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Sharp Convergence to Steady States of Allen-Cahn

Dong Li[1,2,*], Chaoyu Quan[2], Tao Tang[3,4]
and Wen Yang[5]

[1] *Department of Mathematics, Southern University of Science and Technology, Shenzhen, P.R. China.*
[2] *SUSTech International Center for Mathematics, Southern University of Science and Technology, Shenzhen, P.R. China.*
[3] *Guangdong Provincial Key Laboratory of Computational Science and Material Design, Southern University of Science and Technology, Shenzhen, P.R. China.*
[4] *Division of Science and Technology, BNU-HKBU United International College, Zhuhai, P.R. China.*
[5] *Wuhan Institute of Physics and Mathematics, Innovation Academy for Precision Measurement Science and Technology, Chinese Academy of Sciences, Wuhan 430071, P.R. China.*

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Abstract. In our recent work we found a surprising breakdown of symmetry conservation: using standard numerical discretization with very high precision the computed numerical solutions corresponding to very nice initial data may converge to completely incorrect steady states due to the gradual accumulation of machine round-off error. We solved this issue by introducing a new Fourier filter technique for solutions with certain band gap properties. To further investigate the attracting basin of steady states we classify in this work all possible bounded nontrivial steady states for the Allen-Cahn equation. We characterize sharp dependence of nontrivial steady states on the diffusion coefficient and prove strict monotonicity of the associated energy. In particular, we establish

*Corresponding author. *Email addresses:* `lid@sustech.edu.cn` (D. Li), `quancy@sustech.edu.cn` (C.Y. Quan), `ttang@uic.edu.cn` (T. Tang), `math.yangwen@gmail.com` (W. Yang)

a certain self-replicating property amongst the hierarchy of steady states and give a full classification of their energies and profiles. We develop a new modulation theory and prove sharp convergence to the steady state with explicit rates and profiles.

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Key words: Allen-Cahn equation, steady state, ground state solution, asymptotic behavior.

1 Introduction

In this paper, we consider the following one-dimensional Allen-Cahn equation posed on the periodic torus $\mathbb{T}=[-\pi,\pi]$:

$$\begin{cases} \partial_t u = \kappa^2 \partial_{xx} u - f(u), \\ u\big|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $\kappa>0$ measures the strength of diffusion, $f(u)=u^3-u=F'(u)$, and $F(u)=\frac{(u^2-1)^2}{4}$ is the usual double-well potential. The function $u:\mathbb{T}\to\mathbb{R}$ represents the concentration difference of phases in an alloy and typically has values in the physical range $[-1,1]$.

In our recent work [13], we find a very surprising breakdown of parity in typical high-precision computation of (1.1) with very smooth initial data. For example take $\kappa=1$ and consider Eq. (1.1) with the initial data $u_0(x)$ being an odd function of x such as $u_0(x)=\sin x$. By simple PDE arguments the smooth solution should preserve the odd symmetry for all time. However numerical discretized solutions turn out to fail to conserve this parity and converge quickly to the spurious states $u=\pm 1$ in not very long time simulations. This striking contradiction is a manifestation of the gradual accumulation of non-negligible machine round off errors over time. To resolve this issue, we introduced a new Fourier filter method which works successfully for a class of initial data with certain symmetry and band-gap properties. By eliminating the unwanted projections into the unstable directions at each iteration, we rigorously show (see [12, 13]) that the filtered solution will converge to the true steady state in long time simulations.

A natural next task is to understand the situation for general solutions without symmetries or band-gap properties. The pivotal step is to categorize the steady states of the elliptic Allen-Cahn equations and analyze in detail their spectral properties. For full generality we shall consider the steady states of (1.1) on the whole real axis, i.e.

$$\kappa^2 u'' + u - u^3 = 0 \quad \text{in} \quad \mathbb{R}. \tag{1.2}$$

In [6] De Giorgi raised the problem about proving that bounded solutions to $\Delta u = F'(u)$ in dimensions $2 \le n \le 8$ which are monotone in one direction, must depend only on one variable in dimension. Since then there are many works in understanding the structure of the solutions. Particularly, in dimension $n=2$ and $n=3$, Ghoussoub and Gui [9] and Ambrosio and Cabré [2] proved the conjecture respectively.

We mention the recent work of Wang [17] which gives a new approach to the De Giorgi conjecture up to dimension 8 under some additional assumptions. One should note that the De Giorgi conjecture is still unsolved without the aforementioned additional assumptions. In a breakthrough work Del Pino *et al.* [7] established the existence of a counterexample in dimensions $n \ge 9$. We refer the readers to [1] for background on De Giorgi's conjecture, [3, 4] for the study on the symmetry properties of the solutions to the fractional Allen-Cahn equation and the recent survey [5] for some related open problems. It is known that the monotone solutions to (1.2) in any dimension are stable solutions, i.e., the second variation of the associated energy is non-negative, where the energy functional is defined as

$$E(u) = \int_{\mathbb{R}} \left(\frac{\kappa^2}{2} |\nabla u|^2 + \frac{1}{4} (1-u^2)^2 \right) dx. \tag{1.3}$$

Recently, there is a new counterpart problem for the stable solutions to Allen-Cahn equation (see, e.g., [14, 16, 17] and references therein). Based on the monotonicity assumption it is natural to consider the two sides limit (without loss of generality we assume the function is monotone in x_n)

$$u^+ := \lim_{x_n \to +\infty} u, \quad u^- := \lim_{x_n \to -\infty} u.$$

It is known that the limit functions only depend on the previous $n-1$ variables. From a general perspective it is of some importance to study the stable solutions and its energy functional (1.3) in order to understand the structure of steady states. In the first part of this paper, we shall classify all the steady states of the 2π-periodic solutions to (1.2). Furthermore we consider the variation on the energy of the ground state with respect to κ.

Theorem 1.1. *Let $0<\kappa<1$ and m_κ be the largest positive integer such that $m_\kappa \kappa < 1$, then Eq.* (1.2) *admits exactly m_κ non-constant 2π periodic solutions up to some translation and odd reflection.*

It is known that any periodic solution of (1.2) is bounded. By Modica's estimate one can get that $|u| \le 1$, see [15]. For convenience of the readers we shall include an elementary proof for the one dimensional case, see Proposition 2.3. By using Proposition 2.3 it suffices for us to consider periodic solutions to (1.2) satisfying $|u|<1$, since $u\equiv 1$ or $u\equiv -1$ provides the trivial global minimizers of (1.2) from an energy perspective.

To state the next result, we define the energy for 2π-periodic functions $u\in H^1(\mathbb{T})$

$$E_\kappa(u)=\int_{\mathbb{T}}\left(\frac{\kappa^2}{2}|\partial_x u|^2+\frac{1}{4}(1-u^2)^2\right)dx. \tag{1.4}$$

Define

$$E_\kappa=\inf_{u\in\mathcal{S}} E_\kappa(u), \tag{1.5}$$

where

$$\mathcal{S}=\Big\{\phi \mid \phi(x)\in H^1(\mathbb{T}) \text{ solves (1.2)}, \ \phi(x)=\phi(x+2\pi) \text{ and } |\phi|<1,\ x\in\mathbb{R}\Big\}.$$

For the 2π-periodic solutions of (1.2), we call u a ground state if u is the least energy solution. For fixed $0<\kappa<1$, by using the classification result in Proposition 2.3, we can prove that the ground state solution is unique up to a translation and reflection. To fix the symmetries it is convenient to introduce the notion of odd zero-up ground states (see Definition 3.2). In particular if a ground state solution u is odd and satisfy $u'(0)>0$, we shall call it an odd zero-up ground state and denote it by U_κ. For any $0<\kappa<1$, define $m_\kappa\ge 1$ as the unique integer such that

$$\frac{1}{m_\kappa+1}\le\kappa<\frac{1}{m_\kappa}.$$

For each $j=1,\dots,m_\kappa$, define (note below that $j\kappa<1$)

$$\tilde{u}_{\kappa,j}(x)=U_{j\kappa}(jx). \tag{1.6}$$

Then $\{\tilde{u}_{\kappa,j}\}_{j=1}^{m_\kappa}$ are all the possible odd zero-up solutions to (3.1). Furthermore the energies of $\tilde{u}_{\kappa,j}$ are given by

$$E_\kappa(\tilde{u}_{\kappa,j})=\int_{\mathbb{T}}\left(\frac{1}{2}(\kappa\partial_x\tilde{u}_{\kappa,j})^2+\frac{1}{4}(\tilde{u}_{\kappa,j}^2-1)^2\right)dx=E_{j\kappa}(U_{j\kappa}).$$

With this notation, we now state the following structure theorem on the energy functional $E_\kappa(u)$ of the 2π-periodic solutions.

Theorem 1.2. *Let E_κ be defined in* (1.5). *Then it can be achieved for any $\kappa > 0$. In addition, we have*

(a) $E_\kappa = \frac{\pi}{2}$ for $\kappa \geq 1$ and it is only achieved by the zero function.

(b) E_κ is achieved by U_κ whenever $\kappa \in (0,1)$.

(c) If $0 < \kappa_1 < \kappa_2 \leq 1$, then there is strict monotonicity $E_{\kappa_1} < E_{\kappa_2}$.

(d) The odd zero-up ground state U_κ satisfies

$$\left| U_\kappa(x) - \tanh\left(\frac{x}{\sqrt{2}\kappa}\right) \right| \leq C\exp\left(-\frac{d}{\kappa}\right)$$

with universal positive constants C and d, and

$$\lim_{\kappa \to 0} \frac{E_\kappa}{\kappa} = \frac{4}{3}\sqrt{2} > 0.$$

(e) For $0 < \kappa < 1$, the 2π-periodic solutions of problem (1.2) *have the following replica property: any 2π-periodic solution u of* (1.2) *which is not identically ± 1 or 0 must coincide (after some shift and odd reflection if necessary) with $\tilde{u}_{\kappa,j}$ for an integer $j < \frac{1}{\kappa}$. Here $\tilde{u}_{\kappa,j}$ is defined in* (1.6). *Furthermore $E_\kappa(u) = E_{m\kappa}(U_{m\kappa})$.*

From Theorem 1.1 we can see that 0 is the only 2π-periodic solution to Eq. (1.2) whenever $\kappa \geq 1$. This is in complete agreement with numerical experiments. Furthermore, when $\kappa > 1$, $u(x,t)$ converges to 0 exponentially as $\mathcal{O}(e^{-(\kappa^2-1)t})$, while the convergence rate becomes $\mathcal{O}(t^{-1/2})$ for $\kappa = 1$. In the second part of this work, we shall rigorously prove these convergence results and identify explicit profiles. Our strategy is quite robust and we shall illustrate it for a general fractional Allen-Cahn equation

$$\begin{cases} \partial_t u = -\kappa^2 \Lambda^\gamma u + u - u^3, & (x,t) \in \mathbb{T} \times (0,\infty), \\ u\big|_{t=0} = u_0, \end{cases} \tag{1.7}$$

where $\Lambda^\gamma = (-\partial_{xx})^{\gamma/2}$ is the fractional Laplacian of order $\gamma \in (0,2]$. When $\gamma = 2$ it coincides with the usual $-\partial_{xx}$. For simplicity of presentation we state below a simple version of the obtained results in Section 4. Sharper results concerning profiles, rates etc. can be found in Section 4.

Theorem 1.3 (Vanishing as $t\to\infty$). *Let $\kappa\ge 1$ and $0<\gamma\le 2$. Assume u_0 is 2π periodic, odd and bounded. Suppose u is the solution to* (1.7) *corresponding to the initial data u_0. If $\kappa>1$, we have*

$$u(x,t)=e^{-(\kappa^2-1)t}\alpha_*\sin x+r(t),\quad \forall t\ge 1, \tag{1.8}$$

where α_* depends on u_0,γ and κ, and

$$\|r(t)\|_{H^{10}(\mathbb{T})}=o\big(e^{-(\kappa^2-1)t}\big)\quad as\quad t\to+\infty.$$

For $\kappa=1$, we have

$$u(x,t)=t^{-\frac{1}{2}}\beta_*\sin x+r_1(t),\quad \forall t\ge 1, \tag{1.9}$$

where β_* depends on u_0,γ and

$$\|r_1(t)\|_{H^{10}(\mathbb{T})}=o\big(t^{-\frac{1}{2}}\big)\quad as\quad t\to+\infty.$$

When $\kappa\in(0,1)$, the corresponding theory of convergence becomes quite involved. Indeed, from Theorem 1.1 we see that the number of steady states (up to identification of symmetry) increases as $\mathcal{O}(\frac{1}{\kappa})$ when κ decays to zero. At the moment there is no general theory for the precise identification of the corresponding steady for arbitrary initial data. However for a class of benign initial data, we have the following precise and definite convergence results.

Theorem 1.4. *Let $0<\kappa<1$. Assume the initial data $u_0:\mathbb{R}\to\mathbb{R}$ is 2π-periodic, odd and non-negative in $[0,\pi]$, then we have $u(x,t)\to U_\kappa$ or 0 as $t\to\infty$. Moreover, if $u_0\neq 0$ and $E_\kappa(u_0)\le\frac{\pi}{2}$, then $u(x,t)\to U_\kappa$ as $t\to\infty$ and the rate of convergence is exponential in time.*

To the best of our knowledge, Theorem 1.4 along with earlier results are the first sharp quantitative convergence results on the Allen-Cahn equation. We plan to develop this program on much more general phase field models in forthcoming works.

The rest of this paper is organized as follows. In Section 2 we introduce preliminary analysis of the steady states and examine in detail the profiles of the ground states. In Section 3 we prove Theorems 1.1 and 1.2, give full classification of the steady states and analyze their profiles and energy monotonicity. In Section 4 we study the convergence of the general parabolic Allen-Cahn equation (1.7), and prove Theorems 1.3 and 1.4. In Section 5 we give concluding remarks. The proof of Proposition 2.3 is given in the appendix.

2 Classification of steady states

The solutions to $\kappa^2 u'' + u - u^3 = 0$ are remarkably rigid, as documented by the following "patching" of nonlinear solutions.

Proposition 2.1 (Patching of nonlinear solutions via reflection)**.** *The following hold.*

- *Even reflection. Suppose $\kappa > 0$, and for some $\epsilon_0 > 0$ we have*

$$\kappa^2 u'' + u - u^3 = 0, \quad \forall -\epsilon_0 < x < 0,$$

where $u \in C^2((-\epsilon_0, 0))$ and we assume $\lim_{x\to 0-} u'(x) = 0$. Define $u(x) = u(-x)$ for $0 < x < \epsilon_0$. Then it holds that $u \in C^\infty((-\epsilon_0, \epsilon_0))$ with $u'(0) = 0$ and solving the same equation on the whole interval.

- *Odd reflection. Suppose $\kappa > 0$, and for some $\epsilon_0 > 0$ we have*

$$\kappa^2 u'' + u - u^3 = 0, \quad \forall 0 < x < \epsilon_0,$$

where $u \in C^2((0, \epsilon_0))$ and we assume $\lim_{x\to 0+} u(x) = 0$. Define $u(x) = -u(-x)$ for $-\epsilon_0 < x < 0$. Then it holds that $u \in C^\infty((-\epsilon_0, \epsilon_0))$ with $u(0) = 0$ and solving the same equation on the whole interval.

Remark 2.1. Proposition 2.1 shows that the solution is remarkably rigid. If we know the profile of u on some interval (a,b) with $u(a) = 0$, $u'(b) = 0$. Then the solution can be uniquely determined on a larger interval.

Proof. We shall only prove the first case as the second case is similar. First it is not difficult to that u has bounded derivatives in $[-\frac{\epsilon_0}{2}, 0)$ which can be extended to 0 from the left. The extended u satisfies the equation on $(-\epsilon_0, 0) \cap (0, \epsilon_0)$. Furthermore the equation also holds at $x = 0$ up to third order derivatives. Then we can bootstrap the regularity of u by using the equation and conclude that $u \in C^\infty$. $\square$

Observe that for $u_0(x) = \sin x$, since $f(u) = u^3 - u$, we have

$$u(x,t) = \sum_{m \geq 1:\ m \text{ is odd}} c_m(t) \sin mx.$$

In particular it follows that the corresponding steady state u_∞ is odd. If 2π is the minimal period (such solution is actually the odd zero-up ground state up to a reflection if necessary, see Definition 3.2), then $u_\infty(0)=u'_\infty(\frac{\pi}{2})=0$. In addition, u_∞ satisfies the steady state equation

$$\kappa^2 u'' - f(u) = 0 \quad \text{on } \mathbb{T}. \tag{2.1}$$

We may look for the steady state such that it is monotonically increasing on $[0,\frac{\pi}{2}]$ with $u_\infty(0)=u'_\infty(\frac{\pi}{2})=0$. Effectively by using reflection symmetry, the whole graph of u_∞ will be determined by its graph on the interval $[0,\frac{\pi}{2}]$.

To simplify the notation we now write $u=u_\infty$ as the desired steady state. We consider the regime $0<\kappa<1$ (for simplicity we suppress the notational dependence of u on κ). Denote $u(\frac{\pi}{2})=N<1$ and observe that we should have $N\to 1$ as $\kappa\to 0$. Multiplying (2.1) by u' and using $u'(\frac{\pi}{2})=0$, we have

$$(u')^2 = \frac{1}{2\kappa^2}\Big((u^2-1)^2 - (N^2-1)^2 \Big).$$

If u is monotonically increasing, it satisfies

$$u'(x) = \frac{1}{\sqrt{2}\kappa}\sqrt{(u^2-1)^2-(N^2-1)^2}$$

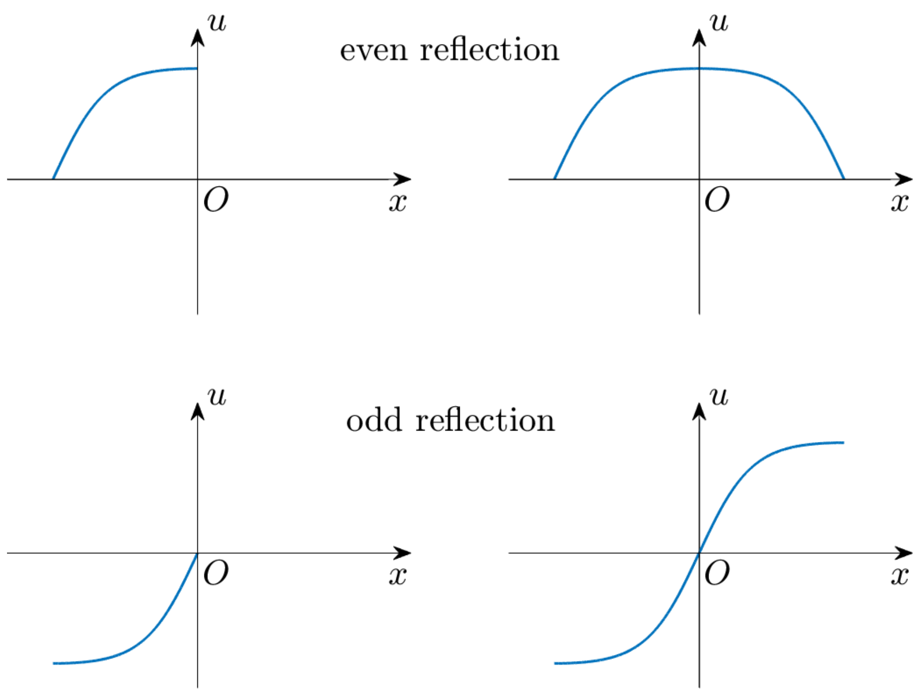

Figure 1: Even reflection and odd reflection.

with $u(0)=0$, $u(\frac{\pi}{2})=N$. We then obtain

$$\int_0^N \frac{1}{\sqrt{(u^2-1)^2-(1-N^2)^2}}du=\frac{\pi}{2\sqrt{2}\kappa}.$$

For each fixed $0<\kappa<1$, there exists a unique $0<N=N(\kappa)<1$ such that the above identity holds. Furthermore one can determine the dependence of N on κ. Indeed by a change of variable $u\to N\sin\theta$, the left-hand side of the above equation is denoted by

$$g(N):=\int_0^N \frac{1}{\sqrt{(u^2-1)^2-(1-N^2)^2}}du=\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2-N^2(1+\sin^2\theta)}}d\theta. \tag{2.2}$$

Here we note that g is monotonically increasing on $[0,1)$, $g(0)=\frac{\pi}{2\sqrt{2}}$ and $g(1)=\infty$. In particular we see the necessity of $\kappa<1$! Otherwise if $\kappa\geq 1$, Eq. (2.1) admits the trivial solution $u\equiv 0$. For $0<\kappa\ll 1$, it is not difficult to check that

$$1-N(\kappa)=\mathcal{O}\big(e^{-\frac{c}{\kappa}}\big).$$

Sharper asymptotics can certainly be derived.

We summarize the above discussion as the following proposition.

Proposition 2.2 (Characterization of a special steady state for $0<\kappa<1$)**.** *The following hold:*

1) *The function g defined in* (2.2) *is monotonically increasing on $[0,1)$, $g(0)=\frac{\pi}{2\sqrt{2}}$ and $g(N)\to\infty$ as $N\to 1$.*

2) *For any $0<\kappa<1$, there exists a unique $0<N_\kappa<1$ such that*

$$g(N_\kappa)=\frac{\pi}{2\sqrt{2}\kappa}. \tag{2.3}$$

Furthermore we have

$$c_1e^{-\frac{c_2}{\kappa}}<1-N_\kappa<c_3e^{-\frac{c_4}{\kappa}}, \tag{2.4}$$

where $c_i>0$, $i=1,\dots,4$ are absolute constants.

3) *For any $0<\kappa<1$, there exists a 2π-periodic C^∞ odd function u_κ such that*

- *u_κ is a steady state, i.e. $\kappa^2u_\kappa''-f(u_\kappa)=0$.*
- *$u_\kappa(0)=u_\kappa'(\frac{\pi}{2})=0$, $u_\kappa(\frac{\pi}{2})=N_\kappa$, and u_κ is monotonically increasing on $[0,\frac{\pi}{2}]$.*

- $u_\kappa(\pi-x)=u_\kappa(x)$ *for* $\frac{\pi}{2}\le x\le\pi$.

Moreover for $0<\kappa\ll 1$, *we have*

$$0\le\tanh\left(\frac{x}{\sqrt{2}\kappa}\right)-u_\kappa(x)\le\exp\left(-\frac{c_5}{\kappa}\right),\quad \forall 0\le x\le\frac{\pi}{2}, \tag{2.5}$$

where $c_5>0$ *is an absolute constant.*

Proof. 1) For fixed $N\in[0,1)$, taking the derivative of $g(N)$ with respect to N, we have

$$g'(N)=\int_0^{\frac{\pi}{2}}\frac{N(1+\sin^2\theta)}{(2-N^2(1+\sin^2\theta))^{\frac{3}{2}}}d\theta.$$

It is not difficult to verify that both the numerator and denominator are positive in $(0,\frac{\pi}{2})$, therefore, we have shown that $g'(N)>0$ and it proves that g is monotonically increasing for $N\in[0,1)$. When $N=0$, we have

$$g(0)=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{2}}d\theta=\frac{\pi}{2\sqrt{2}}.$$

While as N is close to 1, we have

$$\begin{aligned}g(N)&=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{2-2N^2+N^2\cos^2\theta}}d\theta=\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{2-2N^2+N^2\sin^2\theta}}d\theta\\&>\int_0^1\frac{1}{\sqrt{2-2N^2+N^2\theta^2}}d\theta>\frac{1}{N}\int_\delta^1\frac{1}{\sqrt{\delta^2+\theta^2}}d\theta\\&>\frac{1}{N}\int_\delta^1\frac{1}{\sqrt{2}\theta}d\theta=-\frac{1}{\sqrt{2}N}\log\delta,\end{aligned}\tag{2.6}$$

where

$$\delta=\sqrt{\frac{2-2N^2}{N^2}}\to 0\quad\text{as}\quad N\to 1.$$

Here, it is assumed implicitly that $N>\sqrt{2/3}$ so that $\delta<1$. Then, it is easy to see that $g(N)\to\infty$ as $N\to 1$.

2) The existence and uniqueness of N_κ follows easily from the behavior of the function $g(\cdot)$. Now we shall show the upper and lower bounds on N_κ. It is clear from Step 1 that $N_\kappa\to 1$ as $\kappa\to 0$. If $N_\kappa\le\sqrt{2/3}$, then κ is bounded away from zero by an absolute constant and the desired estimate (2.4) clearly

holds in this case. Thus we only need to consider the situation $N_\kappa > \sqrt{2/3}$. To ease the notation we denote $N = N_\kappa$ with $N > \sqrt{2/3}$. From (2.6), we have

$$\frac{\pi}{2\sqrt{2}\kappa} = g(N) > -\frac{1}{\sqrt{2}N}\log\delta,$$

which yields directly

$$1-N > \frac{N^2}{2+2N}e^{-\frac{N\pi}{\kappa}}.$$

On the other hand, using $\sin\theta > \frac{2}{\pi}\theta$, we have

$$\begin{aligned}
\frac{\pi}{2\sqrt{2}\kappa} = g(N) &= \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2-2N^2+N^2\sin^2\theta}}d\theta \\
&< \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{2-2N^2+N^2\left(\frac{2}{\pi}\theta\right)^2}}d\theta = \int_0^1 \frac{\pi}{2N\sqrt{\delta^2+\tilde{\theta}^2}}d\tilde{\theta} \\
&= \int_0^{\delta} \frac{\pi}{2N\sqrt{\delta^2+\tilde{\theta}^2}}d\tilde{\theta} + \int_{\delta}^1 \frac{\pi}{2N\sqrt{\delta^2+\tilde{\theta}^2}}d\tilde{\theta} \\
&< \frac{\pi}{2N} - \frac{\pi}{2N}\log\delta,
\end{aligned}$$

which yields

$$1-N < \frac{N^2}{2+2N}e^{2-\frac{\sqrt{2}N}{\kappa}}.$$

Therefore, when $N > \sqrt{2/3}$, we have

$$\frac{1}{3+\sqrt{6}}e^{-\frac{\pi}{\kappa}} < \frac{N^2}{2+2N}e^{-\frac{N\pi}{\kappa}} < 1-N < \frac{N^2}{2+2N}e^{2-\frac{\sqrt{2}N}{\kappa}} < \frac{e^2}{4}e^{-\frac{2}{\sqrt{3}\kappa}}. \tag{2.7}$$

3) Fix $0 < \kappa < 1$ and consider the function

$$h(u) = \int_0^u \frac{\sqrt{2}\kappa}{\sqrt{(y^2-1)^2-(1-N^2)^2}}dy, \quad 0 \le u \le N.$$

Clearly $h: [0,N] \to [0,\frac{\pi}{2}]$ is strictly monotonically increasing and bijective. The inverse map of $h(u)$ then defines the desired function u_κ on the interval $[0,\frac{\pi}{2}]$. It is known that $u(\frac{\pi}{2}) = N$ and $u'(\frac{\pi}{2}) = 0$, by Proposition 2.1 we derive that u is even respect to $x = \frac{\pi}{2}$, i.e., $u(x) = u(\pi - x)$ for $x \in (0,\frac{\pi}{2})$.

Finally to show (2.5), we denote $\theta(x)=\tanh(\frac{x}{\sqrt{2}\kappa})$. Clearly

$$\begin{cases} \dfrac{du_\kappa}{dx}=\dfrac{1}{\sqrt{2}\kappa}\sqrt{(u_\kappa^2-1)^2-(N^2-1)^2}, \\ \dfrac{d\theta}{dx}=\dfrac{1}{\sqrt{2}\kappa}(1-\theta^2), \\ u_\kappa(0)=\theta(0)=0. \end{cases}$$

Observe that

$$\frac{du_\kappa}{dx}<\frac{1}{\sqrt{2}\kappa}\big(1-u_\kappa^2\big).$$

Denote $\eta(x)=u_\kappa(x)-\theta(x)$. Clearly $\eta(0)=0$ and

$$\eta'<-\frac{1}{\sqrt{2}\kappa}(u_\kappa+\theta)\eta.$$

This implies that $\eta(x)\le 0$ for all $0\le x\le\frac{\pi}{2}$. Thus

$$u_\kappa(x)-\theta(x)\le 0,\quad \forall 0\le x\le\frac{\pi}{2}.$$

We now show the lower bound. By (2.4) we can choose an absolute constant $\delta_0>0$ sufficiently small such that

$$1-N^2<c_3e^{-\frac{c_4}{\kappa}}<\frac{1}{100}\big(1-\theta(x)^2\big),\quad \forall 0\le x\le\delta_0.$$

This implies

$$\begin{aligned} &\sqrt{(1-u_\kappa(x)^2)^2-(1-N^2)^2}+1-\theta(x)^2 \\ \ge\, & C\big(1-u_\kappa(x)^2+1-\theta(x)^2\big) \end{aligned}$$

for $x\in[0,\delta_0]$ and some generic constant $C>0$. Now observe that

$$\frac{d}{dx}\eta=\frac{1}{\sqrt{2}\kappa}\frac{(1-u_\kappa^2)^2-(1-\theta^2)^2-(1-N^2)^2}{\sqrt{(1-u_\kappa^2)^2-(1-N^2)^2}+1-\theta^2}.$$

Thus for $0<x\le\delta_0$, we have

$$\frac{d\eta}{dx}=\mathcal{O}(\kappa^{-1})\cdot\eta+\mathcal{O}\big(e^{-\frac{b_0}{\kappa}}\big),$$

where $b_0>0$ is an absolute constant. Then for $0<x\le b_1$ ($b_1>0$ is a sufficiently small absolute constant), we have

$$\sup_{0\le x\le b_1}|\eta(x)|\le e^{-\frac{b_2}{\kappa}},$$

where $b_2>0$ is an absolute constant. Thus for $0\le x\le b_1$, we have

$$u_\kappa(x)-\theta(x)\ge -e^{-\frac{b_2}{\kappa}}.$$

For $b_1\le x\le \frac{\pi}{2}$, by using monotonicity, we have

$$\begin{aligned} u_\kappa(x)-\theta(x)&\ge u_\kappa(b_1)-\theta\left(\frac{\pi}{2}\right)\\ &=u_\kappa(b_1)-\theta(b_1)+\theta(b_1)-\theta\left(\frac{\pi}{2}\right)\ge -e^{-\frac{\tilde{c}_5}{\kappa}}, \end{aligned}$$

where $\tilde{c}_5$ is an absolute constant. The desired result then follows easily by collecting the estimates. □

A periodic steady state on $\mathbb{T}$ can be extended naturally to the whole space $\mathbb{R}$. Therefore, such periodic steady states can be seen as a special solution to the following steady state equation defined in $\mathbb{R}$

$$\kappa^2u''+u-u^3=0,\quad x\in\mathbb{R}. \tag{2.8}$$

Multiplying (2.8) by u', we derive that

$$\kappa^2(u')^2+u^2-\frac{1}{2}u^4 \text{ is a constant, denoted by } C, \tag{2.9}$$

which can be rewritten as

$$\kappa^2(u')^2=\frac{1}{2}\left(u^2-1\right)^2+C-\frac{1}{2}. \tag{2.10}$$

Concerning the solution of (2.8), we have the following result.

Proposition 2.3. *Let u be a bounded solution to* (2.8) *and C be the constant defined in* (2.9)*, then the following hold:*

(1) If $C>\frac{1}{2}$, u can not exist.

(2) If $C=\frac{1}{2}$, then $u=\pm\tanh\frac{x+c}{\sqrt{2}\kappa}$ or $u\equiv\pm1$.

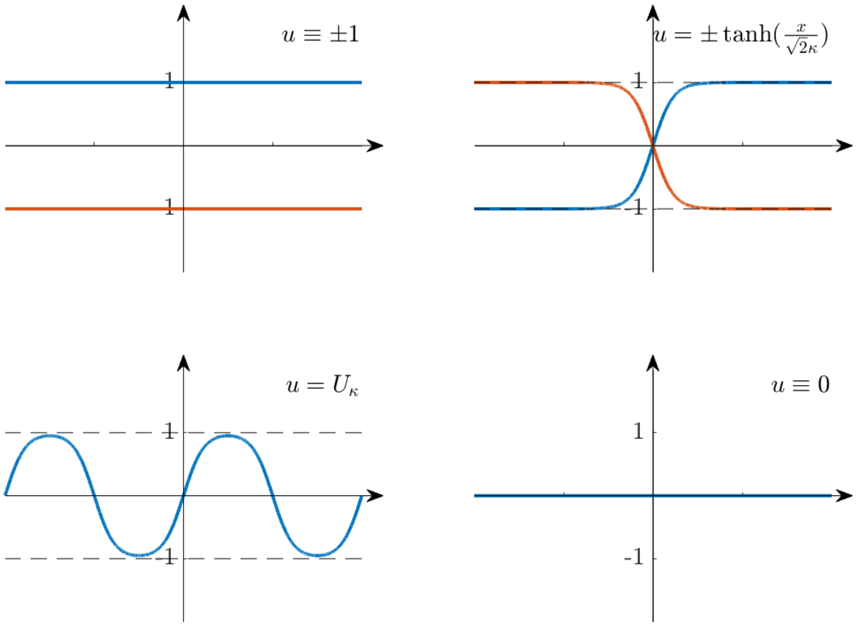

Figure 2: Classification of bounded steady states of Allen-Cahn ($\kappa = \frac{1}{2}$).

(3) *If* $0 < C < \frac{1}{2}$, *then* u *is a periodic function and* $|u| < 1$.

(4) *If* $C = 0$, *then* $u \equiv 0$.

Proof. Although the above conclusion is a folklore, we provide the proof for the sake of completeness in Appendix A and a graphical illustration in Fig. 2. In the case of $0 < C < \frac{1}{2}$, an odd periodic steady state which has its period precisely given by 2π is characterized by Proposition 2.2. □

3 Classification of steady state energy

In this section, we consider the energy $E_\kappa(u)$ (see (1.4)) of solutions to (1.2). From the discussion of Section 2, we see that a nontrivial bounded steady state is a 2π-periodic function. Therefore, we focus on the following problem:

$$\kappa^2 u'' + u - u^3 = 0, \quad u(x) = u(x+2\pi) \quad \text{for} \quad x \in \mathbb{R}. \tag{3.1}$$

For simplicity of presentation, we introduce the following definition.

Definition 3.1 (Odd zero-up solution)**.** *We shall say that* u *is an odd zero-up solution to* (3.1) *if the solution* u *is odd and* $u'(0) > 0$.

Definition 3.2 (Odd zero-up ground states)**.** *For each $0<\kappa<1$, we define $U_\kappa=u_\kappa$, where u_κ is obtained in Proposition* 2.2 *as the odd zero-up ground state solution to* (3.1)*. We also define the odd zero-up ground state energies $E_\kappa^{(0)}$ as*

$$E_\kappa^{(0)}=\int_{\mathbb{T}}\left(\frac{1}{2}\kappa^2\big(U_\kappa'(x)\big)^2+\frac{1}{4}\big(U_\kappa(x)^2-1\big)^2\right)dx \\ =\int_{\mathbb{T}}\left(\frac{1}{2}\big(U_\kappa(x)^2-1\big)^2-\frac{1}{4}\big(N_\kappa^2-1\big)^2\right)dx, \tag{3.2}$$

where $0<N_\kappa<1$ is the unique number satisfying

$$\int_0^{\frac{\pi}{2}}\frac{1}{\sqrt{2-N_\kappa^2(1+\sin^2\theta)}}d\theta=\frac{\pi}{2\sqrt{2}\kappa}. \tag{3.3}$$

For any solution of (3.1), we assume that its minimal period is $\frac{2\pi}{m}$ for some suitable positive integer m. From the proof of Proposition 2.3, it is not difficult to see that $u(x)$ has $2m$ zero points in $[x,x+2\pi)$ for any $x\in\mathbb{R}$ and $u(x)$ has odd symmetry with respect to any zero point. In addition, we can easily prove that the distance between any two consecutive zero points is same and equals to $\frac{2\pi}{m}$. After a suitable shift of the solution, we may assume $u(0)=0$ and u is odd. On the other hand, if $u(x)$ is a solution to (3.1), obviously $u(-x)$ is also a solution. Hence, we may assume that $u'(0)>0$ after reflection if necessary. Therefore, in this section we shall restrict our discussion on the odd zero-up solutions of Eq. (3.1). Concerning all the odd zero-up solutions to (3.1), we have the following classification result.

Theorem 3.1 (Classification of odd zero-up solutions to (3.1))**.** *For any $0<\kappa<1$, define $m_\kappa\geq 1$ as the unique integer such that*

$$\frac{1}{m_\kappa+1}\leq\kappa<\frac{1}{m_\kappa}.$$

Then there are only m_κ odd zero-up solutions to (3.1)*. More precisely the following hold: For each $j=1,\dots,m_\kappa$, define (note below that $j\kappa<1$)*

$$\tilde{u}_{\kappa,j}(x)=U_{j\kappa}(jx).$$

Then $\{\tilde{u}_{\kappa,j}\}_{j=1}^{m_\kappa}$ are all the possible odd zero-up solutions to (3.1)*. Furthermore the energies of $\tilde{u}_{\kappa,j}$ are given by*

$$E_{\kappa,j}=\int_{\mathbb{T}}\left(\frac{1}{2}(\kappa\partial_x\tilde{u}_{\kappa,j})^2+\frac{1}{4}\big(\tilde{u}_{\kappa,j}^2-1\big)^2\right)dx=E_{j\kappa}^{(0)}, \tag{3.4}$$

where $E_{j\kappa}^{(0)}$ was defined in (3.2)*.*

Proof. Suppose u is a possible odd zero-up solution to (3.1). The crucial observation is that we must have u achieves its first peak at $x=\frac{\pi}{2j}$ for some integer $j\geq 1$. Now make a change of variable $y=jx$, and $\tilde{u}(y)=u(x)$. Then clearly

$$j^2\kappa^2\frac{d^2}{dy^2}\tilde{u}-\tilde{u}+\tilde{u}^3=0,$$

$\tilde{u}(0)=0$, $\tilde{u}'(0)>0$, and $\tilde{u}'(\frac{\pi}{2})=0$. From the proof in Step 3 of Proposition 2.2, there exists a unique solution u with $|u|<1$ solving the equation

$$u'=\frac{1}{\sqrt{2}\kappa}\sqrt{(1-u^2)^2-(1-N_\kappa^2)^2}$$

with $u(0)=0$, $u'(\frac{\pi}{2})=0$. As a consequence, we obtain that $\tilde{u}=U_{j\kappa}$. Now note that $j\kappa<1$ and this gives the constraint $j\leq m_\kappa$. The characterization (3.4) follows from the fact that

$$\begin{aligned}E_{\kappa,j}&=\int_{\mathbb{T}}\left(\frac{1}{2}\left(\tilde{u}_{\kappa,j}(x)^2-1\right)^2-\frac{1}{4}\left(N_{j\kappa}^2-1\right)^2\right)dx\\&=\int_{\mathbb{T}}\left(\frac{1}{2}\left(U_{j\kappa}(jx)^2-1\right)^2-\frac{1}{4}\left(N_{j\kappa}^2-1\right)^2\right)dx,\end{aligned}$$

and the fact that $U_{j\kappa}$ is 2π-periodic. □

By Theorem 3.1 one can easily get Theorem 1.1. We notice that the C^0 estimate in the point (d) of Theorem 1.2 follows easily by (2.5). While for the point (e), one can easily prove it by some direct computations. For the left conclusions in Theorem 1.2, we rephrase it as the following result for the odd zero-up solutions.

Theorem 3.2 (Monotonicity and asymptotics of odd zero-up ground state energies)**.** *For any $\kappa>0$, define*

$$\tilde{E}_\kappa=\inf_{u\in\mathcal{S}_O}\int_{\mathbb{T}}\left(\frac{1}{2}(\kappa\partial_x u)^2+\frac{1}{4}\left(u^2-1\right)^2\right)dx,$$

where

$$\mathcal{S}_O:=\left\{\phi\,|\,\phi:\mathbb{T}\to\mathbb{R}\text{ is odd and }C^1,\ \phi'(0)>0\right\}.$$

Then we have

(a) $\tilde{E}_\kappa=\frac{\pi}{2}$ *for* $\kappa\geq 1$. *Furthermore*

$$\int_{\mathbb{T}}\left(\frac{1}{2}(\kappa\partial_x u)^2+\frac{1}{4}\left(u^2-1\right)^2\right)dx>\frac{\pi}{2}$$

for any u not identically zero.

(b) $\tilde{E}_\kappa = E_\kappa^{(0)}$ *for* $0<\kappa<1$. *Moreover the infimum is only achieved by* U_κ.

(c) *If* $0<\kappa_1<\kappa_2\le 1$, *then* $\tilde{E}_{\kappa_1}<\tilde{E}_{\kappa_2}$.

Furthermore

$$\lim_{\kappa\to 0}\frac{E_\kappa^{(0)}}{\kappa}=\gamma_*=\frac{4}{3}\sqrt{2}>0. \tag{3.5}$$

Before proving Theorem 3.2, we establish the following important lemma.

Lemma 3.1. *Let* U_κ *be the odd zero-up ground state to* (3.1). *Suppose that* $0<\kappa_1<\kappa_2<1$, *then we have*

$$U_{\kappa_1}(x)>U_{\kappa_2}(x), \quad x\in(0,\frac{\pi}{2}].$$

Proof. At first, we notice that $U_\kappa(0)=0$, U_κ' is monotone increasing for $x\in(0,\frac{\pi}{2})$. By Eq. (2.10) we have

$$U_\kappa'(x)=\frac{\sqrt{(1-U_\kappa^2(x))^2-(1-N_\kappa^2)^2}}{\sqrt{2}\kappa} \quad \text{for} \quad x\in(0,\frac{\pi}{2}), \tag{3.6}$$

where N_κ is the maximal value of u_κ in $[0,\frac{\pi}{2}]$, i.e., $N_\kappa=U_\kappa(\frac{\pi}{2})$. By (3.6) we have

$$\int_0^x \frac{U_\kappa'(x)}{\sqrt{U_\kappa^4(x)-2U_\kappa^2(x)+2N_\kappa^2-N_\kappa^4}}dx=\frac{x}{\sqrt{2}\kappa}, \quad x\in(0,\frac{\pi}{2}),$$

which is equivalent to

$$\int_0^{U_\kappa(x)} \frac{\kappa}{\sqrt{s^4-2s^2+2N_\kappa^2-N_\kappa^4}}ds=\frac{x}{\sqrt{2}}, \quad x\in(0,\frac{\pi}{2}). \tag{3.7}$$

If $0<\kappa_1<\kappa_2<1$, we have $1>N_{\kappa_1}>N_{\kappa_2}>0$ by Eq. (3.3). This implies that

$$1>2N_{\kappa_1}^2-N_{\kappa_1}^4>2N_{\kappa_2}^2-N_{\kappa_2}^4>0.$$

Therefore for any $s\in(0,\min\{N_{\kappa_1},N_{\kappa_2}\})$ we have

$$\frac{\kappa_1}{\sqrt{s^4-2s^2+2N_{\kappa_1}^2-N_{\kappa_1}^4}}<\frac{\kappa_2}{\sqrt{s^4-2s^2+2N_{\kappa_2}^2-N_{\kappa_2}^4}}. \tag{3.8}$$

Together with (3.7) we derive that $U_{\kappa_1}(x)>U_{\kappa_2}(x)$ for $x\in(0,\frac{\pi}{2}]$. This proves the lemma. $\square$

Proof of Theorem 3.2. We shall prove Theorem 3.2 point by point. For point (a), we notice that $u\equiv 0$ is the only odd zero-up solution to (3.1) whenever $\kappa\geq 1$. Then it is easy to verify that $\tilde{E}_\kappa=\frac{\pi}{2}$ for $\kappa\geq 1$.

Next, we consider the point (b). For any 2π-periodic odd zero-up solution of (3.1) which is different by U_κ, we denote its minimal period by $\frac{2\pi}{m}$ and the solution by u_m, $m\geq 2$. Consider the function

$$v(y)=u_m(x),\quad y=mx.$$

Then it is not difficult to verify that

$$v(x)=U_{m\kappa}(x),\quad x\in(0,\frac{\pi}{2}).$$

By Lemma 3.1, for $m\geq 2$ we have

$$U_\kappa(x)>U_{m\kappa}(x)\quad \text{for}\quad x\in(0,\frac{\pi}{2}). \tag{3.9}$$

On the other hand, we notice that

$$\begin{aligned} E_\kappa^{(0)}&=\int_0^{\frac{\pi}{2}}\left(2\kappa^2U_\kappa'^2+(1-U_\kappa^2)^2\right)dx\\ &=2\kappa^2U_\kappa U_\kappa'\big|_{x=0}^{x=\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\left(-2\kappa^2U_\kappa U_\kappa''+(1-U_\kappa^2)^2\right)dx\\ &=\int_0^{\frac{\pi}{2}}(1-U_\kappa^4)dx. \end{aligned}\tag{3.10}$$

Using (3.9) we have

$$E_\kappa^{(0)}<E_{m\kappa}^{(0)}. \tag{3.11}$$

By Eq. (3.4) we get

$$E_{m\kappa}^{(0)}=E(u_m)=\int_{\mathbb{T}}\left(\frac{1}{2}(\kappa u_m)^2+\frac{1}{4}(u_m^2-1)^2\right)dx.$$

Together with (3.11) we obtain that

$$E_\kappa^{(0)}=\tilde{E}_\kappa,$$

and it proves the point (b).

The assertion (c) follows from assertion (b), Lemma 3.1 and Eq. (3.10).

In the end, we shall show the asymptotics as $\kappa\to 0$. By Proposition 2.2, the main part of U_κ on $[-\frac{\pi}{2},\frac{\pi}{2}]$ is given by $\tanh(\frac{x}{\sqrt{2\kappa}})$. The result (3.5) then follows from a simple computation

$$\gamma_*=\sqrt{2}\int_{\mathbb{R}}\left(\tanh^2 y-1\right)^2 dy=\sqrt{2}\int_{\mathbb{R}}\left(1-\tanh^2 y\right)d\tanh y=\frac{4}{3}\sqrt{2},$$

where $y=\frac{x}{\sqrt{2\kappa}}$. $\square$

Corollary 3.1. *For any $0<\kappa<1$, if u_0 is odd, 2π-periodic (monotonicity of u_0 is not required), and $E(u_0)<E_{2\kappa}^{(0)}$, then the steady state of* (1.1) *is $\pm U_\kappa$, where U_κ be the odd zero-up ground state to* (3.1).

Proof. Obviously, we have $E_\kappa^{(0)}\le E(u_0)<E_{2\kappa}^{(0)}$. From the energy dissipation property of (1.1), we can claim that 2π is the minimal period of the steady state for the initial condition u_0. $\square$

4 Convergence to the steady state

In this section, we investigate the convergence rate of the solution and characterize the detailed profiles as $t\to\infty$.

4.1 Case of $0<\kappa<1$

We start this subsection with the following result on the spectrum analysis. This is crucial in showing the convergence rate is exponential.

Lemma 4.1. *Let $0<\kappa<1$. Assume U_κ is the odd zero-up ground state. Then for any 2π-periodic odd function $\phi\in H^1(\mathbb{T})$ we have*

$$\int_{\mathbb{T}}\kappa^2|\phi'|^2dx+\int_{\mathbb{T}}\left(3U_\kappa^2-1\right)|\phi|^2dx\ge C\|\phi\|^2_{H^1(\mathbb{T})} \tag{4.1}$$

for some universal constant $C>0$.

Proof. First of all, we notice that $C\ge 0$ due to the fact U_κ is the odd zero-up ground state. Next, we shall prove that $C>0$ by contradiction. Suppose that $C=0$ then we can find a sequence of odd functions ϕ_n such that $\|\phi_n\|_{H^1(\mathbb{T})}=1$ and

$$\int_{\mathbb{T}}\kappa^2|\phi_n'|^2dx+\int_{\mathbb{T}}\left(3U_\kappa^2-1\right)|\phi_n|^2dx\le\frac{1}{n}. \tag{4.2}$$

Passing to a subsequence if necessary, we obtain there exists a nontrivial odd function $\phi_* \in H^1(\mathbb{T})$ such that ϕ_n weakly converges to ϕ_* in $H^1(\mathbb{T})$ and

$$\kappa^2 \phi_*'' + (1-3U_\kappa^2)\phi_* = 0 \quad \text{on } \mathbb{T}. \tag{4.3}$$

After direct computations we see that

$$\begin{aligned} E_\kappa(U_\kappa + c\phi_*) = &E_\kappa(U_\kappa) + \frac{c^2}{2}\int_{\mathbb{T}} \left(\kappa^2 |\phi_*'|^2 + (3U_\kappa^2 - 1)|\phi_*|^2\right) dx \\ &+ \int_{\mathbb{T}} \left(c^3 U_\kappa \phi_*^3 + \frac{c^4}{4}\phi_*^4\right) dx \end{aligned} \tag{4.4}$$

for any real number c. Here we have used U_κ is the odd zero-up ground state. Using (4.3) we see that the second term on the right hand side of (4.4) vanishes, then together with $E_\kappa(U_\kappa + c\phi_*) \geq E_k(U_\kappa)$ for any c, we see that

$$\int_{\mathbb{T}} U_k \phi_*^3 = 0.$$

It implies that ϕ_* must possess a zero point in $(0,\pi)$, denoted by x_*. By the well-known Strum Comparison Theorem (see [10, Theorem VI-1-1] for instance) we derive that any solution of the following equation must have a zero point in $(0,x_*)$:

$$\kappa^2 \phi'' + (1-U_\kappa^2)\phi = 0. \tag{4.5}$$

However, we notice that U_k is a solution of (4.5) and positive in $(0,\pi)$. Hence, we arrive at a contradiction and the lemma is proved. □

With above lemma, we are now able to establish the proof of Theorem 1.4.

Proof of Theorem 1.4. By smoothing estimates we may assume with no loss that $u_0 \in C^\infty$. It is not difficult to check that $u(x,t)$ is a 2π-periodic odd function and also odd symmetric with respect to $x = \pi$. Therefore

$$u(0,t) = u(\pi,t) \equiv 0, \quad \forall t \geq 0. \tag{4.6}$$

Together with that $u_0(x)$ is non-negative in $[0,\pi]$, we conclude that $u(x,t) \geq 0$ for $x \in [0,\pi]$ by Maximum Principle, see [8, Section 2] for instance. Similarly, we have $u(x,t) \leq 0$ for $x \in [-\pi,0]$. Now by using the energy conservation we have

$$\frac{d}{dt}\left(\frac{1}{2}\|\partial_x u\|_2^2 + \int_{\mathbb{T}} F(u)dx\right) = -\|\partial_t u\|_2^2, \tag{4.7}$$

where $F(u)=\frac{1}{4}(u^2-1)^2$. It follows that $\|\partial_t u\|_{L^1_t L^2_x}<\infty$ and one can extract a subsequence such that $\partial_t u(t_n)\to 0$ in L^2. By using higher uniform Sobolev estimates one can obtain convergence in higher norms. In particular we can obtain $u(t_n)\to u_\infty$ for some steady state of (3.1). In addition, u_∞ is a 2π-periodic odd function and non-negative for $x\in[0,\pi]$. By the proof of Theorem 1.1 we see that 0 and U_κ are the only steady states which are non-negative in $[0,\pi]$. As a consequence, we derive that u_∞ could be either U_κ or the trivial solution 0.

If $E_\kappa(u_0)\le\frac{\pi}{2}$ and $u_0(x)\ne 0$, using (4.7) we see that

$$E_\kappa(u_\infty)\le E_\kappa(u_0)\le\frac{\pi}{2}.$$

The equality sign holds only $u_\infty=u_0$. While it is known that $E_\kappa(0)=\frac{\pi}{2}$ and $u_0\ne 0$. Then we get $u_\infty=U_\kappa$. To obtain exponential convergence, we can take t_n sufficiently large such that $u(t_n)$ is sufficiently close to the steady state u_κ. Combined with Lemma 4.1 we then obtain the exponential convergence. Thus, we finish the whole proof. $\square$

Remark 4.1. If the initial data u_0 is 2π-periodic and satisfies

$$\begin{cases} u_0(x)=-u_0(-x), & \forall x\in\mathbb{R}, \\ u_0'(x)>0,\ \ u_0(x)=u_0(\pi-x), & \forall x\in\left(0,\frac{\pi}{2}\right). \end{cases} \tag{4.8}$$

Then by Theorem 1.4 we can prove that $u(x,t)\to U_\kappa$ as $t\to\infty$ whenever $E_\kappa(u_0)\le\frac{\pi}{2}$. A typical example of the initial data satisfying (4.8) is $u_0(x)=\sin x$. More examples can be easily constructed along these lines.

Before we end the study for the case $\kappa\in(0,1)$, we present the following result establishing an useful property of the odd zero-up ground state. This part is of independent interest.

Lemma 4.2. *Fix $\kappa\in[\frac{1}{2},1)$ and assume U_κ is the unique odd zero-up ground state for the equation*

$$\kappa^2 u''+u-u^3=0 \quad on \ \ \mathbb{T}=[-\pi,\pi].$$

Suppose that $u\in H^1(\mathbb{T})$ is an odd function on $\mathbb{T}$ such that

$$E_\kappa(u)=\int_{\mathbb{T}}\left(\frac{1}{2}\kappa^2(u')^2+\frac{1}{4}\left(u^2-1\right)^2\right)dx<\frac{\pi}{2}-C_\kappa,$$

where C_κ is a positive constant depending on κ. Then we have

$$\min\left\{\|u-U_\kappa\|_{H^1(\mathbb{T})},\ \|u+U_\kappa\|_{H^1(\mathbb{T})}\right\}\le C\sqrt{E(u)-E(U_\kappa)}, \tag{4.9}$$

where $C>0$ is an absolute constant.

Remark 4.2. In the special case $\kappa=0.9$, we can take $C_\kappa=0.001$. For the case $\kappa\in(0,\frac{1}{2})$, there are multiple steady-states and we shall address this issue elsewhere.

Proof. We first claim that for odd $u\in H^1(\mathbb{T})$, when $E(u)-E(U_\kappa)\to 0$, we must have

$$\min\Big\{\|u-U_\kappa\|_{H^1(\mathbb{T})},\ \|u+U_\kappa\|_{H^1(\mathbb{T})}\Big\}\to 0.$$

We shall prove this by contradiction. Suppose the statement is not true, then for some $c_0>0$, there exists a sequence of odd functions $\{u_n\}$ such that

$$E(u_n)-E(U_\kappa)\le\frac{1}{n}, \tag{4.10}$$

and

$$\min\Big\{\|u_n-U_\kappa\|_{H^1(\mathbb{T})},\ \|u_n+U_\kappa\|_{H^1(\mathbb{T})}\Big\}\ge c_0>0. \tag{4.11}$$

Using (4.10) we can find a universal constant C such that

$$\int_{\mathbb{T}}|u_n'|^2dx+\int_{\mathbb{T}}(1-u_n^2)^2dx\le C, \tag{4.12}$$

which implies that $\{u_n\}$ is a sequence of odd functions, and bounded in $H^1(\mathbb{T})$. Then we could select a subsequence, still denoted by $\{u_n\}$, which converges weakly to u_* in $H^1(\mathbb{T})$ for some odd function $u_*\in H^1(\mathbb{T})$. By the Rellich lemma and lower semi-continuity of weak convergence, we have

$$E(U_\kappa)\le E(u_*)\le\liminf_{n\to+\infty}E(u_n)=E(U_\kappa).$$

This implies that $E(U_\kappa)=E(u_*)$. Then we conclude that u_* is either U_κ or $-U_\kappa$. Thus u_n strongly converges to U_κ or $-U_\kappa$ in $H^1(\mathbb{T})$. This contradicts to (4.11). Therefore, the claim holds.

By using the claim, to establish (4.9), it suffices for us to consider the situation

$$\min\Big\{\|u_n-U_\kappa\|_{H^1(\mathbb{T})},\ \|u_n+U_\kappa\|_{H^1(\mathbb{T})}\Big\}\ll 1.$$

In this case, without loss of generality we assume that $u'(0)\ge 0$ and denote $\eta=u-U_\kappa$. Then it is not difficult to check that

$$E(u)-E(U_\kappa)=\frac{1}{2}\int_{\mathbb{T}}\Big[\kappa^2(\partial_x\eta)^2+(3U_\kappa^2-1)\eta^2\Big]dx+\mathcal{O}(\|\eta\|_{H^1}^3).$$

By Lemma 4.1, we have

$$\frac{1}{2}\int_{\mathbb{T}}\Big[\kappa^2(\partial_x\eta)^2+(3U_\kappa^2-1)\eta^2\Big]dx\ge C\|\eta\|_{H^1}^2,$$

where $C>0$ is an absolute constant. The desired conclusion follows easily. $\square$

4.2 Case of $\kappa \geq 1$

In the case of $\kappa \geq 1$, we consider the a general Allen-Cahn equation

$$\begin{cases} \partial_t u = -\kappa^2 \Lambda^\gamma u - (u^3 - u), \quad (x,t) \in \mathbb{T} \times (0,\infty), \\ u\big|_{t=0} = u_0, \end{cases} \tag{4.13}$$

where $\Lambda^\gamma = (-\partial_{xx})^{\gamma/2}$ is the fractional Laplacian of order $\gamma \in (0,2)$. When $\gamma = 2$ it coincides with $-\partial_{xx}$.

Proposition 4.1 (Preliminary properties of steady states for $0<\gamma<1$)**.** *Let $0<\gamma<2$ and $\kappa>0$. Suppose $\phi: \mathbb{T} \to \mathbb{R}$ is $C^{1,1}$ and satisfies*

$$-\kappa^2 \Lambda^\gamma \phi - (\phi^2 - 1)\phi = 0.$$

Then $\phi \in C^\infty(\mathbb{T})$, and only one of the following occur:

- $\phi \equiv 1$;
- $\phi \equiv -1$;
- $\|\phi\|_\infty < 1$.

Proof. This follows from the usual maximum principle argument using the expression

$$(\Lambda^\gamma \phi)(x) = C_\gamma \sum_{n \in \mathbb{Z}} \mathrm{PV} \int_{|y|<\pi} \frac{\phi(x) - \phi(y)}{|x - y + 2n\pi|^{1+\gamma}} dy.$$

Alternatively one can also derive the result using harmonic extension. ☐

To state the next result, we introduce the Fourier projection operators Π_1, $\Pi_{\geq 2}$ such that for $f = \sum_{m \geq 1} f_m \sin mx$ (assume the series converges sufficiently fast),

$$\Pi_1 f = f_1 \sin x, \quad \Pi_{\geq 2} f = \sum_{m \geq 2} f_m \sin mx. \tag{4.14}$$

In other words, Π_1 is the projection to the first sine-mode, and $\Pi_{\geq 2}$ simply removes the first Fourier mode in the sine series expansion.

Theorem 4.1. *Let $\kappa \geq 1$ and $0 < \gamma \leq 2$. Assume u_0 is 2π periodic, odd and bounded. Suppose u is the solution to* (4.13) *corresponding to the initial data u_0. If $\kappa > 1$, we have exponential decay*

$$\|u(t,\cdot)\|_2 \leq \|u_0\|_2 e^{-(\kappa^2 - 1)t}, \quad \forall t \geq 0, \tag{4.15}$$

$$\|u(t,\cdot)\|_{H^{10}} \le \beta_1 e^{-(\kappa^2-1)t}, \quad \forall t \ge \frac{1}{2}, \tag{4.16}$$

$$\|\Pi_{\ge 2} u(t,\cdot)\|_{H^{10}} \le \beta_2 e^{-\eta_1 t}, \quad \forall t \ge \frac{1}{2}, \tag{4.17}$$

where $\beta_1 > 0$, $\beta_2 > 0$ depend on (u_0, γ, κ), and $\Pi_{\ge 2}$ is defined in (4.14)*. The constant $\eta_1 > \kappa^2 - 1$ is given by*

$$\eta_1 = \min\Big\{\kappa^2 2^\gamma - 1, 3\big(\kappa^2 - 1\big)\Big\}.$$

For $\kappa = 1$, we have algebraic decay

$$\|u(t,\cdot)\|_2 \le \frac{\sqrt{\pi}\|u_0\|_2}{\sqrt{t\|u_0\|_2^2 + \pi}}, \quad \forall t \ge 0, \tag{4.18}$$

$$\|u(t,\cdot)\|_{H^{10}} \le \beta_3 t^{-\frac{1}{2}}, \quad \forall t \ge \frac{1}{2}, \tag{4.19}$$

$$\|\Pi_{\ge 2} u(t,\cdot)\|_{H^{10}} \le \beta_4 t^{-\frac{3}{2}}, \quad \forall t \ge \frac{1}{2}, \tag{4.20}$$

where $\beta_3 > 0$, $\beta_4 > 0$ depend on (u_0, γ).

Remark 4.3. For $\kappa > 1$, higher (i.e. $H^m, m > 10$) Sobolev norms of u also decay exponentially but we shall not dwell on this issue here. Note that we state the decay result for $t \ge \frac{1}{2}$ to allow the smoothing effect to kick in. The number $\frac{1}{2}$ is for convenience only and it can be replaced by any other $t_0 > 0$ with suitable adjustment of the corresponding pre-factors in the estimates.

Proof. First we note that for bounded initial data, local and global wellposedness is not an issue and we focus solely on the decay estimates.

For the L^2 decay estimates, first we assume u_0 is smooth, and in particular has a finite sine-series expansion. It follows that $u(t)$ must have a spectral gap. By using the Poincaré inequality we have

$$\big\|\Lambda^{\frac{\gamma}{2}} u\big\|_{L^2(\mathbb{T})} \ge \|u\|_{L^2(\mathbb{T})}.$$

By using the above estimate and the fact that $\kappa \ge 1$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\|u\|_2^2\big) &= -\kappa^2\big\|\Lambda^{\frac{\gamma}{2}} u\big\|_2^2 + \|u\|_2^2 - \|u\|_4^4 \le -\big(\kappa^2 - 1\big)\|u\|_2^2 - \|u\|_4^4 \\ &\le -\big(\kappa^2 - 1\big)\|u\|_2^2 - \frac{1}{2\pi}\|u\|_2^4,\end{aligned}$$

where in the last step we have used the Hölder's inequality. Then, we derive that in the case of $\kappa > 1$,

$$\|u\|_2 \le \|u_0\|_2 e^{-(\kappa^2-1)t},$$

while in the case of $\kappa = 1$,

$$\|u\|_2 \le \frac{\sqrt{\pi}\|u_0\|_2}{\sqrt{t\|u_0\|_2^2+\pi}}.$$

By a simple approximation argument, both estimates also hold under the assumption that $u_0 \in L^\infty$.

We now show (4.16). First by smoothing estimates and interpolation, we have

$$\left\|\partial_x^{10}(u(t,\cdot))\right\|_2 \le \alpha_1 e^{-\kappa_1 t}, \quad \forall t \ge \frac{1}{2},$$

where $\alpha_1 > 0$ depends on (u_0, γ, κ), and $\kappa_1 > 0$ depends only on κ. It follows easily that

$$\left\|\partial_x^{10}(u^3(t,\cdot))\right\|_2 \le \alpha_2 e^{-\kappa_1 t}\left\|\partial_x^{10}u(t,\cdot)\right\|_2, \quad \forall t \ge \frac{1}{2},$$

where $\alpha_2 > 0$ depends on (u_0, γ, κ). We now compute for $t \ge \frac{1}{2}$,

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\|\partial_x^{10}u(t,\cdot)\|_2^2\right) &\le -\kappa^2\left\|\Lambda^{\frac{\gamma}{2}}\partial_x^{10}u\right\|_2^2 + \left\|\partial_x^{10}u\right\|_2^2 + \left\|\partial_x^{10}(u^3)\right\|_2\left\|\partial_x^{10}u\right\|_2 \\ &\le \left(-(\kappa^2-1)+\alpha_2 e^{-\kappa_1 t}\right)\left\|\partial_x^{10}u\right\|_2^2.\end{aligned}$$

Integrating in time then yields (4.16).

The proof of (4.17) is similar. Note that for all $t \ge \frac{1}{2}$,

$$\left\|\partial_x^{10}\Pi_{\ge 2}(u^3(t,\cdot))\right\|_2 \le \left\|\partial_x^{10}(u^3(t,\cdot))\right\|_2 \le \alpha_3 e^{-3(\kappa^2-1)t},$$

where $\alpha_3 > 0$ depends on (u_0, γ, κ). With this we compute

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\partial_x^{10}\Pi_{\ge 2}u(t,\cdot)\|_2^2\right) \\ &\le -\kappa^2\left\|\Lambda^{\frac{\gamma}{2}}\partial_x^{10}\Pi_{\ge 2}u\right\|_2^2 + \left\|\partial_x^{10}\Pi_{\ge 2}u\right\|_2^2 + \left\|\partial_x^{10}(u^3)\right\|_2\left\|\partial_x^{10}\Pi_{\ge 2}u\right\|_2 \\ &\le \left(-\left(\kappa^2 2^\gamma - 1\right)\right)\left\|\partial_x^{10}\Pi_{\ge 2}u\right\|_2^2 + \alpha_3 e^{-3(\kappa^2-1)t}\left\|\partial_x^{10}\Pi_{\ge 2}u\right\|_2.\end{aligned}$$

Thus (4.17) follows from a simple ODE argument.

Finally (4.20) follows from working with the system

$$\partial_t \Pi_{\ge 2}u = -\kappa^2\Lambda^\gamma\Pi_{\ge 2}u + \Pi_{\ge 2}u - \Pi_{\ge 2}(u^3),$$

and bootstrapping estimates using (4.18). The estimate (4.19) is obvious. We omit the details. $\square$

We turn now to some (by now) standard log-convexity results.

Proposition 4.2 (Log convexity for an almost-linear model)**.** *Suppose* $\mathbb{H}$ *is a real Hilbert space with inner product* $\langle\cdot,\cdot\rangle$ *and norm* $\|\cdot\|$*. Let* A *be a symmetric operator on* $\mathbb{H}$ *with domain* $\mathcal{D}(A)$*. Let* $T>0$ *and* $u\in C_t^1([0,T],\mathbb{H})$ *satisfy* $u(t)\in\mathcal{D}(A)$ *for each* $0<t<T$*, and*

$$\|\partial_t u+Au\|\le\alpha(t)\|u(t)\|,\quad \forall 0<t<T,$$

where $\alpha(t)\ge 0$ *satisfies*

$$\int_0^T\alpha(t)^2dt<\infty.$$

Denote $m(t)=\|u(t)\|^2$*. Then* m *is log-convex*

$$m(t)\le e^{\int_0^T(4\alpha(s)+s\alpha(s)^2)ds}m(0)^{1-\frac{t}{T}}m(T)^{\frac{t}{T}},\quad \forall 0\le t\le T.$$

It follows that either $m(t)\equiv 0$ *on* $[0,T]$ *or* $m(t)>0$ *for all* $t\in[0,T]$*.*

Proof. First we assume that $m(t)>0$ for all $0\le t\le T$. Denote $f=\partial_t u+Au$ so that

$$\partial_t u=-Au+f.$$

Denote

$$b_1=\frac{2}{m}\langle u,f\rangle.$$

Then clearly

$$\frac{d}{dt}\left(\ln m+\int_t^T b_1(s)ds\right)=\frac{2}{m}\langle u,-Au\rangle$$

and

$$\begin{aligned}&\frac{1}{4}\frac{d^2}{dt^2}\left(\ln m+\int_t^T b_1(s)ds\right)\\&=\frac{\langle u_t,-Au\rangle}{m}-\frac{\langle u,-Au\rangle\langle u,-Au+f\rangle}{m^2}\\&=\frac{m\|Au\|^2-|\langle u,-Au\rangle|^2}{m^2}+\frac{\langle f,-Au\rangle}{m}+\frac{\langle u,Au\rangle\langle u,f\rangle}{m^2}.\end{aligned}$$

We decompose $Au=c_1u+cu^{\perp}$ where $c_1,c\in\mathbb{R}$ and $u^{\perp}$ is a unit vector orthogonal to u. Plugging this into the last expression, we obtain

$$\frac{1}{4}\frac{d^2}{dt^2}\left(\ln m+\int_t^T b_1(s)ds\right)=\frac{c^2}{m}-\frac{c\langle f,u^{\perp}\rangle}{m}$$

$$\geq \frac{(|c|-\frac{1}{2}|\langle f,u^{\perp}\rangle|)^2}{m}-\frac{1}{4}\frac{|\langle f,u^{\perp}\rangle|^2}{m}\geq -\frac{1}{4}\alpha(t)^2.$$

It follows that $r(t)$ is convex, where

$$r(t)=\ln m(t)+\underbrace{\int_t^T b_1(s)ds+\int_t^T\int_\tau^T \alpha^2(s)dsd\tau}_{=:b(t)}.$$

From the convexity of $r(t)$ we deduce

$$r(t)\leq \left(1-\frac{t}{T}\right)r(0)+\frac{t}{T}r(T).$$

This implies that

$$\ln m(t)\leq \left(1-\frac{t}{T}\right)\ln m(0)+\frac{t}{T}\ln m(T)+\left(1-\frac{t}{T}\right)b(0)+\frac{t}{T}b(T)-b(t).$$

Since $b(T)=0$ and

$$\int_t^T\int_s^T \alpha(\tau)^2 d\tau ds\geq 0,$$

we obtain

$$\begin{aligned}&\left(1-\frac{t}{T}\right)b(0)+\frac{t}{T}b(T)-b(t)=\left(1-\frac{t}{T}\right)b(0)-b(t)\\ \leq{}& 2\int_0^T |b_1(s)|ds+\int_0^T\int_\tau^T \alpha^2(s)dsd\tau\\ \leq{}& \int_0^T \left(4\alpha(s)+s\alpha(s)^2\right)ds.\end{aligned}$$

Thus the desired inequality holds under the assumption that $m(t)>0$ for all $t\in[0,T]$.

Now we show how to remove this assumption. Assume that $m(t)$ is not identically zero. Since $m(t)\geq 0$ is a continuous function of t and $m(t)$ is not identically zero, we may assume that there exists $t_0\in[0,T]$ such that $m(t_0)>0$. By a continuity argument we can assume $t_0\in(0,T)$. Now denote

$$\begin{aligned}&t_+=\sup\{t:t>t_0 \text{ such that } m(s)>0 \text{ for all } t_0\leq s\leq t\},\\ &t_-=\inf\{t:t<t_0 \text{ such that } m(s)>0 \text{ for all } t\leq s\leq t_0\}.\end{aligned}$$

If $t_+ < T$, then we have $m(t_+) = 0$ with $m(t) > 0$ for all $t_0 \le t < t_+$. By using a version of the proved inequality on the interval $[t_0, t_+ - \eta]$ (note that $m(t) > 0$ for all $t_0 \le t \le t_+ - \eta$ and thus we can use the proved inequality with the interval $[0,T]$ now replaced by $[t_0, t_+ - \eta]$) and sending $\eta \to 0$, we clearly obtain a contradiction. If $t_+ = T$ and $m(T) = 0$, we also obtain a contradiction by a similar argument. By a similar reasoning we obtain $t_- = 0$ and $m(0) > 0$. Thus we have proved that $m(t) > 0$ for all $t \in [0,T]$. $\square$

Lemma 4.3. *For any* $0 < \gamma \le 2$, *there exits* $\eta_0 = \eta_0(\gamma) > 0$ *such that the following hold for any smooth* 2π*-periodic* odd *function* u *on* $\mathbb{T}$*:*

$$\int_{\mathbb{T}} u^3 (-\partial_{xx})^{\frac{\gamma}{2}} u dx \ge \eta_0 \|u\|_4^4.$$

For $\gamma = 2$, *we can take* $\eta_0 = \frac{3}{4}$.

Proof. It follows from a general result in [11]. For $\gamma = 2$ we give a direct proof as follows (below we write $\int_{\mathbb{T}} dx$ as $\int$)

$$\begin{aligned} \int u^3(-\partial_{xx} u) &= 3\int (\partial_x u)^2 u^2 = \frac{3}{4}\int \left(\partial_x(|u|u)\right)^2 dx \\ &\ge \frac{3}{4} \||u|u\|_2^2 = \frac{3}{4}\|u\|_4^4. \end{aligned}$$

Note that in the above we took advantage of the odd symmetry since the function $v = |u|u$ is still odd on $[-\pi, \pi]$. Note that regularity is not an issue here since the function $g(z) = |z|z$ is nice. $\square$

Theorem 4.2 (Log convexity of L^2 mass for the nonlinear case)**.** *Let* $\kappa > 0$ *and* $0 < \gamma \le 2$. *Assume* u_0 *is* 2π *periodic, odd and bounded. To avoid triviality assume* $\|u_0\|_2 > 0$ *so that* u_0 *is not identically zero. Suppose* u *is the solution to* (4.13) *corresponding to the initial data* u_0. *Denote* $m(t) = \|u(t)\|^2_{L^2_x(\mathbb{T})}$. *Then the following hold:*

- *If* $0 < \kappa < 1$, *then* $m(t)$ *is log-convex on any interval* $0 \le t_1 < t_2$
$$m(t) \le e^{c_1 \cdot (t_2 - t_1)^2} m(t_1)^{1 - \frac{t - t_1}{t_2 - t_1}} m(t_2)^{\frac{t - t_1}{t_2 - t_1}}, \quad \forall t \in (t_1, t_2),$$
where $c_1 > 0$ *is a constant depending only on* $(\|u_0\|_\infty, \gamma, \kappa)$.
- *If* $\kappa > 1$, *then* $m(t)$ *is log-convex on any interval* $0 \le t_1 < t_2$
$$m(t) \le c_2 m(t_1)^{1 - \frac{t - t_1}{t_2 - t_1}} m(t_2)^{\frac{t - t_1}{t_2 - t_1}}, \quad \forall t \in (t_1, t_2),$$
where $c_2 > 0$ *is a constant depending only on* $(\|u_0\|_\infty, \gamma, \kappa)$.

- *If $\kappa=1$, then $m(t)$ is log-convex on any interval $0\le t_1<t_2$*

$$m(t)\le (1+t_2-t_1)^{c_3} m(t_1)^{1-\frac{t-t_1}{t_2-t_1}} m(t_2)^{\frac{t-t_1}{t_2-t_1}}, \quad \forall t\in(t_1,t_2),$$

where $c_3>0$ is a constant depending only on $(\|u_0\|_\infty,\gamma,\kappa)$.

- *For each $0<\gamma\le 2$, there is $\kappa_0=\kappa_0(\gamma)>0$ such that if $\kappa\ge\kappa_0$, then we have sharp log-convexity, i.e. on any interval $0\le t_1<t_2$*

$$m(t)\le m(t_1)^{1-\frac{t-t_1}{t_2-t_1}} m(t_2)^{\frac{t-t_1}{t_2-t_1}}, \quad \forall t\in(t_1,t_2). \tag{4.21}$$

Furthermore for $\gamma=2$, we can choose $\kappa_0(2)=2/\sqrt{3}$.

Proof. First we consider the case $0<\kappa<1$. Observe that

$$\left\|\partial_t u+\kappa^2\Lambda^\gamma u-u\right\|_2\le \|u(t)\|_{L_x^\infty}^2\|u(t)\|_2.$$

It is not difficult to check that

$$\sup_{0\le t<\infty}\|u(t)\|_{L_x^\infty}\le \tilde{c}_1,$$

where $\tilde{c}_1>0$ depends only on $(\|u_0\|_\infty,\gamma,\kappa)$. Thus the result follows from Proposition 4.2.

Now for $\kappa>1$, we observe that by using Theorem 4.1 (note that for $0<s\le\frac{1}{2}$ we have uniform control of L^∞-norm), it holds that

$$\|u(s)\|_{L_x^\infty}\le \tilde{c}_2 e^{-\theta_1 s}, \quad \forall s\ge 0,$$

where $\tilde{c}_2>0$ depends only on $(\|u_0\|_\infty,\gamma,\kappa)$ and θ_1 depends only on (κ,γ). Thus

$$\tilde{\alpha}(s):=\|u(s)\|_{L_x^\infty}^2\le \tilde{c}_2^2 e^{-2\theta_1 s}, \quad \forall s\ge 0.$$

On any time interval $[t_1,t_2]$ with $0\le t_1<t_2$, in order to apply Proposition 4.2, we note for $t\ge 0$,

$$\alpha(t)=\tilde{\alpha}(t_1+t)\le \tilde{c}_2^2 e^{-2\theta_1 t}.$$

Thus the desired result follows for $\kappa>1$.

The case for $\kappa=1$ follows similarly from Theorem 4.1 and Proposition 4.2. The main observation is that

$$\|u(s)\|_{L_x^\infty}=\mathcal{O}\big((1+s)^{-\frac{1}{2}}\big) \quad \text{for} \quad s\ge 0.$$

Finally we turn to the proof of (4.21). We shall appeal to a more "nonlinear" proof as follows. Denote

$$m(t)=\|u(t)\|^2_{L^2_x}, \quad A=\kappa^2(-\partial_{xx})^{\frac{\gamma}{2}}-1.$$

Thus we have

$$\partial_t u=-Au-u^3.$$

Denote

$$m'=\frac{d}{dt}m, \quad m''=\frac{d^2}{dt^2}m.$$

It is not difficult to check that (below $\langle\cdot,\cdot\rangle$ denotes the usual L^2 inner product, and $u_t=\partial_t u$)

$$\begin{aligned}
m'&=2\langle u,u_t\rangle=-2\langle u,Au\rangle-2\|u\|_4^4,\\
m''&=-4\langle u_t,Au\rangle-8\langle u^3,u_t\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
&\frac{1}{2}m'=\langle u,u_t\rangle,\\
&\frac{1}{4}m''=\|u_t\|_2^2-\langle u^3,u_t\rangle,\\
&\frac{1}{4}m''m-\frac{1}{4}(m')^2=\|u_t\|_2^2\|u\|_2^2-|\langle u,u_t\rangle|^2+\Big(\langle u^3,Au\rangle+\|u\|_6^6\Big)\|u\|_2^2.
\end{aligned}$$

It remains for us to verify

$$\langle u^3,Au\rangle=\kappa^2\int_{\mathbb{T}}u^3(-\partial_{xx})^{\frac{\gamma}{2}}u\,dx-\|u\|_4^4\geq 0.$$

This in turn follows from Lemma 4.3. □

Corollary 4.1 (No finite time extinction of L^2 mass)**.** *Let $\kappa>0$ and $0<\gamma\leq 2$. Assume u_0 is 2π periodic, odd and bounded. To avoid triviality assume $\|u_0\|_2>0$ so that u_0 is not identically zero. Suppose u is the solution to* (4.13) *corresponding to the initial data u_0. Then $\|u(t)\|_{L^2_x}>0$ for any $0\leq t<\infty$.*

Proof. This follows easily from Proposition 4.2. □

Theorem 4.3 (Profiles as $t\to\infty$)**.** *Let $\kappa\geq 1$ and $0<\gamma\leq 2$. Assume u_0 is 2π periodic, odd and bounded. To avoid triviality assume $\|u_0\|_2>0$ so that u_0 is not identically zero. Suppose u is the solution to* (4.13) *corresponding to the initial data u_0. Then the following hold:*

- *Case* $\kappa>1$. *For all* $t\geq 1$, *we have*

$$u(x,t)=e^{-(\kappa^2-1)t}\alpha_*\sin x+r(t), \tag{4.22}$$

where the constant α_* *depends on* (u_0,γ,κ). *The remainder term* $r(t)$ *has the estimate*

$$\|r(t)\|_{H^{10}}\leq\tilde{\alpha}e^{-\eta_1 t},\quad \forall t\geq 1$$

with $\tilde{\alpha}>0$ *depends only on* (u_0,γ,κ), *and*

$$\eta_1=\min\Big\{\kappa^2 2^\gamma-1,3\big(\kappa^2-1\big)\Big\}>\kappa^2-1.$$

- *Case* $\kappa=1$. *For all* $t\geq 1$, *we have*

$$u(x,t)=t^{-\frac{1}{2}}\beta_*\sin x+r_1(t), \tag{4.23}$$

where the constant β_* *depends on* $(u_0,\ \gamma)$. *If* $\beta_*=0$, *then the remainder term* $r_1(t)$ *has the estimate*

$$\|r_1(t)\|_{H^{10}}\leq\tilde{\beta}t^{-1}\sqrt{\ln(t+2)},\quad \forall t\geq 1$$

with $\tilde{\beta}>0$ *depends only on* (u_0,γ). *If* $\beta_*\neq 0$, *then the remainder term* $r_1(t)$ *has the estimate*

$$\|r_1(t)\|_{H^{10}}\leq\tilde{\beta}t^{-\frac{3}{2}}\ln(t+2),\quad \forall t\geq 1$$

with $\tilde{\beta}>0$ *depends only on* (u_0,γ).

Remark 4.4. Clearly, Theorem 1.3 follows from above result immediately. Note that for $\kappa>1$ and generic nontrivial odd periodic u_0 we could have $\alpha_*=0$. An easy example is $u_0(x)=\sin 2x$. On the other hand, a further interesting question is to investigate whether the following scenario is possible: namely if we denote

$$\alpha_1(t)=\int_{\mathbb{T}}u(t,x)\sin x dx,$$

then for some $t=t_c$, $\alpha_1(t)=0$ for $t\geq t_c$, and $\alpha_1(t)\neq 0$ for $t<t_c$ with t_c-t sufficiently small.

Proof. We first consider $\kappa>1$. Write

$$u=\Pi_1 u+\Pi_{\geq 2}u,$$

where the operators $\Pi_1,\Pi_{\ge 2}$ were defined in (4.14). By Theorem 4.1 the term $\Pi_{\ge 2}u$ has the desired decay for $t\ge 1$ and can be included in the remainder $r(t)$. Thus we only need to treat the single-mode part $\Pi_1 u$. Denote

$$\Pi_1 u(t)=a(t)\sin x, \qquad a(t)=\frac{1}{\pi}\int_{\mathbb{T}}\Pi_1 u(t,x)\sin x dx,$$
$$\Pi_1\big(-u^3(t)\big)=b(t)\sin x, \quad b(t)=\frac{1}{\pi}\int_{\mathbb{T}}\Pi_1\big(-u^3(t,x)\big)\sin x dx.$$

By Theorem 4.1, we have for some $\tilde{C}>0$ depending only on (u_0,γ,κ),

$$|b(t)|\le \tilde{C}e^{-3(\kappa^2-1)t}, \quad \forall t\ge \frac{1}{2}.$$

Clearly we have

$$\frac{d}{dt}a(t)=-(\kappa^2-1)a(t)+b(t).$$

We then write for $t\ge 1$,

$$\begin{aligned} a(t)&=e^{-(\kappa^2-1)(t-\frac{1}{2})}a\left(\frac{1}{2}\right)+\int_{\frac{1}{2}}^{t}e^{-(\kappa^2-1)(t-s)}b(s)ds \\ &=e^{-(\kappa^2-1)t}\left(e^{\frac{1}{2}(\kappa^2-1)}a\left(\frac{1}{2}\right)+\int_{\frac{1}{2}}^{\infty}e^{(\kappa^2-1)s}b(s)ds\right)+\tilde{r}(t), \end{aligned}$$

where

$$|\tilde{r}(t)|\le e^{-(\kappa^2-1)t}\int_t^{\infty}e^{(\kappa^2-1)s}|b(s)|ds=\mathcal{O}\big(e^{-3(\kappa^2-1)t}\big).$$

Clearly then (4.22) follows.

The proof of (4.23) is slightly more intricate. We only need to treat the piece $\Pi_1 u$ since the part $\Pi_{\ge 2}u$ can be included in the remainder term $r_1(t)$. Observe that for $t\ge \frac{1}{2}$, by Theorem 4.1 we have

$$u(t)^3=\big(\Pi_1 u(t)+\Pi_{\ge 2}u(t)\big)^3=(\Pi_1 u)^3+\tilde{r}(t),$$

where

$$\|\tilde{r}(t)\|_{H^{10}}=\mathcal{O}\big(t^{-\frac{5}{2}}\big), \quad \forall t\ge \frac{1}{2}.$$

Denote $\Pi_1 u(t)=a(t)\sin x$, clearly

$$\Pi_1\big((\Pi_1 u(t))^3\big)=\frac{3}{4}a(t)^3\sin x.$$

For $a(t)$ we have the ODE

$$\frac{d}{dt}a(t) = -\frac{3}{4}a(t)^3 + b(t), \quad t \ge \frac{1}{2},$$

where $|b(t)| = \mathcal{O}(t^{-\frac{5}{2}})$.

Denote $\theta(t) = a(t)^2$. We clearly have

$$\frac{d}{dt}\theta(t) = -\frac{3}{2}\theta(t)^2 + \mathcal{O}(t^{-3}).$$

By Proposition 4.3 proved below, we have for $t \ge 3$,

$$\theta(t) = \frac{\theta_*}{t} + \mathcal{O}\big(t^{-2}\ln t\big).$$

Note that $\theta_* \ge 0$ since $\theta(t)$ is always nonnegative.

Now if $\theta_* = 0$ we can take $\beta_* = 0$ and the desired result follows easily. If $\theta_* > 0$, then $|a(t)| \sim t^{-1/2}$ for t large. By continuity it can only take one sign. Thus we obtain $\beta_* = \sqrt{\theta_*}$ or $\beta_* = -\sqrt{\theta_*}$. The estimate for the remainder term is trivial. We omit the details. $\square$

Lemma 4.4. *Assume $T_0 \ge 1$. Suppose $\theta : [T_0, \infty) \to [0, \infty)$ is continuously differentiable and satisfy*

$$0 < \limsup_{t\to\infty}\{t\theta(t)\} < \infty, \quad \theta'(t) \ge -\frac{3}{2}\theta^2(t) - t^{-2.2}, \quad \forall t > T_0.$$

Then we have

$$\liminf_{t\to\infty}\{t\theta(t)\} > 0.$$

Proof. It is natural to appeal to a maximum principle argument. Denote

$$2\theta_0 = \limsup_{t\to\infty}\{t\theta(t)\} > 0. \tag{4.24}$$

Let

$$\Omega(t) = \theta(t) - \eta_0 t^{-1},$$

where $\eta_0 > 0$ satisfies

$$\eta_0 \le \min\left\{\frac{1}{2}\theta_0, \frac{1}{10}\right\}.$$

By (4.24), we can choose $t_0>0$ sufficiently large such that

$$\theta(t_0)\geq\frac{\theta_0}{t_0},\quad \frac{\eta_0}{2t_0^2}-t_0^{-2.2}>0.$$

Note that the second condition above guarantees that

$$\frac{\eta_0}{2t^2}-t^{-2.2}>0,\quad \forall t\geq t_0.$$

Now consider $\Omega(t)$ on the time interval $[t_0,\infty)$. If $\Omega(t)>0$ for all $t\geq t_0$ we are done. Otherwise there exists some time $t_1>t_0$ such that $\Omega(t_1)=0$. But then clearly

$$\theta(t_1)=\frac{\eta_0}{t_1},\quad \Omega'(t_1)=-\frac{3}{2}\frac{\eta_0^2}{t_1^2}+\frac{\eta_0}{t_1^2}-t_1^{-2.2}\geq\frac{\eta_0}{2t_1^2}-t_1^{-2.2}>0.$$

Thus $\Omega(t)$ continuous to be positive a little bit past t_1. This argument then guarantees that $\Omega(t)\geq 0$ for all $t\geq t_0$. $\square$

Lemma 4.5. *Assume $T_0\geq 1$ and $0<\kappa_0<1$. Suppose $\theta:[T_0,\infty)\to[0,\infty)$ satisfies*

$$\theta(t)\leq\frac{\kappa_0}{t},\quad \theta(t)\leq\frac{1}{2}\int_t^\infty\theta(s)^2ds+\frac{1}{2t^2},\quad \forall t\geq T_0.$$

Then there exists a constant $C_1>0$ depending on κ_0 and T_0 such that

$$\theta(t)\leq\frac{C_1}{t^2},\quad \forall t\geq T_0.$$

Proof. We begin by noting that, if we assume

$$\theta(t)\leq\frac{\alpha}{t},\quad \forall t\geq\max\left\{T_0,\frac{1}{\alpha}\right\},$$

then we obtain

$$\theta(t)\leq\frac{1}{2}\cdot\frac{\alpha^2}{t}+\frac{1}{2t^2}\leq\frac{\alpha^2}{t},\quad \forall t\geq\max\left\{T_0,\frac{1}{\alpha^2}\right\}.$$

Now define $\alpha_0=\kappa_0<1$, and $\alpha_{k+1}=\alpha_k^2$. Note that

$$\alpha_k=e^{2^k\ln\kappa_0}.$$

Clearly it holds that
$$\theta(t) \leq \frac{\alpha_k}{t}, \quad \forall t \geq \max\left\{T_0, \frac{1}{\alpha_k}\right\}.$$
Consider $t \in [\frac{1}{\alpha_k}, \frac{1}{\alpha_{k+1}}] = [\frac{1}{\alpha_k}, \frac{1}{\alpha_k^2}]$. Clearly it holds that

$$\alpha_k \leq t^{-\frac{1}{2}}.$$

Thus we have for all $t \in [\frac{1}{\alpha_k}, \frac{1}{\alpha_{k+1}}]$ with $\frac{1}{\alpha_k} \geq T_0$, it holds that

$$\theta(t) \leq t^{-\frac{3}{2}}.$$

Thus we have for all t sufficiently large

$$\theta(t) \leq t^{-\frac{3}{2}}.$$

Iterating this estimate again we obtain $\theta(t) \leq \mathcal{O}(t^{-2})$. □

Proposition 4.3. *Assume $T \geq 3$. Suppose $\theta: [T,\infty) \to [0,\infty)$ is continuously differentiable and satisfy*
$$\sup_{t \geq T_0} \big(t\theta(t)\big) < \infty, \quad \theta'(t) = -\frac{3}{2}\theta^2(t) + F(t), \quad \forall t > T,$$
where for some $K_0 > 0$
$$|F(t)| \leq K_0 t^{-3}, \quad \forall t \geq T.$$
Then there exists $\theta_* \in \mathbb{R}$, such that

$$\theta(t) = \frac{\theta_*}{t} + R(t),$$

where
$$\sup_{t \geq T} \frac{|R(t)|}{\ln t / t^2} < \infty.$$

Proof. Note that we only need to investigate the regime $t \gg 1$. We shall discuss two cases:

Case 1. $\limsup_{t\to\infty}\{t\theta(t)\} > 0$. In this case we use Lemma 4.4. Clearly for T_0 sufficiently large we have

$$\theta(t)t \sim 1, \quad \forall t \geq T_0.$$

From the ODE we obtain

$$\frac{d}{dt}\left(\frac{1}{\theta}\right)=\frac{3}{2}+\mathcal{O}(t^{-1}).$$

It follows that for T_0' sufficiently large and all $t\ge T_0'+2$,

$$\theta(t)=\frac{1}{d_1+d_2(t-T_0')+\mathcal{O}(\ln(t-T_0'))},$$

where $d_1>0, d_2>0$ are constants. The desired asymptotics then follows easily.

Case 2. $\limsup_{t\to\infty}\{t\theta(t)\}=0$. In this case we make a change of variable

$$t=N\tau,\quad \theta(t)=\gamma\Theta(\tau),\quad \frac{N}{\gamma}F(t)=\tilde{F}(\tau).$$

Clearly

$$\frac{d}{d\tau}\Theta(\tau)=-\frac{3}{2}\gamma N\Theta^2+\tilde{F}(\tau),\quad |\tilde{F}(\tau)|\le K_0\frac{1}{\gamma N^2}\tau^{-3}.$$

Thus if we take $\gamma=\frac{1}{3N}$ and N sufficiently large, we obtain

$$\Theta(\tau)\le\frac{1}{2}\int_\tau^\infty\Theta^2(s)ds+\frac{1}{2\tau^2},\quad \forall\tau\ge\tau_0,$$

where τ_0 is sufficiently large. We then use Lemma 4.5 to conclude that $\Theta(\tau)=\mathcal{O}(\tau^{-2})$. Thus in this case $\theta(t)=\mathcal{O}(t^{-2})$. □

5 Concluding remarks

In this work we considered the classification of steady states to the one-dimensional periodic Allen-Cahn equation with standard double well potential. We gave a full classification of all possible steady states and identified their precise dependence on the diffusion coefficient in terms of energy and profiles. We found a novel self-replicating property of steady state solutions amongst the hierarchy solutions organized according to the diffusion parameter. We developed a new modulation theory around these steady states and proved sharp convergence results. We discuss below a few possible future directions.

(1) Classification for other models with different linear dissipations and nonlinearities. Even for the classical Allen-Cahn case, one can investigate the singular potential functions such as the logarithmic function given by

$$f(u) = -u + \frac{1}{8}\log\frac{1+u}{1-u},$$

or the sine-Gordon type

$$f(u) = -\sin u.$$

Besides the one dimensional theory, we also expect some generalizations to higher dimensions.

(2) Patching and extension and solutions across general interfaces. It is natural to consider extending solutions of

$$\mathcal{L}u - f(u) = 0 \quad \text{in } \Omega,$$

where $\mathcal{L}$ is the linear part and f denotes the nonlinear part with appropriate boundary conditions. The task is to investigate under what conditions we can extend the solution across a portion of the boundary of Ω which is assumed to be a hypersurface or even some lower dimensional interface. In one dimension the situation is simple via reflection, but the general situation certainly merits further investigation.

(3) Convergence theory for general initial data, and also for other equations and phase field models. These include nonlocal Allen-Cahn equations driven by general polynomials or logarithmic or even mildly singular nonlinearities, also one can investigate Cahn-Hillard equations, Molecular Beam epitaxy equation, time fractional equations and so on.

Appendix A. Proof of Proposition 2.3

We prove Proposition 2.3 step by step.

(1) In the case of $C > \frac{1}{2}$, we can see that u' never changes sign and it implies that u is either an increasing or a decreasing function. In addition $|u'|$ has a positive lower bound, it implies u is unbounded. Thus, there is no bounded solution.

(2) In the case of $C=\frac{1}{2}$, Eq. (2.10) becomes

$$\kappa^2(u')^2=\frac{1}{2}(u^2-1)^2.$$

It is easy to check that $u=1$ or -1 is always a solution to the above equation. In the range $\{x||u(x)|<1\}$ we solve the above ODE and get $u=\pm\tanh(\frac{x+c}{\sqrt{2}\kappa})$, it defines an entire solution of (2.8). While if $\{x\,|\,|u(x)|>1\}$ is not empty, then we can use the ODE $u'=\frac{1}{\sqrt{2}}(u^2-1)$ to derive that the solution must be unbounded. Therefore, $\{x\,|\,|u(x)|>1\}=\emptyset$. As a result, we get either $|u|\equiv 1$ or $u=\tanh(\frac{x+c}{\sqrt{2}\kappa})$ in this case.

(3) In the case of $0<C<\frac{1}{2}$, according to (2.10), we have

$$\frac{1}{2}(u^2-1)^2+C-\frac{1}{2}\geq 0,$$

i.e.,

$$|u|\geq\sqrt{1+\sqrt{1-2C}}\quad\text{or}\quad|u|\leq\sqrt{1-\sqrt{1-2C}}.$$

If there exists some point x_1 such that $u(x_1)>\sqrt{1+\sqrt{1-2C}}$ (multiplying by -1 if necessary), then we see from (2.10), that either $u(x)$ is strictly increasing for $x\in(x_1,+\infty)$ and $u'(x)$ has a positive lower bound, or strictly decreasing for $x\in(-\infty,x_1)$ and $u'(x)$ has a negative upper bound. Consequently, $u(x)$ is unbounded, which implies that such x_1 can not exist. Further, $u\equiv\sqrt{1+\sqrt{1-2C}}$ is obviously not the solution to (2.8). Therefore, we can claim that

$$|u|\leq\sqrt{1-\sqrt{1-2C}}. \tag{A.1}$$

Now we show that u has local maxima and minima not at infinity. Otherwise, $u(x)$ is monotonic when $|x|$ is sufficiently large. Without loss of generality, suppose that $u(x)$ is monotone increasing for x large. Then by (A.1) and (2.10) we have

$$\lim_{x\to\infty}u(x)=\sqrt{1-\sqrt{1-2C}},\quad\lim_{x\to+\infty}u'(x)=0.$$

Using (2.8) we derive that $u''(x)\neq 0$ when $u(x)$ is around $\sqrt{1-\sqrt{1-2C}}$. Contradiction arises. Thus, $u(x)$ have both local maxima and minima. Let

$x=a$ be a local maximum point and $x=b$ be the closest local minimum point to a, then by (2.10) we get

$$u(a)=\sqrt{1-\sqrt{1-2C}}, \quad u(b)=-\sqrt{1-\sqrt{1-2C}}.$$

By reflection symmetry, $u(2b-a)=\sqrt{1-\sqrt{1-2C}}$. Repeating the reflection process we could see that $u(x)$ is a periodic function with minimal period equals to $|2b-2a|$.

(4) Finally, for the last statement, it can be obtained $|u|\geq\sqrt{2}$ or $u\equiv 0$ from (2.10). From a discussion similar to the above, $|u|\geq\sqrt{2}$ will lead to contradiction. Then we have $u\equiv 0$.

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