

where we have used the fact that

$$
\left\|\frac{\mu}{v}\right\|_\infty \lesssim \|\theta\|_\infty^{\max\{(\ell_1+1)\varsigma_1,(\ell_2-1)\varsigma_2\}} = \|\theta\|_\infty^{(\ell_1+1)\varsigma_1}.
$$

The term $\mathcal{J}_5$ can be controlled as follows:

$$
\begin{aligned}
|\mathcal{J}_5| &\leq C\int_0^t \int_{\mathbb{R}} \left|\frac{\kappa v_x\theta_x\theta_t}{v^3}\right| \mathrm{d}x\mathrm{d}\tau \\
&\leq C\left\|\frac{1}{\sqrt{\kappa v\mu}}\right\|_\infty \int_0^t \left\|\frac{\sqrt{\kappa}}{\sqrt{v}}\theta_t\right\| \cdot \left\|\frac{\mu v_x}{v}\right\| \cdot \left\|\frac{\kappa\theta_x}{v}\right\|_{L^\infty(\mathbb{R})} \mathrm{d}\tau \\
&\leq \frac{c_v}{8}X(T)+C\|\theta\|_\infty \int_0^t \int_{\mathbb{R}} \left|\frac{\kappa\theta_x}{v} \cdot \left(c_v\theta_t+pu_x-\frac{\mu u_x^2}{v}\right)\right| \mathrm{d}x\mathrm{d}\tau \\
&\leq \frac{c_v}{8}X(T)+C\|\theta\|_\infty \left(\int_0^t \int_{\mathbb{R}} \frac{\kappa\theta^2}{v}\left(\theta_t^2+\frac{\theta^2u_x^2}{v^2}+\frac{\mu^2u_x^4}{v^2}\right)\mathrm{d}x\mathrm{d}\tau\right)^{\frac{1}{2}}
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{c_v}{8}X(T)+C\|\theta\|_\infty^2 X(T)^{\frac{1}{2}}+\kappa_2^{\frac{1}{2}}\|\theta\|^{\frac{b+7}{2}}+\kappa_2^{\frac{1}{2}}\left(1+Z(T)^{\frac{3}{8}}\right)\|\theta\|^{\frac{b+5+(\ell_1+1)\varsigma_1}{2}}\\
&\leq \frac{c_v}{4}X(T)+CY(T)^{\frac{4}{2b+3}}+\kappa_2^{-\frac{11}{4b+6}}Y(T)^{\frac{b+7}{4b+6}}+\kappa_2^{-\frac{8+2\varsigma_1}{2b+3}}Y(T)^{\frac{b+\frac{11}{2}+\varsigma_1}{2b+3}}+CZ(T)^{\frac{3}{4}}.
\end{aligned}
$$

For the last term $\mathcal{J}_6$, since

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x}{v}K_v u_{xx}\mathrm{d}x\mathrm{d}\tau\\
=&\int_0^t\int_{\mathbb{R}}\left(\frac{\kappa\theta_x}{v}\right)_x K_v u_x\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x}{v}\left(K_{vv}v_x+\left(\frac{\kappa}{v}\right)_v\theta_x\right)u_x\mathrm{d}x\mathrm{d}\tau,
\end{aligned}
$$

we have

$$
\mathcal{J}_6=\int_0^t\int_{\mathbb{R}}\left(\frac{\kappa\theta_x}{v}\right)_x K_v u_x\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x}{v}\left(\frac{\kappa}{v}\right)_v\theta_x u_x\mathrm{d}x\mathrm{d}\tau.
$$

Thus

$$
\begin{aligned}
|\mathcal{J}_6|&\leq C\int_0^t\int_{\mathbb{R}}\left|\frac{\sqrt{\mu}u_x}{\sqrt{v\theta}}\right|\cdot\left|\left(\frac{\kappa\theta_x}{v}\right)_x\right|\cdot\sqrt{\frac{\theta^3}{\mu v^3}}\mathrm{d}x\mathrm{d}\tau+C\int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v\theta^2}\cdot\frac{\theta^2}{v^2}|u_x|\mathrm{d}x\mathrm{d}\tau\\
&\leq C\left(\int_0^t\int_{\mathbb{R}}\frac{\theta^3}{\mu v^3}\left(\theta_t^2+\frac{\theta^2u_x^2}{v^2}+\frac{\mu^2u_x^4}{v^2}\right)\mathrm{d}x\mathrm{d}\tau\right)^{\frac{1}{2}}+\left(1+Y(T)^{\frac{2+2\varsigma_1}{2b+3}}\right)\left(1+Z(T)^{\frac{3}{8}}\right)\\
&\leq C\left(1+Y(T)^{\frac{3+\varsigma_1}{2(2b+3)}}\right)X(T)^{\frac{1}{2}}+Y(T)^{\frac{3+\varsigma_1}{2b+3}}+\left(1+Y(T)^{\frac{2+2\varsigma_1}{2b+3}}\right)\left(1+Z(T)^{\frac{3}{8}}\right)\\
&\leq\frac{c_v}{4}X(T)+C+CY(T)^{\frac{4+4\varsigma_1}{2b+3}}+Z(T)^{\frac{3}{4}}.
\end{aligned}
$$

Combining all the above estimates and utilizing Young's inequality, we can complete the proof of our lemma. □

Our next step is to show that $Z(T)$ can be controlled by $X(T)$ and $Y(T)$ conversely. Before that, we give the following estimates for $\int_0^t\int_{\mathbb{R}}\frac{\phi_{xt}^2}{v}\mathrm{d}x\mathrm{d}\tau$ and $\int_{\mathbb{R}}u_t^2\mathrm{d}x$, which will be used to derive the bound of $Z(T)$.

Lemma 3.4. *Under the assumptions of Theorem* 1.1 *we have*

$$
\int_0^T\int_{\mathbb{R}}\left(\frac{\phi_{xt}^2}{v}+v\phi_t^2\right)(t,x)\mathrm{d}x\mathrm{d}t\lesssim 1+Y(T)+Z(T)^{\lambda_2},\tag{3.8}
$$

where

$$
\lambda_2=\frac{3}{4}\cdot\frac{2b+3}{2b+2-2\varsigma_1}<1.
$$

Proof. Differentiating (1.1d) with respect to t yields

$$\left(\frac{\phi_x}{v}\right)_{xt} = -(ve^{-\phi})_t.$$

Hence

$$\begin{aligned}\left(\left(\frac{\phi_x}{v}\right)_t \phi_t\right)_x &= \left(\frac{\phi_x}{v}\right)_t \phi_{xt} + \left(\frac{\phi_x}{v}\right)_{xt} \phi_t \\ &= \left(\frac{\phi_{xt}}{v} - \frac{\phi_x u_x}{v^2}\right)\phi_{xt} - u_x e^{-\phi}\phi_t + ve^{-\phi}\phi_t^2.\end{aligned}$$

Integrating the above identity over $(0,t)\times\mathbb{R}$ and using Hölder's inequality, Young's inequality, and Corollary 2.2 yields

$$\begin{aligned}&\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_{xt}^2}{v} + ve^{-\phi}\phi_t^2\right)\mathrm{d}x\mathrm{d}\tau \\ =&\int_0^t\int_{\mathbb{R}}\frac{u_x}{v^2}\phi_x\phi_{xt}\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}} u_x e^{-\phi}\phi_t\mathrm{d}x\mathrm{d}\tau \\ \leq&\left\|\frac{u_x}{v}\right\|_\infty \cdot \left\|\frac{\phi_x}{\sqrt{v}}\right\|_{L^2([0,T],L^2(\mathbb{R}))} \cdot \left\|\frac{\phi_{xt}}{\sqrt{v}}\right\|_{L^2([0,T],L^2(\mathbb{R}))} \\ &\quad + \left\|\sqrt{\frac{\theta}{\mu}e^{-\phi}}\right\|_\infty \cdot \left\|\sqrt{ve^{-\phi}}\phi_t\right\|_{L^2([0,T],L^2(\mathbb{R}))} \cdot \left\|\frac{\sqrt{\mu}u_x}{\sqrt{v\theta}}\right\|_{L^2([0,T],L^2(\mathbb{R}))} \\ \leq&\frac{1}{2}\int_0^T\int_{\mathbb{R}}\frac{\phi_{xt}^2}{v}\mathrm{d}x\mathrm{d}\tau + C\left(1+Y(T)^{\frac{1+2\varsigma_1}{2b+3}}\right)\left(1+Z(T)^{\frac{3}{4}}\right) \\ &\quad + \frac{1}{2}\int_0^T\int_{\mathbb{R}} ve^{-\phi}\phi_t^2\mathrm{d}x\mathrm{d}\tau + C\left(1+Y(T)^{\frac{1}{2b+3}}\right) \\ \leq&\frac{1}{2}\int_0^T\int_{\mathbb{R}}\frac{\phi_{xt}^2}{v}\mathrm{d}x\mathrm{d}\tau + \frac{1}{2}\int_0^T\int_{\mathbb{R}} ve^{-\phi}\phi_t^2\mathrm{d}x\mathrm{d}\tau + C + CY(T) + CZ(T)^{\lambda_2}.\end{aligned}$$

Hence we have finished the proof of Lemma 3.4. □

Lemma 3.5. *Under the assumptions of Theorem* 1.1 *we have for* $0\leq t\leq T$ *that*

$$\int_{\mathbb{R}} u_t^2(t,x)\mathrm{d}x + \int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xt}^2\mathrm{d}x\mathrm{d}\tau \lesssim 1 + X(T) + Y(T) + Z(T)^{\lambda_3}, \tag{3.9}$$

where

$$\lambda_3 = \frac{3}{4}\cdot\frac{2b+3}{2b+3/2-\varsigma_1} < 1.$$

Proof. Differentiating (1.1b) with respect to t and multiplying the resulting identity by u_t, we have

$$\left(\frac{1}{2}u_t^2\right)_t+\frac{\mu u_{xt}^2}{v}=\left(u_t\left(\frac{\mu u_x}{v}\right)_t-u_tp_t\right)_x+u_{xt}\left(p_t-\left(\frac{\mu}{v}\right)_tu_x\right)+\left(\frac{\phi_x}{v}\right)_tu_t.$$

Integrating this identity over $(0,t)\times\mathbb{R}$ yields

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}}u_t^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xt}^2\mathrm{d}x\mathrm{d}\tau\\=&\frac{1}{2}\int_{\mathbb{R}}u_{0t}^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}}p_tu_{xt}\mathrm{d}x\mathrm{d}\tau\\&-\int_0^t\int_{\mathbb{R}}\left(\frac{\mu}{v}\right)_vu_x^2u_{xt}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_x}{v}\right)_tu_t\mathrm{d}x\mathrm{d}\tau\\=:&\frac{1}{2}\int_{\mathbb{R}}u_{0x}^2\mathrm{d}x+\sum_{j=1}^3\mathcal{K}_j.\end{aligned}$$

Noticing that (1.6) implies

$$\left|\left(\frac{\mu}{v}\right)_v\right|\lesssim\sqrt{\frac{\mu^3}{v^3}}+\frac{\mu}{v^2}=\sqrt{\frac{\mu^3}{v^3}}\left(1+\frac{1}{\sqrt{\mu v}}\right)\lesssim\sqrt{\frac{\mu^3}{v^3}},\tag{3.10}$$

$\mathcal{K}_1$ and $\mathcal{K}_2$ can be controlled by combining Hölder's and Young's inequalities with Lemmas 2.3, 3.1, and 3.2. That is

$$\begin{aligned}|\mathcal{K}_1|\le&\frac{1}{4}\int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xt}^2\mathrm{d}x\mathrm{d}\tau+C\int_0^t\int_{\mathbb{R}}\frac{v}{\mu}\left(\frac{\theta_t^2}{v^2}+\frac{\theta^2u_x^2}{v^4}\right)\mathrm{d}x\mathrm{d}\tau\\\le&\frac{1}{4}\int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xt}^2\mathrm{d}x\mathrm{d}\tau+CX(T)+CY(T)^{\frac{3}{2b+3}},\\|\mathcal{K}_2|\le&C\int_0^t\int_{\mathbb{R}}\left|\frac{\sqrt{\mu}}{\sqrt{v}}u_{xt}\right|\cdot\left|\frac{\sqrt{\mu}u_x}{\sqrt{v\theta}}\right|\cdot\left|\left(\frac{\mu}{v}\right)_v\cdot\frac{v\sqrt{\theta}}{\mu}\cdot u_x\right|\mathrm{d}x\mathrm{d}\tau\\\le&\frac{1}{4}\int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xt}^2\mathrm{d}x\mathrm{d}\tau+C\left(1+Y(T)^{\frac{(\ell_1+1)\varsigma_1+1}{2b+3}}\right)\left(1+Z(T)^{\frac{3}{4}}\right)\\\le&\frac{1}{4}\int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xt}^2\mathrm{d}x\mathrm{d}\tau+C+CY(T)+CZ(T)^{\lambda_3}.\end{aligned}$$

For the last term $\mathcal{K}_3$, we can get by utilizing (1.1b) and by splitting it into three terms:

$$\mathcal{K}_3=-\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_x}{v}\right)_tp_x\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_x}{v}\right)_t\left(\frac{\mu u_x}{v}\right)_x\mathrm{d}x\mathrm{d}\tau$$

$$+\frac{1}{2}\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_x^2}{v^2}\right)_t \mathrm{d}x\mathrm{d}\tau =: \sum_{j=1}^3 \mathcal{K}_{3j}.$$

Utilizing Lemma 3.4, we obtain

$$\begin{aligned}
|\mathcal{K}_{31}| \le& R\int_0^t\int_{\mathbb{R}}\left|\frac{\phi_{xt}}{v}\left(\frac{\theta_x}{v}-\frac{\theta v_x}{v^2}\right)\right|\mathrm{d}x\mathrm{d}\tau + R\int_0^t\int_{\mathbb{R}}\left|\frac{\phi_x u_x}{v^2}\left(\frac{\theta_x}{v}-\frac{\theta v_x}{v^2}\right)\right|\mathrm{d}x\mathrm{d}\tau \\
\le& \int_0^t\int_{\mathbb{R}}\frac{\phi_{xt}^2}{v}\mathrm{d}x\mathrm{d}\tau + \left\|\frac{\theta^2}{\kappa v^2}\right\|_\infty \int_0^t\int_{\mathbb{R}}\frac{\kappa\theta_x^2}{v\theta^2}\mathrm{d}x\mathrm{d}\tau + \left\|\frac{\theta}{\mu v^2}\right\|\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_x^2}{v^3}\mathrm{d}x\mathrm{d}\tau \\
&+\left\|\frac{\theta u_x}{\sqrt{\kappa}v^2}\right\|_\infty \left\|\frac{\phi_x}{\sqrt{v}}\right\|_{L^2([0,T],L^2(\mathbb{R}))}\left\|\frac{\sqrt{\kappa}\theta_x}{\sqrt{v}\theta}\right\|_{L^2([0,T],L^2(\mathbb{R}))} \\
&+\left\|\frac{\sqrt{\theta}u_x}{v^2\sqrt{\mu}}\right\|_\infty \left\|\frac{\phi_x}{\sqrt{v}}\right\|_{L^2([0,T],L^2(\mathbb{R}))}\left\|\frac{\sqrt{\mu\theta}v_x}{\sqrt{v^3}}\right\|_{L^2([0,T],L^2(\mathbb{R}))} \\
\le& C+CY(T)+CZ(T)^{\lambda_2}+\left(1+Z(T)^{\frac{3}{8}}\right)\left(1+Y(T)^{\frac{\frac{3}{2}+2\varsigma_1}{2b+3}}\right) \\
&+\left(1+Z(T)^{\frac{3}{8}}\right)\left(1+Y(T)^{\frac{\frac{3}{2}+\frac{3}{2}\varsigma_1}{2b+3}}\right) \\
\le& C+CY(T)+CZ(T)^{\lambda_2}.
\end{aligned}$$

The term $\mathcal{K}_{32}$ can be estimated similarly as follows:

$$\begin{aligned}
|\mathcal{K}_{32}| =& \int_0^t\int_{\mathbb{R}}\left|\left(ve^{-\phi}\right)_t\frac{\mu u_x}{v}\right|\mathrm{d}x\mathrm{d}\tau \\
\le& \int_0^t\int_{\mathbb{R}} e^{-\phi}\frac{\mu u_x^2}{v}\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{\mathbb{R}}\left|\mu u_x e^{-\phi}\phi_t\right|\mathrm{d}x\mathrm{d}\tau \\
\le& C\|\theta\|_\infty + C\|\sqrt{\mu\theta}\|_\infty\cdot\|\sqrt{v}\phi_t\|_{L^2([0,T],L^2(\mathbb{R}))}\cdot\left\|\frac{\sqrt{\mu}u_x}{\sqrt{v\theta}}\right\|_{L^2([0,T],L^2(\mathbb{R}))} \\
\lesssim& 1+Y(T)^{\frac{1}{2b+3}}+Y(T)^{\frac{1+(\ell_1+1)\varsigma_1}{2b+3}}+Y(T)+Z(T)^{\lambda_2} \\
\lesssim& 1+Y(T)+Z(T)^{\lambda_2}.
\end{aligned}$$

It is obvious that

$$|\mathcal{K}_{33}| \lesssim 1+Y(T)^{\frac{\varsigma_1}{2b+3}}.$$

Thus we have

$$|\mathcal{K}_3| \lesssim 1+Y(T)+Z(T)^{\lambda_2}. \tag{3.11}$$

Since $\lambda_2 < \lambda_3$, we can complete the proof of our lemma. $\square$

With Lemmas 3.4 and 3.5 in hand, now we start to bound $Z(T)$.

Lemma 3.6. *Under the assumptions of Theorem* 1.1*, we have*

$$Z(T)\lesssim 1+X(T)+Y(T). \tag{3.12}$$

Proof. We can conclude from (1.1b) that

$$u_{xx}=\frac{v}{\mu}\left(u_t+p_x-\left(\frac{\mu}{v}\right)_v v_x u_x-\frac{\phi_x}{v}\right).$$

By direct calculations, we obtain

$$\begin{aligned}Z(T)&\leq\int_{\mathbb{R}}\frac{v^2}{\mu^2}u_t^2\mathrm{d}x+\int_{\mathbb{R}}\frac{\theta_x^2}{\mu^2}\mathrm{d}x+\int_{\mathbb{R}}\frac{\theta^2 v_x^2}{\mu^2 v^2}\mathrm{d}x+\int_{\mathbb{R}}\frac{\mu}{v}v_x^2u_x^2+\int_{\mathbb{R}}\frac{\phi_x^2}{\mu^2}\mathrm{d}x\mathrm{d}x\\&\lesssim 1+X(T)+Y(T)+Z(T)^{\lambda_3}+Y(T)^{\frac{2}{2b+3}}+Z(T)^{\frac{3}{4}\cdot\frac{2b+3}{2b+2}}\\&\lesssim 1+X(T)+Y(T)+Z(T)^{\lambda_3},\end{aligned}$$

where we have used Lemma 3.5. Then our lemma can be proved by Young's inequality. ☐

We are now in a position to derive the upper bound of $\theta(t,x)$. In fact, Lemmas 3.3 and 3.6 tell us that

$$X(T)+Y(T)\leq C_0\left(1+\kappa_2^{-\frac{14\varsigma_1}{2b+3}}Y(T)^{\lambda_1}\right), \tag{3.13}$$

where

$$C_0=C_0\left(\Pi_0,V_0,\kappa_1,\|\phi_{0x}\|,\|\sqrt{1-\phi_0e^{-\phi_0}-e^{-\phi_0}}\|\right).$$

Thus we obtain

$$X(T)+Y(T)+Z(T)\lesssim 1 \tag{3.14}$$

as long as κ_2 is chosen sufficiently large. To be specific, a sufficient condition to guarantee the above estimate is that κ_2 is assumed to satisfy (1.10).

Recalling the definition of $X(T)$,$Y(T)$ and $Z(T)$, and by combining Lemmas 2.1 -3.6, we have the following lemma.

Lemma 3.7. *Under the assumptions of Theorem* 1.1*, there exist positive constants C_1 and C_2, which depend only on Π_0, V_0, $\|\phi_{0x}\|$, and $\|\sqrt{1-\phi_0e^{-\phi_0}-e^{\phi_0}}\|$, such that*

$$\theta(t,x)\leq C_1,\qquad \forall(t,x)\in[0,T]\times\mathbb{R}, \tag{3.15}$$

$$C_2\leq v(t,x)\leq C_2^{-1},\quad \forall(t,x)\in[0,T]\times\mathbb{R}. \tag{3.16}$$

Moreover, we have

$$\left\|\left(v-1,u,\theta-1,v_x,u_t,u_x,\theta_x,\phi_x,u_{xx},\sqrt{1-\phi e^{-\phi}-e^{-\phi}}\right)(t)\right\|^2 \\ +\int_0^t\left\|\left(\sqrt{\theta}v_x,u_x,\theta_x,\theta_t,\phi_x,\phi_t,u_{xt},\phi_{xt}\right)(\tau)\right\|^2\mathrm{d}\tau\lesssim 1. \tag{3.17}$$

Now we obtain uniform bounds for $\int_0^t\int_{\mathbb{R}}u_{xx}^2\mathrm{d}x\mathrm{d}\tau$ and $\int_0^t\int_{\mathbb{R}}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau$. In fact, we have the following lemma.

Lemma 3.8. *Under the assumptions of Theorem* 1.1*, for any* $0\le t\le T$*, we have*

$$\|u_x(t)\|^2+\int_0^t\int_{\mathbb{R}}u_{xx}^2(\tau,x)\mathrm{d}x\mathrm{d}\tau\lesssim 1, \tag{3.18}$$

$$\|\theta_x(t)\|^2+\int_0^t\int_{\mathbb{R}}\theta_{xx}^2(\tau,x)\mathrm{d}x\mathrm{d}\tau\lesssim 1. \tag{3.19}$$

Proof. Firstly, multiplying (1.1b) by u_{xx} and integrating the resultant identity over $[0,t]\times\mathbb{R}$, we obtain

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}}u_x^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xx}^2\mathrm{d}x\mathrm{d}\tau\\
=&\frac{1}{2}\int_{\mathbb{R}}u_{0x}^2\mathrm{d}x+\int_0^t\int_{\mathbb{R}}p_xu_{xx}\mathrm{d}x\mathrm{d}\tau\\
&-\int_0^t\int_{\mathbb{R}}\left(\frac{\mu}{v}\right)_v v_xu_xu_{xx}\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}}\frac{\phi_x}{v}u_{xx}\mathrm{d}x\mathrm{d}\tau.\\
\le&\frac{1}{4}\int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xx}^2\mathrm{d}x\mathrm{d}\tau+C\int_0^t\|u_x\|_\infty^2\mathrm{d}\tau+C\\
\le&\frac{1}{4}\int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xx}^2\mathrm{d}x\mathrm{d}\tau+C\int_0^t\int_{\mathbb{R}}u_xu_{xx}\mathrm{d}x\mathrm{d}\tau+C\\
\le&\frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{\mu}{v}u_{xx}^2\mathrm{d}x\mathrm{d}\tau+C.
\end{aligned}$$

Here we used (2.1), Lemma 3.7, and Sobolev's inequality.

Secondly, multiplying (1.1c) by θ_{xx} and integrating the result with respect to x over $\mathbb{R}$, one has

$$\begin{aligned}
&\frac{c_v}{2}\int_{\mathbb{R}}\theta_x^2(t,x)\mathrm{d}x+\int_0^t\int_{\mathbb{R}}\frac{\kappa}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau\\
=&\frac{c_v}{2}\int_{\mathbb{R}}\theta_x^2(0,x)\mathrm{d}x+\int_0^t\int_{\mathbb{R}}pu_x\theta_{xx}\mathrm{d}x\mathrm{d}\tau
\end{aligned}$$

$$
\begin{aligned}
&-\int_0^t\int_{\mathbb{R}}\frac{\mu u_x^2}{v}\theta_{xx}\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_{\mathbb{R}}\left(\frac{\kappa}{v}\right)_v v_x\theta_x\theta_{xx}\mathrm{d}x\mathrm{d}\tau\\
\le&\frac{1}{4}\int_0^t\int_{\mathbb{R}}\frac{\kappa}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau+C\int_0^t\|\theta_x(\tau)\|_\infty^2\mathrm{d}\tau+C\\
\le&\frac{1}{2}\int_0^t\int_{\mathbb{R}}\frac{\kappa}{v}\theta_{xx}^2\mathrm{d}x\mathrm{d}\tau+C.
\end{aligned}
$$

This completes the proof of Lemma 3.8. □

It is easy to see that the constant C_0 in (3.13), the constants C_i, $i=1,2$ appeared in the estimates (3.15), (3.16) obtained in Lemma 3.7, and the right-hand sides of the estimates (3.17)-(3.19) obtained in Lemmas 3.7 and 3.8 depend on Π_0, V_0, $\|\phi_{0x}\|$, and $\|\sqrt{1-\phi_0e^{-\phi_0}-e^{-\phi_0}}\|$, the main purpose of our next lemma is to show that $\|\phi_{0x}\|$ and $\|\sqrt{1-\phi_0e^{-\phi_0}-e^{-\phi_0}}\|$ can be controlled by $\|v_0-1\|$ and V_0.

Lemma 3.9. *Under the assumptions of Theorem* 1.1*, there exists a generic positive constant C_3, which is independent of v and ϕ, such that the following estimates hold for all $0\le t\le T$:*

$$
\begin{aligned}
&\left\|\frac{\phi_x(t)}{\sqrt{v(t)}}\right\|^2+\left\|\sqrt{v(t)\left(1-\phi(t)e^{-\phi(t)}-e^{-\phi(t)}\right)}\right\|^2\\
\le&C_3\left(\left\|\frac{v(t)-1}{\sqrt{v(t)}}\right\|^2+\left\|\frac{v(t)-1}{\sqrt{v(t)}}\right\|^6\right).
\end{aligned}\tag{3.20}
$$

Consequently, one can deduce that

$$
\|\phi_{0x}\|^2+\left\|\sqrt{(1-\phi_0e^{-\phi_0}-e^{-\phi_0})}\right\|^2\le C_3V_0^{-4}\left(\|v_0-1\|^2+\|v_0-1\|^6\right).\tag{3.21}
$$

Proof. Multiplying the first equation of (1.1d) by $\phi(t,x)$ and integrating the result with respect to x over $\mathbb{R}$, we can get that

$$
\int_{\mathbb{R}}\left(\frac{\phi_x^2}{v}+v\phi\left(1-e^{-\phi}\right)\right)dx=-\int_{\mathbb{R}}\phi(1-v)dx.\tag{3.22}
$$

Noticing that there exist generic positive constants D_1 and D_2 independent of ϕ such that

$$
|\phi|\le\begin{cases}D_1\sqrt{\phi(1-e^{-\phi})}, & \phi\le 1,\\ D_2\sqrt{\phi}\cdot\sqrt{\phi(1-e^{-\phi})}, & \phi\ge 1,\end{cases}
$$

thus if $\phi \le 1$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} \phi\Big|_{\chi_{\phi\le 1}}(1-v)dx\right| &\le D_1\int_{\mathbb{R}}|1-v|\sqrt{\phi(1-e^{-\phi})}dx \\ &\le \frac{1}{4}\int_{\mathbb{R}} v\phi\left(1-e^{-\phi}\right)dx + D_1^2\int_{\mathbb{R}}\frac{(1-v)^2}{v}dx,\end{aligned} \tag{3.23}$$

while if $\phi \ge 1$, we can conclude that

$$\begin{aligned}\left|\int_{\mathbb{R}} \phi\Big|_{\chi_{\phi\ge 1}}(1-v)dx\right| &\le D_2\int_{\mathbb{R}}\left(\sqrt{\phi}\cdot\sqrt{\phi(1-e^{-\phi})}\right)\Big|_{\chi_{\phi\ge 1}}|1-v|dx \\ &\le \frac{1}{4}\int_{\mathbb{R}} v\phi\left(1-e^{-\phi}\right)dx + D_2^2\left\|\phi|_{\chi_{\phi\ge 1}}\right\|_{L^\infty(\mathbb{R})}\int_{\mathbb{R}}\frac{(1-v)^2}{v}dx.\end{aligned} \tag{3.24}$$

To derive an estimate on the last term in the right-hand side of (3.24), we set

$$\Omega(t,x) := \int_0^{\phi(t,x)}\sqrt{z(1-e^{-z})}dz,$$

which, for any $\phi \ge 1$, satisfies

$$\Omega(t,x) \ge \frac{1}{3}\left|\phi(t,x)|_{\chi_{\phi(t,x)\ge 1}}\right|^{\frac{3}{2}} - D_3 \tag{3.25}$$

for some generic positive constant $D_3 > 0$ independent of $\phi(t,x)$. On the other hand, due to

$$\begin{aligned}\Omega(t,x) &= \int_{-\infty}^{x}\frac{\partial\Omega(t,y)}{\partial y}dy \\ &= \int_{-\infty}^{x}\sqrt{\phi(t,y)(1-e^{-\phi(t,y)})}\frac{\partial\phi(t,y)}{\partial y}dy \\ &\le \left\|\frac{\phi_x}{\sqrt{v}}\right\|\left\|\sqrt{v(1-\phi e^{-\phi})}\right\|\end{aligned}$$

we can get from (3.25) that there exists a generic positive constant D_4 independent of ϕ such that

$$\left|\phi(t,x)|_{\chi_{\phi(t,x)\ge 1}}\right| \le D_4\left(1+\left\|\frac{\phi_x}{\sqrt{v}}\right\|^{\frac{2}{3}}\left\|\sqrt{v(1-\phi e^{-\phi})}\right\|^{\frac{2}{3}}\right). \tag{3.26}$$

Thus we get from (3.24) and (3.26) that

$$
\begin{aligned}
&\left|\int_{\mathbb{R}}\phi|_{\chi_{\phi\geq 1}}(1-v)dx\right| \qquad (3.27)\\
\leq&\frac{1}{4}\int_{\mathbb{R}}v\phi\left(1-e^{-\phi}\right)dx+D_2^2D_4\left(1+\left\|\frac{\phi_x}{\sqrt{v}}\right\|^{\frac{2}{3}}\left\|\sqrt{v(1-\phi e^{-\phi})}\right\|^{\frac{2}{3}}\right)\int_{\mathbb{R}}\frac{(v-1)^2}{v}dx\\
\leq&\frac{1}{2}\int_{\mathbb{R}}v\phi\left(1-e^{-\phi}\right)dx+\frac{1}{2}\int_{\mathbb{R}}\frac{\phi_x^2}{v}dx+D_5\left(\int_{\mathbb{R}}\frac{(v-1)^2}{v}dx+\left(\int_{\mathbb{R}}\frac{(v-1)^2}{v}dx\right)^3\right).
\end{aligned}
$$

Here D_5 is a generic positive constant independent of v and θ. Inserting (3.23), (3.27) into (3.22) and noticing that

$$
\begin{aligned}
\int_{\mathbb{R}}v\phi(1-e^{-\phi})dx&=\int_{\mathbb{R}}v(\phi-1+e^{-\phi})dx+\int_{\mathbb{R}}v(1-\phi e^{-\phi}-e^{-\phi})dx\\
&\geq\int_{\mathbb{R}}v(1-\phi e^{-\phi}-e^{-\phi})dx,
\end{aligned}
$$

which follows from the fact that

$$
\int_{\mathbb{R}}v(\phi-1+e^{-\phi})dx\geq 0,
$$

we can deduce (3.20), from which one can get (3.21) immediately. This completes the proof of Lemma 3.9. □

As a result of Lemmas 2.1-3.9, we can obtain the following corollary.

Corollary 3.1. *Under the assumptions of Theorem* 1.1*, there exists a positive constant* C_4*, which depends only on* Π_0 *and* V_0*, such that*

$$
\begin{aligned}
&\|(v-1,u,\theta-1,\phi)(t)\|_1^2\\
&\quad+\int_0^t\left(\left\|\sqrt{\theta}v_x(\tau)\right\|^2+\|(\theta_x,u_x)(\tau)\|_1^2+\|\phi_x(\tau)\|^2\right)\mathrm{d}\tau\leq C_4^2. \qquad (3.28)
\end{aligned}
$$

We now consider the local-in-time estimate on the lower bound on the absolute temperature $\theta(t,x)$. To this end, we repeat the method used in [28] to obtain the following lemma.

Lemma 3.10. *Under the assumptions stated in Theorem* 1.1*, for each* $0\leq s<t\leq T$ *and* $x\in\mathbb{R}$*, there exists a positive constant* C_5 *depending only on* Π_0 *and* V_0 *such that*

$$
\theta(t,x)\geq\frac{\inf_{x\in\mathbb{R}}\theta(s,x)}{C_5\inf_{x\in\mathbb{R}}\theta(s,x)(t-s)+1}. \qquad (3.29)
$$

4 Estimates of higher order derivatives

In this section, to simplify the presentation, we introduce $A\lesssim_h B$ if $A\lesssim C_h B$ holds uniformly for some constant C_h, depending only on Π_0, V_0, and $H(C_2)$, where C_2 is given in Lemma 3.7. $C(m_2)$ will be employed to denote some positive constant which depends only on m_2,Π_0,V_0, and $H(C_2)$. Firstly, we give the second-order derivatives of (v,u,θ) in the next two lemmas.

Lemma 4.1. *Assume that the conditions of Theorem* 1.1 *hold, then one can get that*

$$\begin{aligned}
&\sup_{t\in[0,T]}\|(u_{xx},\theta_{xx})(t)\|^2+\int_0^T\|(u_{xxx},\theta_{xxx})(t)\|^2\mathrm{d}t\\
\lesssim_h\, & C(m_2)+\sup_{t\in[0,T]}\|v_{xx}(t)\|^2+\int_0^T\|v_{xx}(t)\|^2\mathrm{d}t.
\end{aligned}\tag{4.1}$$

Proof. The proof is divided into the following steps:

Step 1. Differentiating (1.1b) with respect to x and multiplying the resulting identity by u_{xxx} gives

$$\begin{aligned}
&\left(\frac{1}{2}u_{xx}^2\right)_t+\frac{\mu u_{xxx}^2}{v}-\left(u_{xt}u_{xx}-\left(\frac{\phi_x}{v}\right)_x u_{xx}\right)_x\\
=\,&p_{xx}u_{xxx}+\Big(\frac{\mu u_{xxx}}{v}-\Big(\frac{\mu u_x}{v}\Big)_{xx}\Big)u_{xxx}+\left(\frac{\phi_x}{v}\right)_{xx}u_{xx}.
\end{aligned}$$

Integrating the above identity over $[0,t]\times\mathbb{R}$, we have

$$\begin{aligned}
&\frac{1}{2}\|u_{xx}(t)\|^2+\int_0^t\int_{\mathbb{R}}u_{xxx}^2\mathrm{d}x\mathrm{d}\tau\\
\leq\,&\frac{1}{2}\|u_{xx}(0)\|^2+\underbrace{\int_0^t\int_{\mathbb{R}}\Big(\frac{\mu u_{xxx}}{v}-\Big(\frac{\mu u_x}{v}\Big)_{xx}\Big)u_{xxx}\mathrm{d}x\mathrm{d}\tau}_{\mathcal{L}_1}\\
&+\underbrace{\int_0^t\int_{\mathbb{R}}p_{xx}u_{xxx}\mathrm{d}x\mathrm{d}\tau}_{\mathcal{L}_2}+\underbrace{\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_x}{v}\right)_{xx}u_{xx}\mathrm{d}x\mathrm{d}\tau}_{\mathcal{L}_3}.
\end{aligned}$$

By repeating the argument used in [28], the terms $\mathcal{L}_1$ and $\mathcal{L}_2$ can be estimated as follows:

$$|\mathcal{L}_1|+|\mathcal{L}_2|\lesssim C(\delta)+C(m_2)+\sup_{\tau\in[0,t]}\|v_{xx}(\tau)\|^2+\delta\int_0^t\|(u_{xxx},\theta_{xxx})(\tau)\|^2d\tau.$$

As for $\mathcal{L}_3$, noticing that

$$\mathcal{L}_3=-\int_0^t\int_{\mathbb{R}}(ve^{-\phi})_x u_{xx}\mathrm{d}x\mathrm{d}\tau=-\int_0^t\int_{\mathbb{R}}v_x e^{-\phi}u_{xx}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}ve^{-\phi}\phi_x u_{xx}\mathrm{d}x\mathrm{d}\tau,$$

we find that

$$|\mathcal{L}_3|\lesssim\int_0^t\int_{\mathbb{R}}\theta v_x^2\mathrm{d}x\mathrm{d}\tau+C(m_2)\int_0^t\int_{\mathbb{R}}u_{xx}^2\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}\phi_x^2\mathrm{d}x\mathrm{d}\tau\lesssim C(m_2).$$

Combining $\mathcal{L}_1$, $\mathcal{L}_2$ with $\mathcal{L}_3$, we obtain

$$\begin{aligned}&\|u_{xx}(t)\|^2+\int_0^t\int_{\mathbb{R}}u_{xxx}^2\mathrm{d}x\mathrm{d}\tau\\ \lesssim_h\ &C(m_2)+\sup_{\tau\in[0,t]}\|v_{xx}(\tau)\|^2+\int_0^t\|v_{xx}(\tau)\|^2\mathrm{d}\tau\\ &+\delta\int_0^t\|(u_{xxx},\theta_{xxx})(\tau)\|^2\mathrm{d}\tau. \end{aligned}\tag{4.2}$$

Step 2. Differentiating (1.1c) with respect to x, multiplying it by θ_{xxx}, and integrating the resultant over $[0,t]\times\mathbb{R}$, we can get by repeating the argument used in [28] to get that

$$\begin{aligned}&\|\theta_{xx}(t)\|^2+\int_0^t\int_{\mathbb{R}}\theta_{xxx}^2\mathrm{d}x\mathrm{d}\tau\\ \lesssim_h\ &C(m_2)+\sup_{\tau\in[0,t]}\|v_{xx}(\tau)\|^2+\int_0^t\|v_{xx}(\tau)\|^2\mathrm{d}\tau\\ &+\delta\int_0^t\|(u_{xxx},\theta_{xxx})(\tau)\|^2\mathrm{d}\tau. \end{aligned}\tag{4.3}$$

Combining (4.2) and (4.3) and by taking δ small enough, we can prove our lemma. □

Next, we derive a m_2-dependent bound for the second derivatives with respect to x of the solution $(v(t,x),u(t,x),\theta(t,x))$.

Lemma 4.2. *Assume that the conditions of Theorem* 1.1 *hold, we have for* $0\leq t\leq T$ *that*

$$\sup_{t\in[0,T]}\|(v_{xx},u_{xx},\theta_{xx})(t)\|^2+\int_0^T\|(v_{xx},u_{xxx},\theta_{xxx})(t)\|^2\mathrm{d}t\leq C(m_2).\tag{4.4}$$

Proof. Differentiating (1.1b) with respect to x and multiplying the result by $(\frac{\mu v_x}{v})_x$ to find

$$\frac{1}{2}\left|\left(\frac{\mu v_x}{v}\right)_x\right|_t^2+R\frac{\mu\theta v_{xx}^2}{v^3}=\left(u_{xt}+R\left(\frac{\theta_x}{v}\right)_x\right)\left(\frac{\mu v_x}{v}\right)_x$$
$$+\left(R\frac{\mu\theta v_{xx}^2}{v^3}-R\left(\frac{\theta v_x}{v^2}\right)_x\left(\frac{\mu v_x}{v}\right)_x\right)-\left(\frac{\phi_x}{v}\right)_x\left(\frac{\mu v_x}{v}\right)_x.$$

Integrating the above identity over $[0,t]\times\mathbb{R}$ gives

$$\begin{aligned}
&\frac{1}{2}\left\|\left(\frac{\mu v_x}{v}\right)_x(t)\right\|+R\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_{xx}^2}{v^3}\mathrm{d}x\mathrm{d}\tau\\
=&\frac{1}{2}\left\|\left(\frac{\mu v_x}{v}\right)_x(0)\right\|+\underbrace{\int_0^t\int_{\mathbb{R}}\left(u_{xt}+R\left(\frac{\theta_x}{v}\right)_x\right)\left(\frac{\mu v_x}{v}\right)_x\mathrm{d}x\mathrm{d}\tau}_{\mathcal{L}_4}\\
&+\underbrace{\int_0^t\int_{\mathbb{R}}\left(R\frac{\mu\theta v_{xx}^2}{v^3}-R\left(\frac{\theta v_x}{v^2}\right)_x\left(\frac{\mu v_x}{v}\right)_x\right)\mathrm{d}x\mathrm{d}\tau}_{\mathcal{L}_5}\\
&-\underbrace{\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_x}{v}\right)_x\left(\frac{\mu v_x}{v}\right)_x\mathrm{d}x\mathrm{d}\tau}_{\mathcal{L}_6}.
\end{aligned}\tag{4.5}$$

The terms $\mathcal{L}_4$ and $\mathcal{L}_5$ can be estimated similarly as in [28]

$$|\mathcal{L}_4|+|\mathcal{L}_5|\lesssim C(m_2)+\frac{R}{2}\int_0^t\int_{\mathbb{R}}\frac{\mu\theta v_{xx}^2}{v^3}\mathrm{d}x\mathrm{d}\tau+C(m_2)\int_0^t\|(\theta_x,v_x)\|^2\|v_{xx}\|^2\mathrm{d}\tau.\tag{4.6}$$

We only give the estimate of term $\mathcal{L}_6$. Since

$$\left(\frac{\phi_x}{v}\right)_{xx}=\left(-ve^{-\phi}\right)_x=-v_xe^{-\phi}+ve^{-\phi}\phi_x,$$

we can obtain

$$\int_0^t\int_{\mathbb{R}}\left|\left(\frac{\phi_x}{v}\right)_{xx}\right|^2\mathrm{d}x\mathrm{d}\tau\lesssim\|\theta^{-1}\|_\infty\int_0^t\int_{\mathbb{R}}\theta v_x^2\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\mathbb{R}}\phi_x^2\mathrm{d}x\mathrm{d}\tau\le C(m_2).\tag{4.7}$$

Thus $\mathcal{L}_6$ can be estimated as

$$|\mathcal{L}_6|=\left|\int_0^t\int_{\mathbb{R}}\left(\frac{\phi_x}{v}\right)_{xx}\cdot\frac{\mu v_x}{v}\mathrm{d}x\mathrm{d}\tau\right|\le C(m_2).\tag{4.8}$$

Combining (4.6) and (4.8), we can derive

$$\|v_{xx}(t)\|^2+\int_0^t\|v_{xx}(\tau)\|^2\mathrm{d}\tau\lesssim_h C(m_2)+\int_0^t\|(\theta_x,v_x)\|^2\|v_{xx}\|^2\mathrm{d}x.$$

We apply the Gronwall inequality to the above estimate to obtain

$$\|v_{xx}(t)\|^2+\int_0^t\|v_{xx}(\tau)\|^2\mathrm{d}\tau\lesssim C(m_2), \tag{4.9}$$

which combined with (4.1) implies (4.4). The proof is completed. □

Secondly, we give the third-order derivatives of (v,u,θ) in Lemmas 4.3 and 4.4. Since the higher-order derivatives of ϕ_x can be converted to lower-order derivatives of v and ϕ_x by using (1.1d), the proof of Lemmas 4.3 and 4.4 is similar to that of Lemmas 4.1-4.2, we thus omit the details for brevity.

Lemma 4.3. *Assume that the conditions of Theorem* 1.1 *hold, we can deduce for* $0\le t\le T$ *that*

$$\begin{aligned}&\sup_{t\in[0,T]}\|(u_{xxx},\theta_{xxx})(t)\|^2+\int_0^T\|(u_{xxxx},\theta_{xxxx})(t)\|^2\mathrm{d}t\\&\lesssim_h C(m_2)+\sup_{t\in[0,T]}\|v_{xxx}(t)\|^2+\int_0^T\|v_{xxx}(t)\|^2\mathrm{d}t.\end{aligned} \tag{4.10}$$

Lemma 4.4. *Assume that the conditions of Theorem* 1.1 *hold, we have for* $0\le t\le T$ *that*

$$\sup_{t\in[0,T]}\|(v_{xxx},u_{xxx},\theta_{xxx})(t)\|^2+\int_0^T\|(v_{xxx},u_{xxxx},\theta_{xxxx})(t)\|^2\mathrm{d}t\le C(m_2). \tag{4.11}$$

By virtue of Lemmas 4.1-4.4, we can get the following corollary.

Corollary 4.1. *Assume that the conditions of Theorem* 1.1 *hold, we have for all* $t\in[0,T]$ *that*

$$\begin{aligned}&\|(v-1,u,\theta-1,\phi)(t)\|_3^2\\&\quad+\int_0^t\Big(\|v_x(\tau)\|_2^2+\|(u_x,\theta_x)(\tau)\|_3^2+\|\phi_x(\tau)\|^2\Big)\mathrm{d}x\le C(m_2).\end{aligned} \tag{4.12}$$

With Corollary 4.1 in hand, Theorem 1.1 follows immediately by combining the local existence of solution $(v(t,x),u(t,x),\theta(t,x),\phi(t,x))$ of the Cauchy problem (1.1)-(1.6) and the continuation argument designed in [28] and we omit the details for brevity.

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