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Well-Posedness of the Free Boundary Problem for the Compressible Euler Equations and the Incompressible Limit

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Abstract. In this paper, we study the free boundary problem of the compressible Euler equations in the Eulerian coordinates. By deriving the evolution equation of the free surface, we relate the Taylor stability condition to the hyperbolicity of this evolution equation. Our approach not only yields exact information of the free surface, but also gives a simple proof of the local well-posedness of the free boundary problem. This approach provides a unified framework to treat both compressible and incompressible free boundary problems. As a byproduct, we can also prove the incompressible limit.

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1 Introduction

1.1 Presentation of the problem

The compressible Euler equations are

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$$\begin{cases} \partial_t \rho + u\cdot\nabla\rho + \rho\nabla\cdot u = 0, \\ \partial_t u + u\cdot\nabla u + \dfrac{\nabla p}{\rho} = 0, \end{cases} \tag{1.1}$$

where ρ is the density and u is the velocity of a compressible liquid. The pressure p is given by the state equation

$$p = p(\rho) = \frac{1}{\epsilon^2}(\rho - 1) \tag{1.2}$$

with $0<\epsilon<1$ as the inverse of the sound speed. The method here also works for more general state equations, but here we choose the linear one (1.2) to simplify our arguments.

We are considering the free boundary problem (FBP) in the domain

$$\Omega_t = \Big\{ x = \big(x^1, x^2, x^3\big) = \big(\overline{x}, x^3\big) : \overline{x} \in \mathbb{T}^2, -1 < x^3 < f(t,\overline{x}) \Big\}$$

with the free boundary given by a two dimensional surface

$$\Gamma_t = \Big\{ \big(\overline{x}, x^3\big) : \overline{x} \in \mathbb{T}^2, x^3 = f(t,\overline{x}) \Big\}.$$

On the free surface, there holds that

$$\begin{cases} u\cdot n = V(t,\overline{x}), \\ p = 0, \end{cases} \tag{1.3}$$

where V is the normal velocity of Γ_t and n is the unit outer normal direction of Γ_t. Since the free surface separates the fluid and the vacuum, we have the evolution equation of the free surface

$$\partial_t f = u\cdot N \tag{1.4}$$

with $N=(-\partial_1 f, -\partial_2 f, 1)^\top$ and $n=\frac{N}{|N|}$. On the bottom of the domain $\Gamma_- = \{(\overline{x}, -1) : \overline{x}\in\mathbb{T}^2\}$, we pose the slip boundary condition

$$u^3 = 0 \quad \text{on } \Gamma_-. \tag{1.5}$$

The initial data in

$$\Omega_0 = \Big\{ \big(\overline{x}, x^3\big) : \overline{x}\in\mathbb{T}^2, -1 < x^3 < f_{\text{in}} \Big\}$$

are given by

$$\rho(0,x) = \rho_{\text{in}}(x), \quad u(0,x) = u_{\text{in}}(x). \tag{1.6}$$

The first goal of this paper is to prove the well-posedness of the problem under the Taylor sign condition

$$-\frac{\partial p}{\partial n} \geq c_0 > 0 \tag{1.7}$$

for a constant $c_0 > 0$. A key ingredient is that for the derivatives $\partial_i f$ $(i=1,2)$ of the graph function, we can derive an evolution equation which is of the form of water wave equations

$$D_t^2 \partial_i f - \partial_n p \cdot \mathcal{G}(\partial_i f) = \cdots \tag{1.8}$$

with $\mathcal{G}$ as the Dirichlet-Neumann operator and $D_t = \partial_t + u \cdot \nabla$ as the material derivative. Therefore, with all the estimates of the free surface, we can just work in the Eulerian coordinates.

As the second goal, we shall study the incompressible limit as $\epsilon \to 0$. In this direction, we need to make sure that all the a priori estimates are independent of ϵ. This is rather nontrivial for the free boundary problem. The pressure term $\partial_n p$ in the free surface equation (1.8) could induce singularities of type $\frac{1}{\epsilon}$ due to the singular state equation (1.2). Here we shall consider the perturbations of the form

$$\rho = 1 + \mathcal{O}(\epsilon^2), \quad p = \mathcal{O}(1), \quad u = \mathcal{O}(1).$$

This will erase the potential singularity in the free surface equation. However, the resulting hyperbolic system for $(\rho - 1, u) = (\epsilon^2 p, u)$ is no longer symmetric. Formally, we have

$$\begin{cases} \epsilon^2 D_t p + \nabla \cdot u = \cdots, \\ D_t u + \nabla p = \cdots. \end{cases}$$

This prevents us from getting uniform-in-ϵ estimates in the interior of the moving domain. To cope with this, we have to treat shear waves of the vorticity and pressure waves due to compressibility separately. The shear waves $\omega = \operatorname{curl} u$ satisfy the transport equation

$$D_t \omega = \cdots,$$

and estimates are easier since there is no boundary term in the energy estimates. For the pressure waves, the pressure p satisfies a wave type equation

$$\epsilon^2 D_t^2 p - \Delta p = \operatorname{tr}(\nabla u)^2 + \cdots. \tag{1.9}$$

Setting the scaled material derivative $\widehat{D}_t = \epsilon D_t$, standard energy estimates show that

$$D_t \frac{|\widehat{D}_t p|^2 + |\nabla p|^2}{2} = \big(\widehat{D}_t p\big) \cdot \frac{1}{\epsilon} \operatorname{tr}(\nabla u)^2 + \cdots.$$

There is a singularity of $\frac{1}{\epsilon}$ and the source term $\mathrm{tr}(\nabla u)^2 = \mathcal{O}(1)$. The key observation is that $D_t u = -\nabla p$ from the momentum equation. That is, the term $D_t u$ is actually a pressure term. By taking two more $\widehat{D}_t$ to the pressure wave equation (1.9), we have

$$D_t \frac{|\widehat{D}_t^3 p|^2 + |\widehat{D}_t^2 \nabla p|^2}{2} = \widehat{D}_t^3 p \cdot \frac{1}{\epsilon} \widehat{D}_t^2 \nabla u \cdot \nabla u + \cdots = \widehat{D}_t^3 p \cdot (\widehat{D}_t \nabla^2 p) \cdot \nabla u + \cdots.$$

The singularity $\frac{1}{\epsilon}$ can be absorbed in the higher order energy estimates! As for lower order energy, we shall use the elliptic estimates to avoid this singularity.

With the a priori estimates independent of ϵ, we can construct the solution in the moving domain and prove that the solution to the FBP (1.1)-(1.6) will converge to the solution to the FBP of the following incompressible system:

$$\begin{cases} \partial_t \widetilde{u} + \widetilde{u} \cdot \nabla \widetilde{u} + \nabla \widetilde{p} = 0 & \text{in } \widetilde{\Omega}_t, \\ \nabla \cdot \widetilde{u} = 0 & \text{in } \widetilde{\Omega}_t, \\ \widetilde{p} = 0 & \text{in } \widetilde{\Gamma}_t, \\ \partial_t \widetilde{f} = \widetilde{u} \cdot \widetilde{N} & \text{in } \widetilde{\Gamma}_t, \\ \widetilde{u}^3 = 0 & \text{on } \Gamma_-, \\ \widetilde{u}(0,x) = \widetilde{u}_{\mathrm{in}}(x) & \text{in } \widetilde{\Omega}_0, \\ \widetilde{f}(0,\overline{x}) = \widetilde{f}_{\mathrm{in}}(\overline{x}) & \text{on } \mathbb{T}^2 \end{cases} \tag{1.10}$$

in the moving domain

$$\widetilde{\Omega}_t = \Big\{ (\overline{x}, x^3) : \overline{x} \in \mathbb{T}^2, -1 < x^3 < \widetilde{f}(t, \overline{x}) \Big\}.$$

1.2 History of related works

The free boundary problems (FBP) have been studied intensively through history. For the water wave problems, since the breakthrough works [30, 31] by Wu, the local well-posedness of water wave problems in different scenarios have been considered: problem with vorticity in [6, 22, 34], effects of surface tension in [4], non-trivial bottom topography in [16]. Many other situations were considered in [3, 9, 27] etc. See also the low regularity results in [1, 2]. There are also results on the global well-posedness, see [13, 17] for a more detailed review.

When taking the compressibility of the fluid into consideration, the FBP is usually investigated in the Lagrangian coordinates where the evolving domain can be fixed to the initial domain. Lindblad proved the local well-posedness for compressible liquids in [20, 21] using the Nash-Moser iteration. In [7], Coutand *et al.* used a specially chosen parabolic regularization to derive the existence and proved local well-posedness and the zero surface tension limit. The incompressible limit was achieved by Lindblad and Luo in [23,25]. See also [7,12] for the discussion of the case with surface tension and [32,33] for other fluid systems. When treating the FBP of compressible gases, more results are involved (cf. [8,10,11,14,15,24]).

For the FBP of the compressible fluid, to the authors' knowledge, this is the first paper to derive the evolution equation of the free surface and relate the stability condition of the problem to the hyperbolicity of the evolution equation. Another advantage of our approach is that the FBP can be decoupled: after we have investigated the evolution of the free surface and got the full regularity estimates of the free surface, the evolution inside the domain can be treated as a problem in a fixed domain. This is a simplification compared with the Lagrangian coordinates methods. Furthermore, our approach provides a unified framework to treat FBP's in both incompressible and compressible fluids. We believe that our framework could also be applied to the long-time incompressible limit, which is our future goal. It is also possible to apply this framework to many other relevant problems, such as the flows with capillary effects and two-phase flows. In [18,19,28,29], the authors used similar approaches to treat the free boundary problems in incompressible elastic and MHD systems. In the compressible case, the estimates of the pressure are more subtle, since it satisfies a wave-type equation. For this, we shall use some good derivatives $D_t\overline{\partial}_i f$ of the free surface and wave-type estimates of the pressure p to deal with the loss of regularity encountered in the FBP with vorticity and compressibility.

The rest of the paper is organized as follows. After laying out some preliminaries in Section 2, we shall derive the evolution equation of the free surface and state the main results in Section 3. The estimates of the pressure waves, the free surface and the shear waves are in Sections 4-6 respectively. The solution will be constructed in Section 7 and the incompressible limit will be proved in Section 8. In the appendix, some analytic tools are listed in Appendix A and some results on the div-curl system are presented in Appendix B.

2 Preliminaries

2.1 Harmonic coordinates

Given a smooth function $f_* = f_*(\overline{x}) > -1$, we define a reference domain

$$\Omega_* = \left\{ \left(\overline{x}, x^3\right) : \overline{x} \in \mathbb{T}^2, -1 < x^3 < f_*(\overline{x}) \right\}$$

with the upper surface

$$\Gamma_* = \Big\{ \big(\overline{x}, f_*(\overline{x})\big) : \overline{x} \in \mathbb{T}^2 \Big\}.$$

We shall consider the FBP that lies in a neighborhood of the reference domain Ω_*. To this end, we define

$$\mathcal{N}(\delta,\kappa) = \Big\{ f \in H^\kappa(\mathbb{T}^2) : \|f - f_*\|_{H^\kappa(\mathbb{T}^2)} \le \delta \Big\}.$$

For a function $f \in \mathcal{N}(\delta,\kappa)$, set

$$\Omega_f = \Big\{ x : \overline{x} \in \mathbb{T}^2, -1 < x^3 < f(\overline{x}) \Big\}, \quad \Gamma_f = \Big\{ x : \overline{x} \in \mathbb{T}^2, x^3 = f(\overline{x}) \Big\}.$$

Then we can introduce the harmonic coordinates. Define $\Phi_f : \Omega_* \to \Omega_f$ as a harmonic map

$$\begin{cases} \Delta \Phi_f = 0 & \text{in } \Omega_*, \\ \Phi_f\big(\overline{x}, f_*(\overline{x})\big) = \big(\overline{x}, f(\overline{x})\big) & \text{on } \Gamma_*, \\ \Phi_f(\overline{x}, -1) = (\overline{x}, -1) & \text{on } \Gamma_-. \end{cases} \tag{2.1}$$

Given f_*, there exists $\delta_0 > 0$ such that Φ_f is a bijection when $\delta \le \delta_0$. Thus we can also define the inverse map $\Phi_f^{-1} : \Omega_f \to \Omega_*$ such that

$$\Phi_f \circ \Phi_f^{-1} = \mathrm{Id}_{\Omega_f}, \quad \Phi_f^{-1} \circ \Phi_f = \mathrm{Id}_{\Omega_{f_*}}.$$

Let us list some basic inequalities about harmonic coordinates without proof.

Lemma 2.1. *Assume that $f \in \mathcal{N}(\delta_0,\kappa)$ with $\kappa \ge 4$. Then there exists a constant $C = C(\delta_0, \|f_*\|_{H^\kappa(\mathbb{T}^2)})$ such that*

1. *If $u \in H^s(\Omega_f)$ with $s \in [0,\kappa]$, then*

$$\|u \circ \Phi_f\|_{H^s(\Omega_*)} \le C \|u\|_{H^s(\Omega_f)}.$$

2. *If $u \in H^s(\Omega_*)$ with $s \in [0,\kappa]$, then*

$$\big\|u \circ \Phi_f^{-1}\big\|_{H^s(\Omega_f)} \le C \|u\|_{H^s(\Omega_*)}.$$

3. *If $u, v \in H^s(\Omega_f)$ with $s \in [2,\kappa]$, then*

$$\|uv\|_{H^s(\Omega_f)} \le C \|u\|_{H^s(\Omega_f)} \|v\|_{H^s(\Omega_f)}.$$

2.2 The Dirichlet-Neumann operator

For a smooth enough function $g=g(\bar{x})$ on $\Gamma_f=\{(\bar{x},f(\bar{x})):\bar{x}\in\mathbb{T}^2\}$, denote the harmonic extension of g to Ω_f by $\mathcal{H}_f g$, that is,

$$\begin{cases}\Delta\mathcal{H}_f g=0 & \text{in } \Omega_f,\\ (\mathcal{H}_f g)\big(\bar{x},f(\bar{x})\big)=g(\bar{x}) & \text{on } \Gamma_f,\\ (\mathcal{H}_f g)(\bar{x},-1)=0 & \text{on } \Gamma_-.\end{cases} \tag{2.2}$$

Here we use the Dirichlet boundary condition on the bottom Γ_- instead of the Neumann boundary condition as in the usual case. This modification is useful in the energy estimates in the following sections.

With the harmonic extension (2.2) at hand, the harmonic map Φ_f in (2.1) can be expressed as

$$\Phi_f(x)=x+\big(0,0,\mathcal{H}_{f_*}(f(\bar{x})-f_*(\bar{x})\big).$$

When the harmonic extension $\mathcal{H}_f$ in (2.2) is defined with the Dirichlet boundary condition on the bottom, the definition of the Dirichlet-Neumann (DN) operator should be modified as well. For a smooth enough function $g=g(\bar{x})$ on $\Gamma_f=\{(\bar{x},f(\bar{x})):\bar{x}\in\mathbb{T}^2\}$, define

$$\mathcal{G}_f g=(\nabla_N\mathcal{H}_f g)\big|_{\Gamma_f}, \tag{2.3}$$

where $N=(-\partial_1 f,-\partial_2 f,1)^\top$ is the scaled normal vector on the surface Γ_f. However, all the regularity properties of the Dirichlet-Neumann operator will be kept in spite of the modification, as discussed in Appendix A. The same arguments in [17] yield the following basic properties of the DN operator.

Lemma 2.2. *Let $f\in\mathcal{N}(\delta_0,\kappa)$ with $\kappa\geq 4$. Then there is a constant $C=C(\delta_0,\|f_*\|_{H^\kappa(\mathbb{T}^2)})$ such that*

1. *$\mathcal{G}_f$ is self-adjoint*

$$(\mathcal{G}_f\phi,\psi)=(\phi,\mathcal{G}_f\psi),\quad \forall\phi,\psi\in H^{\frac{1}{2}}(\mathbb{T}^2).$$

2. *$\mathcal{G}_f$ is positive*

$$(\mathcal{G}_f\phi,\phi)\geq C\|\phi\|_{\dot{H}^{\frac{1}{2}}(\mathbb{T}^2)}.$$

Further quantitative discussion on the DN operator is included in Appendix A.

2.3 Notations

In the domain Ω_t, we use $(\partial_t,\partial_1,\partial_2,\partial_3)$ as Eulerian derivatives and $D_t=\partial_t+u\cdot\nabla$ as the material derivative. We shall also use the following tangential derivatives:

$$\begin{aligned}\overline{\partial}_t&=\partial_t+\mathcal{H}_f(\partial_t f)\partial_3,\\ \overline{\partial}_j&=\partial_j+\mathcal{H}_f(\partial_j f)\partial_3,\quad j=1,2.\end{aligned}\tag{2.4}$$

It is direct to verify that $D_t=\overline{\partial}_t+u^1\overline{\partial}_1+u^2\overline{\partial}_2$ on the free surface by (1.4). The derivatives $\overline{\partial}=(\overline{\partial}_1,\overline{\partial}_2)$ are tangent to both Γ_t and Γ_-. We denote by $\Lambda=\langle\nabla\rangle=(1-\Delta)^{1/2}$ and $\Upsilon=\langle\overline{\partial}\rangle=(1+|\overline{\partial}|^2)^{1/2}$ for high order derivatives. The scaled material derivative $\widehat{D}_t=\epsilon D_t$ is also used to take care of the sound speed $\frac{1}{\epsilon}$.

The minuscule indices i,j,k are in $\{1,2\}$ and the capital indices J,K are in $\{1,2,3\}$. We shall use the Einstein summation convention, i.e. a repeated index in a term means summation of terms over the index. To simplify the arguments, we also omit all the binomial coefficients

$$\binom{l}{m}=\frac{l!}{m!(l-m)!}.$$

The brackets $\lfloor s\rfloor$ denotes the largest integer less or equal to $s\in\mathbb{R}$.

In the following, to simplify the notations, we shall use $\mathcal{H}=\mathcal{H}_f$ and $\mathcal{G}=\mathcal{G}_f$ if there is no confusion of the function f. Since many elliptic estimates are dependent on the regularity of the free surface f, we shall use the notation $L=\|f\|_{H^{\kappa-1/2}(\Gamma_t)}$. The function $C(L)$ means some continuous function depending only on L.

3 Reformulation and the main results

Given the state equation (1.2), one can define the enthalpy $h=h(\rho)$ as

$$h=h(\rho):=\int_1^\rho\frac{p'(\lambda)}{\lambda}\mathrm{d}\lambda=\frac{1}{\epsilon^2}\log\rho,\tag{3.1}$$

which is a strictly increasing function of $\rho>0$ such that

$$h\big|_{\Gamma_t}=0,\quad \nabla h=\frac{\nabla p}{\rho}.$$

The inverse function $\rho=\rho(h)=e^{\epsilon^2 h}$ is well-defined. Therefore, we can rewrite the system (1.1) as a first order system of (u,h)

$$\begin{cases} \epsilon^2 D_t h+\nabla\cdot u=0, \\ D_t u+\nabla h=0, \\ h|_{\Gamma_t}=0,\ \ u^3|_{\Gamma_-}=0. \end{cases} \tag{3.2}$$

3.1 Evolution of the enthalpy

For the pressure parts of the system, we have a second order wave equation for the enthalpy h from (3.2) as

$$\begin{cases} \epsilon^2 D_t^2 h-\Delta h=\mathrm{tr}\,(\nabla u)^2, \\ h|_{\Gamma_t}=0,\ \ \partial_3 h|_{\Gamma_-}=0, \end{cases} \tag{3.3}$$

where we have used the boundary condition $u^3|_{\Gamma_-}=0$.

3.2 Evolution of the vorticity

For the shear parts of the system, we define the vorticity of the flow $\omega=\mathrm{curl}\, u=\nabla\times u$. Then the vorticity ω satisfies the following transport equation:

$$D_t\omega=\omega\cdot\nabla u-\omega(\nabla\cdot u). \tag{3.4}$$

3.3 Evolution of the surface

Given the evolution equation of the free surface (1.4) and the definition of N, we can rewrite the evolution of Γ_t as

$$D_t f=u^3. \tag{3.5}$$

Taking another material derivative and using (3.2) yields

$$D_t^2 f=D_t u^3=-\partial_3 h. \tag{3.6}$$

A direct computation shows that

$$\begin{aligned} D_t^2\overline{\partial}_i f &=\overline{\partial}_i D_t^2 f+\left[D_t,\overline{\partial}_i\right]D_t f+D_t\left[D_t,\overline{\partial}_i\right]f \\ &=\overline{\partial}_i D_t^2 f-\overline{\partial}_i u^j\overline{\partial}_j D_t f-D_t\left(\overline{\partial}_i u^j\overline{\partial}_j f\right) \\ &=\overline{\partial}_i D_t^2 f-\overline{\partial}_i D_t u^j\overline{\partial}_j f-2\overline{\partial}_i u^j D_t\overline{\partial}_j f \\ &=\overline{\partial}_i D_t^2 f-\overline{\partial}_i D_t u^j\overline{\partial}_j f+Q_i \end{aligned} \tag{3.7}$$

with

$$Q_i := -2\overline{\partial}_i u^j D_t \overline{\partial}_j f. \tag{3.8}$$

By (3.6), the first two terms on the last line of (3.7) are

$$\begin{aligned}
&\overline{\partial}_i D_t^2 f - \overline{\partial}_i D_t u^j \overline{\partial}_j f = -\overline{\partial}_i \partial_3 h + \overline{\partial}_i \partial_j h \overline{\partial}_j f \\
&= -\partial_i \partial_3 h - \partial_i f \partial_3^2 h + \partial_i \partial_j h \partial_j f + \partial_i f \partial_3 \partial_j h \partial_j f \\
&= -\nabla_N \partial_i h + \partial_i f \nabla_N \partial_3 h \\
&= -\nabla_N q_i + \partial_3 h \nabla_N \mathcal{H}(\partial_i f),
\end{aligned} \tag{3.9}$$

where

$$q_i := \partial_i h + \mathcal{H}(\partial_i f)\partial_3 h, \quad i = 1,2 \tag{3.10}$$

with the harmonic extension $\mathcal{H}$ given in (2.2). Using the boundary conditions in (3.3), we can derive an elliptic equation of q_i as follows:

$$\begin{cases}
\Delta q_i = \partial_i \Delta h + \mathcal{H}(\partial_i f)\partial_3 \Delta h + 2\nabla \mathcal{H}(\partial_i f) \cdot \nabla \partial_3 h & \text{in } \Omega_t, \\
q_i = 0 & \text{on } \Gamma_t, \\
\partial_3 q_i = 0 & \text{on } \Gamma_-.
\end{cases} \tag{3.11}$$

The resulting elliptic system has homogeneous boundary conditions. This is where we need the modified harmonic extension.

Substituting (3.9) back into (3.7) gives us the evolution equation of the free surface

$$D_t^2 \overline{\partial}_i f - \partial_3 h \mathcal{G}(\partial_i f) = -\nabla_N q_i + Q_i, \tag{3.12}$$

where q_i, Q_i are given in (3.10) and (3.8). The Dirichlet-to-Neumann (DN) operator $\mathcal{G}$ is defined in (2.3) as $\mathcal{G}g = (\nabla_N \mathcal{H} g)|_{\Gamma_t}$.

Remark 3.1. When considering the evolution surface f, we shall use $\partial_3 h$ in (3.12) instead of $\partial_n h$ in the Taylor sign condition. The relation between them is given by

$$\partial_3 h = \frac{1}{1+|\nabla f|^2}\left(\nabla_N h + \partial_i f \overline{\partial}_i h\right) = \frac{1}{\sqrt{1+|\nabla f|^2}} \partial_n h,$$

where the conditions $h=0$ and $\overline{\partial}_i h = 0$ on Γ_f are used.

3.4 Main result

To study the free boundary problem, we need the compatibility condition on the free surface Γ_t

$$D_t^l h\big|_{\Gamma_t} = 0, \quad l = 0, \ldots, \kappa+1 \tag{3.13}$$

with $D_t = \partial_t + u \cdot \nabla$ as the material derivative.

The energy of the system is given by

$$\begin{aligned}\mathcal{E}_\kappa(t) = &\|D_t f\|_{H^{\kappa-\frac{1}{2}}(\Gamma_t)} + \|f\|_{H^\kappa(\Gamma_t)} + \|u\|_{H^\kappa(\Omega_t)} \\ &+ \sum_{l=2}^{\kappa} \frac{1}{\epsilon} \left\| \left(\widehat{D}_t^{l+1} h, \widehat{D}_t^l \nabla h \right) \right\|_{L^2(\Omega_t)}. \end{aligned} \tag{3.14}$$

For some $0 < T < 1$, denote by

$$M = \sup_{0<t<T} \mathcal{E}_\kappa(t),$$

and initially

$$M_0 = \mathcal{E}_\kappa(0).$$

Now we can state the existence result.

Theorem 3.1. *Let $\kappa \geq 4$ be an integer. Assume that the initial data satisfies the bound $\mathcal{E}_\kappa(0) = M_0 < \infty$. Furthermore, assume that there exists $c_0 > 0$ such that*

1. $f_{in} \geq -1 + c_0$;
2. $-\frac{N_{f_{in}}}{|N_{f_{in}}|} \cdot \nabla p \geq c_0$.

Then there exists $\epsilon_0 > 0$ and $T > 0$ such that, when $0 < \epsilon < \epsilon_0$, the system (1.1)-(1.6) *admits a unique solution (f,u,h) in $[0,T]$ under the compatibility condition* (3.13) *satisfying*

1. $M \leq C(c_0, M_0) + (T+\epsilon) C(c_0, M)$;
2. $f \geq -1 + \dfrac{c_0}{2}$;
3. $-\dfrac{N_f}{|N_f|} \cdot \nabla p \geq \dfrac{c_0}{2}$.

As for the incompressible limit, we put back the upper index ϵ to indicate the dependence of ϵ.

Theorem 3.2. *Fix a reference domain Ω_*. Under the assumptions of Theorem* 3.1 *for $(f_{in}^\epsilon, u_{in}^\epsilon, h_{in}^\epsilon)$, assume further that*

1. $f_{in}^\epsilon \longrightarrow \widetilde{f}_{in}$ *in* $H^\kappa(\mathbb{T}^2)$;
2. $u_{in}^\epsilon \circ \Phi_{f_{in}^\epsilon} \longrightarrow \widetilde{u}_{in} \circ \Phi_{\widetilde{f}_{in}}$ *in* $H^\kappa(\Omega_*)$.

Then $(f^\epsilon, u^\epsilon\circ\Phi_{f^\epsilon_{in}}, h^\epsilon\circ\Phi_{f^\epsilon_{in}})$ *converges weakly in* $L^\infty(0,T;H^\kappa(\mathbb{T}^2)\times(H^\kappa(\Omega_*))^4)$ *and strongly in* $C^{\kappa-3}([0,T]\times\Omega_*)$ *to a limit* $(\widetilde{f},\widetilde{u}\circ\Phi_{\widetilde{f}},\widetilde{p}\circ\Phi_{\widetilde{f}})$*, where* $(\widetilde{f},\widetilde{u},\widetilde{p})$ *is the unique solution in* $C([0,T];H^\kappa(\mathbb{T}^2)\times(H^\kappa(\Omega_*))^4)$ *of the incompressible system* (1.10) *with the initial data* $(\widetilde{f}_{in},\widetilde{u}_{in})$*.*

In the following lemma, we shall show that there exist a lot of initial data satisfying our assumptions in Theorems 3.1-3.2. Therefore, the limit of the initial data $(\widetilde{f}_{\rm in},\widetilde{u}_{\rm in},\widetilde{h}_{\rm in})$ in Theorem 3.2 can be non-trivial.

Lemma 3.1. *If there holds that*

$$\begin{aligned}&\|f_{in}\|_{H^\kappa(\mathbb{T}^2)}+\|u_{in}\|_{H^{\kappa+1}(\Omega_0)}+\|h_{in}\|_{H^{\kappa+1}(\Omega_0)}\\&\quad+\frac{1}{\epsilon}\big\|\Delta h_{in}+\mathrm{tr}(\nabla u_{in})^2\big\|_{H^{\kappa-1}(\Omega_0)}+\frac{1}{\epsilon^2}\|\nabla\cdot u_{in}\|_{H^\kappa(\Omega_0)}\le\frac{M_0}{C}\end{aligned}\tag{3.15}$$

for some constant C independent on ϵ*, then* $\mathcal{E}_\kappa(0)\le M_0$*.*

Proof. From Eq. (3.5) we have

$$\big\|D_tf(0)\big\|_{H^{\kappa-\frac12}(\mathbb{T}^2)}\le\big\|u^3\big\|_{H^{\kappa-\frac12}(\mathbb{T}^2)}\lesssim M_0.$$

Next we consider the time derivatives of h

$$\frac{1}{\epsilon}\Big(\widehat{D}_t^{l+1}h,\widehat{D}_t^l\nabla h\Big)=\frac{1}{\epsilon^2}\Big(\widehat{D}_t^{l+1}(\epsilon h),-\widehat{D}_t^{l+1}u\Big),\quad 2\le l\le\kappa.$$

Since

$$\begin{cases}\widehat{D}_t(\epsilon h)=-\nabla\cdot u,\\ \widehat{D}_tu=-\epsilon\nabla h,\end{cases}$$

and

$$\begin{cases}\widehat{D}_t^2(\epsilon h)=\epsilon\Delta h+\epsilon\,\mathrm{tr}\,(\nabla u)^2,\\ \widehat{D}_t^2u=\nabla(\nabla\cdot u)+\epsilon^2(\nabla u)^\top\nabla h.\end{cases}$$

One has by (3.15) that

$$\frac{1}{\epsilon}\big\|\big(\widehat{D}_t(\epsilon h),\widehat{D}_tu\big)(0)\big\|_{H^\kappa(\Omega_0)}+\frac{1}{\epsilon^2}\big\|\big(\widehat{D}_t^2(\epsilon h),\widehat{D}_t^2u\big)(0)\big\|_{H^{\kappa-1}(\Omega_0)}\lesssim M_0.$$

By induction, assume that for some $l\ge2$,

$$\sum_{m=2}^{l}\frac{1}{\epsilon^2}\big\|\big(\widehat{D}_t^m(\epsilon h),\widehat{D}_t^mu\big)(0)\big\|_{H^{\kappa+1-m}(\Omega_0)}\lesssim M_0.$$

Setting

$$A(\nabla)=\begin{pmatrix} 0 & -\nabla \\ -\nabla^\top & 0_{3\times 3}\end{pmatrix}$$

and $w=(\epsilon h,u)^\top$, one has $\widehat{D}_t w=A(\nabla)w$ and

$$\begin{aligned}\widehat{D}_t^{l+1}w&=A(\nabla)\widehat{D}_t^l w+\big[\widehat{D}_t^l,A(\nabla)\big]w\\&=A(\nabla)\widehat{D}_t^l w+l\big[\widehat{D}_t,A(\nabla)\big]\widehat{D}_t^{l-1}w\\&\quad+\sum_{m=1}^{l-1}\Big[\widehat{D}_t^m,\big[\widehat{D}_t,A(\nabla)\big]\Big]\widehat{D}_t^{l-1-m}w.\end{aligned}$$

By induction assumption and the fact $\widehat{D}_t=\epsilon D_t$, we can prove the lemma. □

4 Estimates of the pressure parts

Since $D_t u=-\nabla h$, the derivatives $\widehat{D}_t^l u$ with $\widehat{D}_t=\epsilon D_t$ can be viewed as pressure parts. In this section, we shall prove the estimates for the pressure parts $(\widehat{D}_t^l u,\widehat{D}_t^l h)$.

The main result of this section is the following proposition.

Proposition 4.1. *There exists some $\epsilon_0>0$ such that, when $0<\epsilon<\epsilon_0$,*

$$\frac{1}{\epsilon}\|\widehat{D}_t u\|_{H^{\kappa-1}(\Omega_t)}+\sum_{l=2}^{\kappa+1}\frac{1}{\epsilon^2}\big\|\widehat{D}_t^l u\big\|_{H^{\kappa+1-l}(\Omega_t)}\le C(M), \tag{4.1}$$

$$\|h\|_{H^\kappa(\Omega_t)}+\sum_{l=1}^{\kappa+1}\frac{1}{\epsilon}\big\|\widehat{D}_t^l h\big\|_{H^{\kappa+1-l}(\Omega_t)}\le C(M), \tag{4.2}$$

$$\sum_{l=2}^{\kappa}\frac{1}{\epsilon}\big\|\big(\widehat{D}_t^{l+1}h,\widehat{D}_t^l\nabla h\big)\big\|_{L^2(\Omega_t)}\le C(M_0)+TC(M). \tag{4.3}$$

The rest of this section is devoted to the proof of Proposition 4.1. The following lemma implies that we only need to bound $\widehat{D}_t^l h$.

Lemma 4.1. *Assume that $0\le s\le\kappa$, there holds that*

$$\frac{1}{\epsilon}\|\widehat{D}_t u\|_{H^{s-1}(\Omega_t)}\le\|h\|_{H^s(\Omega_t)}, \tag{4.4}$$

and for $2\le l\le s+1$,

$$\frac{1}{\epsilon^2}\big\|\widehat{D}_t^l u\big\|_{H^{s+1-l}}\le\frac{1}{\epsilon}\big\|\widehat{D}_t^{l-1}h\big\|_{H^{s+2-l}(\Omega_t)}+C\left(\sum_{m=0}^{l-2}\big\|\widehat{D}_t^m h\big\|_{H^{s+2-l}(\Omega_t)}\right). \tag{4.5}$$

Proof. The momentum equation $\widehat{D}_t u = -\epsilon\nabla h$ yields (4.4). If $l \geq 2$, then

$$\widehat{D}_t^l u = -\epsilon \widehat{D}_t^{l-1}\nabla h = -\epsilon\nabla\widehat{D}_t^{l-1}h + \epsilon^2\sum_{m=0}^{l-2}\widehat{D}_t^m[\nabla, D_t]\widehat{D}_t^{l-2-m}h.$$

Therefore,

$$\begin{aligned}\frac{1}{\epsilon^2}\|\widehat{D}_t^l u\|_{H^{s+1-l}(\Omega_t)} \leq & \frac{1}{\epsilon}\|\nabla\widehat{D}_t^{l-1}h\|_{H^{s+1-l}(\Omega_t)}\\ & + C\left(\sum_{m=0}^{l-2}\|(\widehat{D}_t^m u, \widehat{D}_t^m h)\|_{H^{s+2-l}(\Omega_t)}\right).\end{aligned}$$

Then a reduction argument proves (4.5). □

To estimate spacial derivatives of $\widehat{D}_t^l h$, we rewrite the wave equation of h in (3.3) as an elliptic system for $\widehat{D}_t^l h$ with $0 \leq l \leq \kappa - 1$ as

$$\begin{cases}\Delta\widehat{D}_t^l h = \widehat{D}_t^{l+2}h - \widehat{D}_t^l \mathrm{tr}(\nabla u)^2 + [\Delta, \widehat{D}_t^l]h & \text{in } \Omega_t,\\ \widehat{D}_t^l h = 0 & \text{on } \Gamma_t,\\ \partial_3\widehat{D}_t^l h = [\partial_3, \widehat{D}_t^l]h & \text{on } \Gamma_-.\end{cases} \tag{4.6}$$

To apply the elliptic theory in the following subsections, we list some computation here

$$\widehat{D}_t^l \mathrm{tr}(\nabla u)^2 = \sum_{m=0}^{\lfloor \frac{l}{2}\rfloor}\widehat{D}_t^m\partial_J u^K\cdot\widehat{D}_t^{l-m}\partial_K u^J, \tag{4.7}$$

$$\begin{aligned}[\Delta, \widehat{D}_t^l]h =& \sum_{m=0}^{l-1}\widehat{D}_t^m[\Delta, \widehat{D}_t]\widehat{D}_t^{l-1-m}h \qquad (4.8)\\ =& \epsilon\sum_{m=0}^{l-1}\widehat{D}_t^m\left\{\Delta u^J\cdot\partial_J\widehat{D}_t^{l-1-m}h + \nabla u^J\cdot\nabla\partial_J\widehat{D}_t^{l-1-m}h\right\}\\ =& \epsilon\sum_{m=0}^{l-1}\sum_{n=0}^{m}\left\{\widehat{D}_t^n\Delta u^J\cdot\widehat{D}_t^{m-n}\partial_J\widehat{D}_t^{l-1-m}h + \widehat{D}_t^n\nabla u^J\cdot\widehat{D}_t^{m-n}\nabla\partial_J\widehat{D}_t^{l-1-m}h,\right\},\end{aligned}$$

$$\begin{aligned}[\partial_3, \widehat{D}_t^l]h =& \sum_{m=0}^{l-1}\widehat{D}_t^m[\partial_3, \widehat{D}_t]\widehat{D}_t^{l-m-1}h = \epsilon\sum_{m=0}^{l-1}\widehat{D}_t^m\left\{\partial_3 u^J\cdot\partial_J\widehat{D}_t^{l-m-1}h\right\}\\ =& \epsilon\sum_{m=0}^{l-1}\sum_{n=0}^{m}\widehat{D}_t^n\partial_3 u^J\cdot\widehat{D}_t^{m-n}\partial_J\widehat{D}_t^{l-m-1}h. \qquad (4.9)\end{aligned}$$

4.1 Estimates of lower order derivatives of h

In this subsection, we shall estimate the lower order derivatives $\|\widehat{D}_t^l h\|_{H^{s+1-l}(\Omega_t)}$ with $0\le l\le s\le 2$. From the compatibility conditions (3.13) $\widehat{D}_t^2 h|_{\Gamma_t}=0$ and

$$\begin{aligned}
&\nabla\widehat{D}_t^2 h=\widehat{D}_t^2\nabla h+\big[\nabla,\widehat{D}_t^2\big]h\\
&=\widehat{D}_t^2\nabla h+2\epsilon\nabla u^J\cdot\partial_J\widehat{D}_t h-\epsilon^2\partial_J h\cdot\nabla\partial_J h-2\epsilon^2\nabla u^J\cdot\partial_J u^K\cdot\partial_K h,
\end{aligned}$$

we have by the Poincaré inequality that

$$\begin{aligned}
\frac{1}{\epsilon}\big\|\widehat{D}_t^2 h\big\|_{H^1(\Omega_t)}&\le\frac{1}{\epsilon}C(L)\big\|\nabla\widehat{D}_t^2 h\big\|_{L^2(\Omega_t)}\\
&\le C(M)+\epsilon C\left(M,\|h\|_{H^3(\Omega_t)},\frac{1}{\epsilon}\big\|\widehat{D}_t h\big\|_{H^2(\Omega_t)}\right).
\end{aligned}\tag{4.10}$$

To estimate $\widehat{D}_t h$, we can use the elliptic system (4.6) with $l=1$ and get that

$$\begin{aligned}
\frac{1}{\epsilon}\big\|\widehat{D}_t h\big\|_{H^2(\Omega_t)}&\le C(L)\left\{\frac{1}{\epsilon}\big\|\widehat{D}_t^3 h\big\|_{L^2(\Omega_t)}+\big\|D_t\mathrm{tr}(\nabla u)^2\big\|_{L^2(\Omega_t)}\right.\\
&\qquad\left.+\big\|[\Delta,D_t]h\big\|_{L^2(\Omega_t)}+\big\|[\partial_3,D_t]h\big\|_{H^1(\Omega_t)}\right\}\\
&\le C(M)+C\big(M,\|h\|_{H^3(\Omega_t)}\big).
\end{aligned}\tag{4.11}$$

Similarly, the elliptic system (4.6) with $l=0$ implies that

$$\begin{aligned}
\|h\|_{H^3(\Omega_t)}&\le C(L)\Big\{\big\|\widehat{D}_t^2 h\big\|_{H^1(\Omega_t)}+\big\|\mathrm{tr}(\nabla u)^2\big\|_{H^1(\Omega_t)}\Big\}\\
&\le C(M)+\epsilon C(M)\cdot\frac{1}{\epsilon}\big\|\widehat{D}_t^2 h\big\|_{H^1(\Omega_t)}.
\end{aligned}\tag{4.12}$$

Then we can take $\epsilon<0$ small and sum over (4.10)-(4.12) to get that

$$\|h\|_{H^3(\Omega_t)}+\frac{1}{\epsilon}\big\|\widehat{D}_t h\big\|_{H^2(\Omega_t)}+\frac{1}{\epsilon}\big\|\widehat{D}_t^2 h\big\|_{H^1(\Omega_t)}\le C(M).\tag{4.13}$$

This proves the lower order estimates of the enthalpy h in (4.2).

4.2 Estimates of higher order derivatives of h

To prove (4.2) for higher order derivatives, we use induction. Assume that, for some $3\le s\le\kappa$, there holds that

$$\|h\|_{H^s(\Omega_t)}+\sum_{l=1}^{s}\frac{1}{\epsilon}\big\|\widehat{D}_t^l h\big\|_{H^{s-l}(\Omega_t)}\le C(M).\tag{4.14}$$

We shall prove the inequality (4.14) with s replaced by $s+1$, that is,

$$\|h\|_{H^{\min\{s+1,\kappa\}}(\Omega_t)}+\sum_{l=1}^{s+1}\frac{1}{\epsilon}\|\widehat{D}_t^l h\|_{H^{s+1-l}(\Omega_t)}\le C(M). \tag{4.15}$$

We remark that one can only get $h\in H^{\kappa+1/2}(\Omega_t)$ due to limited smoothness of the free surface. That is why we modify the index from $s+1$ to $\min\{s+1,\kappa\}$.

When $l=s+1$, we have by the definition of M that

$$\frac{1}{\epsilon}\|\widehat{D}_t^{s+1}h\|_{L^2(\Omega_t)}\le M. \tag{4.16}$$

When $l=s$, we have

$$\begin{aligned}\frac{1}{\epsilon}\|\widehat{D}_t^s h\|_{H^1(\Omega_t)}&\le C(L)\cdot\frac{1}{\epsilon}\|\nabla\widehat{D}_t^s h\|_{L^2(\Omega_t)}\\&\le C(L)\left\{\frac{1}{\epsilon}\|\widehat{D}_t^s\nabla h\|_{L^2(\Omega_t)}+\frac{1}{\epsilon}\|[\nabla,\widehat{D}_t^s]h\|_{L^2(\Omega_t)}\right\}\le C(M),\end{aligned} \tag{4.17}$$

by using the definition of M and the assumption (4.14).

To estimate $\|\widehat{D}_t^l h\|_{H^{s+1-l}(\Omega_t)}$ with $1\le l\le s-1$, we need another induction over l. Assume that we have proved that

$$\sum_{m=l+1}^{s+1}\frac{1}{\epsilon}\|\widehat{D}_t^m h\|_{H^{s+1-m}(\Omega_t)}\le C(M). \tag{4.18}$$

Consider $\widehat{D}_t^l h$ in the elliptic system (4.6),

$$\begin{aligned}\|\widehat{D}_t^l h\|_{H^{s+1-l}(\Omega_t)}\le C(L)\Big\{&\|\widehat{D}_t^{l+2}h\|_{H^{s-1-l}(\Omega_t)}+\|\widehat{D}_t^l\mathrm{tr}(\nabla u)^2\|_{H^{s-1-l}(\Omega_t)}\\&+\|[\Delta,\widehat{D}_t^l]h\|_{H^{s-1-l}(\Omega_t)}+\|[\partial_3,\widehat{D}_t^l]h\|_{H^{s-l}(\Omega_t)}\Big\}.\end{aligned} \tag{4.19}$$

Since $l\ge 1$, we have from the computation (4.7)-(4.9) and (4.14) that

$$\begin{aligned}&\frac{1}{\epsilon}\|\widehat{D}_t^l\mathrm{tr}(\nabla u)^2\|_{H^{s-1-l}(\Omega_t)}\\&\le\sum_{m=0}^{\lfloor\frac{l}{2}\rfloor}\left\|\widehat{D}_t^m\partial_J u^K\cdot\frac{1}{\epsilon}\widehat{D}_t^{l-m}\partial_K u^J\right\|_{H^{s-1-l}(\Omega_t)}\le C(M),\end{aligned} \tag{4.20}$$

$$
\begin{aligned}
&\frac{1}{\epsilon}\big\|\big[\Delta,\widehat{D}_t^l\big]h\big\|_{H^{s-1-l}(\Omega_t)}\\
\leq &\sum_{m=0}^{l-1}\sum_{n=0}^{m}\Big\{\big\|\widehat{D}_t^n\Delta u^J\cdot\widehat{D}_t^{m-n}\partial_J\widehat{D}_t^{l-1-m}h\big\|_{H^{s-1-l}(\Omega_t)}\\
&\qquad\qquad+\big\|\widehat{D}_t^n\nabla u^J\cdot\widehat{D}_t^{m-n}\nabla\partial_J\widehat{D}_t^{l-1-m}h\big\|_{H^{s-1-l}(\Omega_t)}\Big\}\leq C(M), \qquad (4.21)\\
&\frac{1}{\epsilon}\big\|\big[\partial_3,\widehat{D}_t^l\big]h\big\|_{H^{s-l}(\Omega_t)}\\
\leq &\sum_{m=0}^{l-1}\sum_{n=0}^{m}\big\|\widehat{D}_t^n\partial_3 u^J\cdot\widehat{D}_t^{m-n}\partial_J\widehat{D}_t^{l-m-1}h\big\|_{H^{s-l}(\Omega_t)}\leq C(M). \qquad (4.22)
\end{aligned}
$$

Taking all the estimates (4.20)-(4.22) back to (4.19), one has from (4.18) that

$$
\frac{1}{\epsilon}\big\|\widehat{D}_t^l h\big\|_{H^{s+1-l}(\Omega_t)}\leq C(M).
$$

By induction in l, we get

$$
\sum_{l=1}^{s+1}\frac{1}{\epsilon}\big\|\widehat{D}_t^l h\big\|_{H^{s+1-l}(\Omega_t)}\leq C(M). \tag{4.23}
$$

Lastly, for $\|h\|_{H^{s+1}(\Omega_t)}$ with $s\leq\kappa-1$, we can derive from the elliptic system (4.6) with $l=0$ that

$$
\begin{aligned}
\|h\|_{H^{s+1}(\Omega_t)}&\leq C(L)\big\|\Delta h\big\|_{H^{s-1}(\Omega_t)}\\
&\leq C(L)\Big\{\big\|\widehat{D}_t^2 h\big\|_{H^{s-1}(\Omega_t)}+\big\|\mathrm{tr}(\nabla u)^2\big\|_{H^{s-1}(\Omega_t)}\Big\}\leq C(M). \qquad (4.24)
\end{aligned}
$$

Together (4.23) and (4.24) yield (4.15). This finishes the induction procedure and proves (4.2).

4.3 Energy estimates of the enthalpy h

In this subsection, we shall derive the energy estimates for $(\widehat{D}_t^{l+1}h,\widehat{D}_t^l\nabla h)$ in (4.3). From the wave equation (3.3), one has for $2\leq l\leq\kappa$ that,

$$
\widehat{D}_t^{l+2}h-\nabla\cdot\widehat{D}_t^l\nabla h=\widehat{D}_t^l\mathrm{tr}(\nabla u)^2-\big[\nabla\cdot,\widehat{D}_t^l\big]\nabla h. \tag{4.25}
$$

Noticing that this conservation formulation is slightly different from the one used in the elliptic estimates (4.6). As the result,

$$
D_t\frac{|\widehat{D}_t^{l+1}h|^2+|\widehat{D}_t^l\nabla h|^2}{2}-\nabla\cdot(\widehat{D}_t^l\nabla h\cdot D_t\widehat{D}_t^l h)
$$

$$=\frac{1}{\epsilon}\widehat{D}_t^{l+1}h\cdot\widehat{D}_t^l\mathrm{tr}(\nabla u)^2-\frac{1}{\epsilon}\widehat{D}_t^{l+1}h\cdot\left[\nabla\cdot,\widehat{D}_t^l\right]\nabla h-\frac{1}{\epsilon}\widehat{D}_t^l\nabla h\cdot\left[\nabla,\widehat{D}_t^{l+1}\right]h.$$

Integrating over Ω and using the compatibility condition (3.13) and $\widehat{D}_t^l\partial_3 h|_{\Gamma_-}=-\widehat{D}_t^l D_t u^3|_{\Gamma_-}=0$, we have the energy identity

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\frac{|\widehat{D}_t^{l+1}h|^2+|\widehat{D}_t^l\nabla h|^2}{2\epsilon^2}\mathrm{d}x\\
=&\int_{\Omega_t}D_t\frac{|\widehat{D}_t^{l+1}h|^2+|\widehat{D}_t^l\nabla h|^2}{2\epsilon^2}\mathrm{d}x+\int_{\Omega_t}(\nabla\cdot u)\frac{|\widehat{D}_t^{l+1}h|^2+|\widehat{D}_t^l\nabla h|^2}{2\epsilon^2}\mathrm{d}x\\
=&\int_{\Omega_t}\frac{1}{\epsilon^3}\widehat{D}_t^{l+1}h\cdot\widehat{D}_t^l\mathrm{tr}(\nabla u)^2\mathrm{d}x-\int_{\Omega_t}\frac{1}{\epsilon^3}\widehat{D}_t^{l+1}h\cdot\left[\nabla\cdot,\widehat{D}_t^l\right]\nabla h\mathrm{d}x\\
&\quad-\int_{\Omega_t}\frac{1}{\epsilon^3}\widehat{D}_t^l\nabla h\cdot\left[\nabla,\widehat{D}_t^{l+1}\right]h\mathrm{d}x+\int_{\Omega_t}(\nabla\cdot u)\frac{|\widehat{D}_t^{l+1}h|^2+|\widehat{D}_t^l\nabla h|^2}{2\epsilon^2}\mathrm{d}x\\
=:&I_1+I_2+I_3+I_4.
\end{aligned}\tag{4.26}$$

For I_1 in (4.26), since $l\geq 2$, we can use (4.7) to get that

$$\begin{aligned}
I_1\leq&\frac{1}{\epsilon}\left\|\widehat{D}_t^{l+1}h\right\|_{L^2(\Omega_t)}\cdot\left\|\partial_J u^K\cdot\frac{1}{\epsilon^2}\widehat{D}_t^l\partial_K u^J\right\|_{L^2(\Omega_t)}\\
&+\sum_{m=1}^{\lfloor\frac{l}{2}\rfloor}\frac{1}{\epsilon}\left\|\widehat{D}_t^{l+1}h\right\|_{L^2(\Omega_t)}\cdot\left\|\frac{1}{\epsilon}\widehat{D}_t^m\partial_J u^K\cdot\frac{1}{\epsilon}\widehat{D}_t^{l-m}\partial_K u^J\right\|_{L^2(\Omega_t)}\leq C(M).
\end{aligned}\tag{4.27}$$

Similar arguments can be applied to I_2 and I_3 in (4.26) to get that

$$\begin{aligned}
I_2=&-\int_{\Omega_t}\frac{1}{\epsilon}\widehat{D}_t^{l+1}h\cdot\frac{1}{\epsilon}\sum_{m=0}^{l-1}\sum_{n=0}^{m}\widehat{D}_t^n\partial_J u^K\cdot\widehat{D}_t^{m-n}\partial_K\widehat{D}_t^{l-1-m}\partial_J h\mathrm{d}x\qquad(4.28)\\
\leq&\frac{1}{\epsilon}\left\|\widehat{D}_t^{l+1}h\right\|_{L^2(\Omega_t)}\cdot\sum_{m=0}^{l-1}\left\|\partial_J u^K\cdot\frac{1}{\epsilon}\widehat{D}_t^m\partial_K\widehat{D}_t^{l-1-m}\partial_J h\right\|_{L^2(\Omega_t)}\\
&+\frac{1}{\epsilon}\left\|\widehat{D}_t^{l+1}h\right\|_{L^2(\Omega_t)}\cdot\sum_{m=0}^{l-1}\sum_{n=1}^{m}\left\|\frac{1}{\epsilon}\widehat{D}_t^n\partial_J u^K\cdot\widehat{D}_t^{m-n}\partial_K\widehat{D}_t^{l-1-m}\partial_J h\right\|_{L^2(\Omega_t)}\leq C(M),
\end{aligned}$$

and

$$I_3=-\int_{\Omega_t}\frac{1}{\epsilon}\widehat{D}_t^l\nabla h\cdot\frac{1}{\epsilon}\sum_{m=0}^{l}\sum_{n=0}^{m}\widehat{D}_t^n\nabla u^J\cdot\widehat{D}_t^{m-n}\partial_J\widehat{D}_t^{l-m}h\mathrm{d}x$$

$$
\begin{aligned}
&\leq \frac{1}{\epsilon}\|\widehat{D}_t^l\nabla h\|_{L^2(\Omega_t)}\cdot\sum_{m=0}^{l}\left\|\nabla u^J\cdot\frac{1}{\epsilon}\widehat{D}_t^m\partial_J\widehat{D}_t^{l-m}h\right\|_{L^2(\Omega_t)}\\
&\quad+\frac{1}{\epsilon}\|\widehat{D}_t^l\nabla h\|_{L^2(\Omega_t)}\cdot\sum_{m=0}^{l}\sum_{n=1}^{m}\left\|\frac{1}{\epsilon}\widehat{D}_t^n\nabla u^J\cdot\widehat{D}_t^{m-n}\partial_J\widehat{D}_t^{l-m}h\right\|_{L^2(\Omega_t)}\leq C(M). \qquad (4.29)
\end{aligned}
$$

For I_4 in (4.26), it is direct to obtain that

$$
I_4\leq M\|\nabla\cdot u\|_{L^\infty(\Omega_t)}\leq C(M). \tag{4.30}
$$

With the estimates (4.27)-(4.30), we have from (4.26) that

$$
\frac{1}{\epsilon}\big\|\big(\widehat{D}_t^{l+1}h,\widehat{D}_t^l\nabla h\big)\big\|_{L^2(\Omega_t)}\leq C(M_0)+TC(M).
$$

This proves (4.3).

5 Estimates of the free surface

In this section, we shall prove the following estimates of the free surface.

Proposition 5.1. *Under the Taylor sign condition* (1.7)*, there holds that*

$$
\|D_tf\|_{H^{\kappa-\frac{1}{2}}(\mathbb{T}^2)}+\|f\|_{H^\kappa(\mathbb{T}^2)}\leq C(c_0,M_0)+TC(c_0,M). \tag{5.1}
$$

Recall the evolution equation of the free surface in (3.12),

$$
D_t^2\overline{\partial}_if-\partial_3h\mathcal{G}\overline{\partial}_if=-\nabla_Nq_i+Q_i. \tag{5.2}
$$

For $0\leq l\leq\kappa-1$, set

$$
F=\Upsilon^{\kappa-\frac{3}{2}}\overline{\partial}_if=\langle\overline{\partial}\rangle^{\kappa-\frac{3}{2}}\overline{\partial}_if.
$$

Taking $\Upsilon^{\kappa-3/2}$ to both sides of (5.2), we have an equation for F,

$$
\begin{aligned}
D_t^2F-\partial_3h\mathcal{G}F=&-\big[\Upsilon^{\kappa-\frac{3}{2}},D_t^2\big]\overline{\partial}_if+\big[\Upsilon^{\kappa-\frac{3}{2}},\partial_3h\mathcal{G}\big]\overline{\partial}_if\\
&-\Upsilon^{\kappa-\frac{3}{2}}\nabla_Nq_i+\Upsilon^{\kappa-\frac{3}{2}}Q_i. \qquad (5.3)
\end{aligned}
$$

A direct computation shows that

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{T}^2}\Big\{|D_tF|^2-\partial_3h\big|\mathcal{G}^{\frac{1}{2}}F\big|^2\Big\}\mathrm{d}\overline{x}
$$

$$
\begin{aligned}
&=\frac{1}{2}\int_{\mathbb{T}^2} D_t\Big\{|D_tF|^2-\partial_3 h\big|\mathcal{G}^{\frac{1}{2}}F\big|^2\Big\}\mathrm{d}\overline{x}\\
&\quad+\frac{1}{2}\int_{\mathbb{T}^2}\big(\overline{\partial}_j u_j\big)\Big\{|D_tF|^2-\partial_3 h\big|\mathcal{G}^{\frac{1}{2}}F\big|^2\Big\}\mathrm{d}\overline{x}\\
&=\int_{\mathbb{T}^2}\Big\{D_t^2F\cdot D_tF-\partial_3 h\cdot D_t\mathcal{G}^{\frac{1}{2}}F\cdot\mathcal{G}^{\frac{1}{2}}F\Big\}\mathrm{d}\overline{x}\\
&\quad+\frac{1}{2}\int_{\mathbb{T}^2}\Big\{\big(\overline{\partial}_j u_j\big)|D_tF|^2-\big(\overline{\partial}_j u_j\partial_3 h+D_t\partial_3 h\big)|\mathcal{G}^{\frac{1}{2}}F|^2\Big\}\mathrm{d}\overline{x}\\
&=\int_{\mathbb{T}^2}\Big\{D_t^2F-\partial_3 h\mathcal{G}F\Big\}D_tF\mathrm{d}\overline{x}+\int_{\mathbb{T}^2}\big[\mathcal{G}^{\frac{1}{2}},\partial_3 hD_t\big]F\cdot\mathcal{G}^{\frac{1}{2}}F\mathrm{d}\overline{x}\\
&\quad+\frac{1}{2}\int_{\mathbb{T}^2}\Big\{\big(\overline{\partial}_j u_j\big)|D_tF|^2-\big(\overline{\partial}_j u_j\partial_3 h+D_t\partial_3 h\big)\big|\mathcal{G}^{\frac{1}{2}}F\big|^2\Big\}\mathrm{d}\overline{x},
\end{aligned}
$$

where we have used the fact that $\mathcal{G}$ is self-adjoint. Therefore,

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big\{|D_tF|^2-\partial_3 h\big|\mathcal{G}^{\frac{1}{2}}F\big|^2\Big\}\mathrm{d}\overline{x}\\
=&\frac{1}{2}\int_{\mathbb{T}^2}\Big\{\big(\overline{\partial}_j u_j\big)|D_tF|^2\mathrm{d}\overline{x}-\big(\overline{\partial}_j u_j\partial_3 h+D_t\partial_3 h\big)\big|\mathcal{G}^{\frac{1}{2}}F\big|^2\Big\}\mathrm{d}\overline{x}\\
&+\int_{\mathbb{T}^2}\big[\mathcal{G}^{\frac{1}{2}},\partial_3 hD_t\big]F\cdot\mathcal{G}^{\frac{1}{2}}F\mathrm{d}\overline{x}-\int_{\mathbb{T}^2}\big[\Upsilon^{\kappa-\frac{3}{2}},D_t^2\big]\overline{\partial}_i f\cdot D_tF\mathrm{d}\overline{x}\\
&+\int_{\mathbb{T}^2}\big[\Upsilon^{\kappa-\frac{3}{2}},\partial_3 h\mathcal{G}\big]\overline{\partial}_i f\cdot D_tF\mathrm{d}\overline{x}-\int_{\mathbb{T}^2}\Upsilon^{\kappa-\frac{3}{2}}\nabla_N q_i\cdot D_tF\mathrm{d}\overline{x}\\
&+\int_{\mathbb{T}^2}\Upsilon^{\kappa-\frac{3}{2}}Q_i\cdot D_tF\mathrm{d}\overline{x}\\
=:&I_1+I_2+I_3+I_4+I_5+I_6.
\end{aligned}
\tag{5.4}
$$

For I_1 in (5.4), since $\kappa\geq 4$, Proposition 4.1 yields that

$$
I_1\lesssim C(M)\big\|\big(D_tF,\mathcal{G}^{\frac{1}{2}}F\big)\big\|_{L^2(\mathbb{T}^2)}^2\lesssim C(M).
\tag{5.5}
$$

For I_2 in (5.4), we split it into two terms,

$$
\begin{aligned}
I_2&=\int_{\mathbb{T}^2}\big[\mathcal{G}^{\frac{1}{2}},\partial_3 h\big]D_tF\cdot\mathcal{G}^{\frac{1}{2}}F\mathrm{d}\overline{x}+\int_{\mathbb{T}^2}\partial_3 h\big[\mathcal{G}^{\frac{1}{2}},D_t\big]F\cdot\mathcal{G}^{\frac{1}{2}}F\mathrm{d}\overline{x}\\
&\leq C(M)\big\|\big[\mathcal{G}^{\frac{1}{2}},\partial_3 h\big]D_tF\big\|_{L^2(\mathbb{T}^2)}+C(M)\big\|\big[\mathcal{G}^{\frac{1}{2}},D_t\big]F\big\|_{L^2(\mathbb{T}^2)}\\
&\leq C(M)\|D_tF\|_{L^2(\mathbb{T}^2)}+C(M)\|F\|_{H^{\frac{1}{2}}(\mathbb{T}^2)}\leq C(M),
\end{aligned}
\tag{5.6}
$$

where we have used the commutator estimates (A.26) and (A.27).

Similarly, for I_3 and I_4 in (5.4), we have

$$\begin{aligned}
I_3 &\lesssim C(M)\big\|\big[\Upsilon^{\kappa-\frac{3}{2}},D_t^2\big]\overline{\partial}_i f\big\|_{L^2(\mathbb{T}^2)}\\
&\lesssim C(M)\Big\{\big\|\big[\Upsilon^{\kappa-\frac{3}{2}},D_t\big]D_t\overline{\partial}_i f\big\|_{L^2(\mathbb{T}^2)}+\big\|D_t\big[\Upsilon^{\kappa-\frac{3}{2}},D_t\big]\overline{\partial}_i f\big\|_{L^2(\mathbb{T}^2)}\Big\}\\
&\lesssim C(M)\Big\{\big\|D_t\overline{\partial}_i f\big\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}+\big\|\overline{\partial}_i f\big\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}\Big\}\lesssim C(M), \qquad (5.7)\\
I_4 &= \int_{\mathbb{T}^2}\big[\Upsilon^{\kappa-\frac{3}{2}},\partial_3 h\big]\mathcal{G}\overline{\partial}_i f\cdot D_t F \mathrm{d}\overline{x}+\int_{\mathbb{T}^2}\partial_3 h\cdot\big[\Upsilon^{\kappa-\frac{3}{2}},\mathcal{G}\big]\overline{\partial}_i f\cdot D_t F\mathrm{d}\overline{x}\\
&\le C(M)\Big\{\big\|\big[\Upsilon^{\kappa-\frac{3}{2}},\partial_3 h\big]\mathcal{G}\overline{\partial}_i f\big\|_{L^2(\mathbb{T}^2)}+\big\|\big[\Upsilon^{\kappa-\frac{3}{2}},\mathcal{G}\big]\overline{\partial}_i f\big\|_{L^2(\mathbb{T}^2)}\Big\}\\
&\le C(M)\Big\{\big\|\mathcal{G}\overline{\partial}_i f\big\|_{H^{\kappa-\frac{5}{2}}(\mathbb{T}^2)}+\big\|\overline{\partial}_i f\big\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}\Big\}\le C(M). \qquad (5.8)
\end{aligned}$$

For I_5 in (5.4), q_i satisfies the elliptic equation (3.11) where one has

$$\Delta q_i=\partial_i\Delta h+\mathcal{H}(\partial_i f)\partial_3\Delta h+2\nabla\mathcal{H}(\partial_i f)\cdot\nabla\partial_3 h.$$

Since $\Delta h=\widehat{D}_t^2 h-\mathrm{tr}(\nabla u)^2$ and

$$\|\Delta h\|_{H^{\kappa-1}(\Omega_t)}\le\big\|\widehat{D}_t^2 h\big\|_{H^{\kappa-1}(\Omega_t)}+\big\|\mathrm{tr}(\nabla u)^2\big\|_{H^{\kappa-1}(\Omega_t)}\le C(M),$$

we have from (4.2) and (A.4) that

$$\begin{aligned}
\|\Delta q_i\|_{H^{\kappa-2}(\Omega_t)}\lesssim{}&\|\partial_i\Delta h\|_{H^{\kappa-2}(\Omega_t)}+\|\mathcal{H}(\partial_i f)\|_{H^{\kappa-\frac{1}{2}}(\Omega_t)}\|\partial_3\Delta h\|_{H^{\kappa-2}(\Omega_t)}\\
&+\|\nabla\mathcal{H}(\partial_i f)\|_{H^{\kappa-\frac{3}{2}}(\Omega_t)}\|\nabla\partial_3 h\|_{H^{\kappa-2}(\Omega_t)}\le C(M).
\end{aligned}$$

As the result, one has

$$\begin{aligned}
I_5&\lesssim\|\nabla_N q_i\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}\|D_t F\|_{L^2(\mathbb{T}^2)}\\
&\lesssim C(c_0,M)\|\Delta q_i\|_{H^{\kappa-2}(\Omega_t)}\lesssim C(c_0,M). \qquad (5.9)
\end{aligned}$$

Using the definition of Q_i in (3.8), I_6 in (5.4) can be estimated in the following:

$$I_6\lesssim\|Q_i\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}\|D_t F\|_{L^2(\mathbb{T}^2)}\lesssim C(M). \qquad (5.10)$$

Combining all the estimates (5.5)–(5.10), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|\big(D_t\overline{\partial}_i f,\mathcal{G}^{\frac{1}{2}}\overline{\partial}_i f\big)\big\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}\lesssim C(c_0,M), \qquad (5.11)$$

where we need the constant c_0 in the Taylor sign condition in (1.7).

As for $\|f\|_{L^2(\mathbb{T}^2)}$, we use (1.4) to get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2_{L^2(\mathbb{T}^2)} \le \|f\|_{L^2(\Omega_t)}\|u\cdot N\|_{L^2(\Omega_t)} \le C(M). \tag{5.12}$$

Together, (5.11) and (5.12) prove (5.1).

6 Estimates of the shear parts

In this section we shall estimate the vorticity $\omega = \nabla\times u$ first, then recover the bound of $\|u\|_{H^\kappa(\Omega_t)}$ by an elliptic estimate. The main result of this section is the following proposition.

Proposition 6.1. *There holds that*

$$\|u\|_{H^\kappa(\Omega_t)} \le C(c_0,M_0)+(T+\epsilon)C(c_0,M). \tag{6.1}$$

6.1 Estimates of the vorticity ω

Take $\Lambda^{\kappa-1}=\langle\nabla\rangle^{\kappa-1}$ to both sides of the vorticity equation (3.4) to get that

$$D_t(\Lambda^{\kappa-1}\omega)=\Lambda^{\kappa-1}(\omega\cdot\nabla u-\omega\nabla\cdot u)+[D_t,\Lambda^{\kappa-1}]\omega. \tag{6.2}$$

Then

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}|\Lambda^{\kappa-1}\omega|^2\mathrm{d}x \qquad (6.3)\\
=&\int_{\Omega_t}\Big\{D_t\Lambda^{\kappa-1}\omega\cdot\Lambda^{\kappa-1}\omega+\frac{1}{2}(\nabla\cdot u)|\Lambda^{\kappa-1}\omega|^2\Big\}\mathrm{d}x\\
=&\int_{\Omega_t}\Lambda^{\kappa-1}(\omega\cdot\nabla u-\omega\nabla\cdot u)\cdot\Lambda^{\kappa-1}\omega\mathrm{d}x\\
&+\int_{\Omega_t}\big[D_t,\Lambda^{\kappa-1}\big]\omega\cdot\Lambda^{\kappa-1}\omega\mathrm{d}x+\frac{1}{2}\int_{\Omega_t}(\nabla\cdot u)|\Lambda^{\kappa-1}\omega|^2\mathrm{d}x\\
\lesssim& C(M)\Big\{\big\|\Lambda^{\kappa-1}(\omega\cdot\nabla u-\omega\nabla\cdot u)\big\|_{L^2(\Omega_t)}+\big\|\big[D_t,\Lambda^{\kappa-1}\big]\omega\big\|_{L^2(\Omega_t)}+\big\|\nabla\cdot u\big\|_{L^\infty(\Omega_t)}\Big\}.
\end{aligned}$$

Since $\kappa\ge 4$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|^2_{H^{\kappa-1}(\Omega_t)}\lesssim C(M),$$

and

$$\|\omega\|_{H^{\kappa-1}(\Omega_t)}\le C(M_0)+TC(M). \tag{6.4}$$

6.2 Estimates of u

For the norm $\|u\|_{L^2(\Omega_t)}$, we use the first order hyperbolic system (3.2). Since

$$D_t\frac{|u|^2+|\epsilon h|^2}{2}+\nabla\cdot(uh)=0,$$

one has

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_t}\frac{|u|^2+|\epsilon h|^2}{2}\mathrm{d}x\\&=\int_{\Omega_t}D_t\frac{|u|^2+|\epsilon h|^2}{2}\mathrm{d}x+\int_{\Omega_t}(\nabla\cdot u)\frac{|u|^2+|\epsilon h|^2}{2}\mathrm{d}x\\&=\int_{\Omega_t}(\nabla\cdot u)\frac{|u|^2+|\epsilon h|^2}{2}\mathrm{d}x\le C(M)\|(u,\epsilon h)\|^2_{L^2(\Omega_t)}.\end{aligned}$$

Therefore, there holds that

$$\|u\|_{L^2(\Omega_t)}\le C(M_0)+TC(M). \tag{6.5}$$

For higher order derivatives of u, since

$$D_t\overline{\partial}_i f=\overline{\partial}_i D_t f+\left[D_t,\overline{\partial}_i\right]f=\overline{\partial}_i u\cdot N, \tag{6.6}$$

the elliptic estimates in (B.3) and (6.6) imply that

$$\begin{aligned}\|u\|_{H^s(\Omega_t)}&\le C(L)\Big\{\|\nabla\cdot u\|_{H^{s-1}(\Omega_t)}+\|\nabla\times u\|_{H^{s-1}(\Omega_t)}+\left\|D_t\overline{\partial}f\right\|_{H^{s-\frac{3}{2}}(\Gamma_t)}+\|u\|_{L^2(\Omega_t)}\Big\}\\&\le C(L)\Big\{\epsilon\left\|\widehat{D}_t h\right\|_{H^{s-1}(\Omega_t)}+\|\omega\|_{H^{s-1}(\Omega_t)}+\left\|D_t\overline{\partial}f\right\|_{H^{s-\frac{3}{2}}(\Gamma_t)}\Big\}\\&\le\Big\{C(c_0,M_0)+(T+\epsilon)C(c_0,M)\Big\}\cdot\Big\{\epsilon^2M+C(M_0)+(T+\epsilon)C(M)\Big\}\\&\le C(c_0,M_0)+(T+\epsilon)C(c_0,M),\end{aligned}$$

where in the second last step we have used (5.1). This proves (6.1).

7 Construction of the solution

In this section, we shall give a construction of the solution to our problem. This relies on solving a regularized ν-system with a parameter $\nu>0$. The first step is to use an iteration scheme to construct the solution to the ν-system, of which the life-span T^ν may depend on the parameter ν. In the next step, we shall write down the ν-system explicitly and derive the uniform-in-ν estimates of the solution to the ν-system. That is, the life-span is actually independent of the parameter ν. In last step, we shall take the limit $\nu\to 0$ and get the solution to our original problem.

7.1 The initial data

Assume that we have the initial data for the original compressible system satisfying

$$f_{\rm in}\in H^\kappa(\mathbb{T}^2),\quad (u_{\rm in},h_{\rm in})\in H^\kappa(\Omega_{f_{\rm in}}),$$

and there exists a constant $c_0>0$ such that

1. $f_{\rm in}\geq -1+c_0$;
2. $-\dfrac{N_{f_{\rm in}}}{|N_{f_{\rm in}}|}\cdot\nabla p_{\rm in}\geq c_0$ with $p_{\rm in}=\dfrac{1}{\epsilon^2}\big(e^{\epsilon^2 h_{\rm in}}-1\big)$.

Let $\nu>0$ be a small constant. We first regularize the free surface by setting

$$f_{\rm in}^\nu=f_{\rm in}*\eta_\nu,$$

where

$$\bar{\eta}_\nu(\overline{x})=\frac{1}{\nu^2}\eta\left(\frac{\overline{x}}{\nu}\right)$$

and η is a mollifier. Define the harmonic mapping $\Phi_{\rm in}^\nu:\Omega_{f_{\rm in}^\nu}\to\Omega_{f_{\rm in}}$ by (2.1) and set

$$u_{\rm in}^\nu=u_{\rm in}\circ\Phi_{\rm in}^\nu,\quad h_{\rm in}^\nu=h_{\rm in}\circ\Phi_{\rm in}^\nu.$$

Then

$$\nu\|f_{\rm in}^\nu\|_{H^{\kappa+\frac12}}\lesssim\|f_{\rm in}\|_{H^\kappa}$$

and

1. $f_{\rm in}^\nu\geq -1+\frac34 c_0$;
2. $-\dfrac{N_{f_{\rm in}^\nu}}{|N_{f_{\rm in}^\nu}|}\cdot\nabla p_{\rm in}^\nu\geq \dfrac34 c_0$ with $p_{\rm in}^\nu=\dfrac{1}{\epsilon^2}\big(e^{\epsilon^2 h_{\rm in}^\nu}-1\big)$,

when ν is small enough.

The regularized initial data are

$$f_I^\nu=f_{\rm in}^\nu,\quad (\partial_t f)_I^\nu=u_{\rm in}^\nu\cdot N_{f_{\rm in}^\nu},\tag{7.1}$$

$$\omega_I^\nu=\nabla\times u_{\rm in}^\nu,\quad \sigma_I^\nu=\nabla\cdot u_{\rm in}^\nu,\quad \alpha_I^j=\int_{\mathbb{T}^2}u_{\rm in}^{\nu,j}(\overline{x},-1)d\overline{x},\quad j=1,2.\tag{7.2}$$

We define $h_I^\nu\in H^{\kappa+1}(\Omega_{f_{\rm in}^\nu})$ by solving the following elliptic problem:

$$\begin{cases}\Delta h_I^\nu=(\Delta h_{\rm in}^\nu)*\eta_\nu,\\ h_I^\nu\big|_{\Gamma_{f_{\rm in}^\nu}}=0,\ \ \partial_3 h_I^\nu\big|_{\Gamma_-}=0,\end{cases}\tag{7.3}$$

where

$$\eta_\nu(x)=\frac{1}{\nu^3}\eta\left(\frac{x}{\nu}\right)$$

and η is the mollifier. In addition, we choose a large constant $M_0>0$ such that

$$\begin{aligned}&\|f_I^\nu\|_{H^\kappa(\mathbb{T}^2)}+\nu\|f_I^\nu\|_{H^{\kappa+\frac12}(\mathbb{T}^2)}+\|(\partial_t f)_I^\nu\|_{H^{\kappa-1}(\mathbb{T}^2)}+\nu\|(\partial_t f)_I^\nu\|_{H^{\kappa-\frac12}(\mathbb{T}^2)}\\&\quad+\left\|\left(\omega_I^\nu,\epsilon^{-2}\sigma_I^\nu\right)\right\|_{H^{\kappa-1}(\Omega_{f_{\rm in}^\nu})}+\|h_I^\nu\|_{H^\kappa(\Omega_{f_{\rm in}^\nu})}\le M_0.\end{aligned}\tag{7.4}$$

7.2 Iteration scheme

We choose $f_*=f_{\rm in}^\nu$ and $\Omega_*=\Omega_{f_{\rm in}^\nu}$ as the reference region. To construct the iteration map, we introduce the following functional space.

Definition 7.1. *Given constants $M_1^\nu,M_2^\nu>0$, we define the space $\mathcal{X}=\mathcal{X}(T^\nu,M_1^\nu,M_2^\nu)$ as the collection of $(f,\omega_*,\sigma_*,\alpha^1,\alpha^2)$ such that*

$$\begin{aligned}&\left(f(0),\partial_t f(0),\omega_*(0),\sigma_*(0),\alpha^1(0),\alpha^2(0)\right)\\&=\left(f_I^\nu,(\partial_t f)_I^\nu,\omega_I^\nu,\sigma_I^\nu,\alpha_I^1,\alpha_I^2\right),\\&\|f-f_*\|_{H^{\kappa-\frac12}(\mathbb{T}^2)}\le\delta_0,\end{aligned}$$

and

$$\begin{aligned}&\sup_{0<t<T_\nu}\Big(\|f(t)\|_{H^{\kappa+\frac12}(\mathbb{T}^2)}+\|\partial_t f(t)\|_{H^{\kappa-\frac12}(\mathbb{T}^2)}\\&\qquad+\left\|\left(\omega_*,\epsilon^{-2}\sigma_*\right)(t)\right\|_{H^{\kappa-1}(\Omega_*)}+\left|\alpha^1(t)\right|+\left|\alpha^2(t)\right|\Big)\le M_1^\nu,\\&\sup_{0<t<T_\nu}\Big(\left\|\partial_t^2 f\right\|_{H^{\kappa-\frac32}(\mathbb{T}^2)}+\left\|(\partial_t\omega_*,\epsilon^{-2}\partial_t\sigma_*)\right\|_{H^{\kappa-2}(\Omega_*)}+\left|\partial_t\alpha^1\right|+\left|\partial_t\alpha^2\right|\Big)\le M_2^\nu.\end{aligned}$$

Given $(f,\omega_*,\sigma_*,\alpha^1,\alpha^2)\in\mathcal{X}(T^\nu,M_1^\nu,M_2^\nu)$, our goal is to construct an iteration map

$$\left(F,W_*,Q_*,A^1,A^2\right)=\mathcal{F}\left(f,\omega_*,\sigma_*,\alpha^1,\alpha^2\right)\in\mathcal{X}(T^\nu,M_1^\nu,M_2^\nu)$$

with suitably chosen constants T^ν,M_1^ν,M_2^ν. Firstly, we pull back (ω_*,σ_*) to the domain Ω_f and set

$$\omega=\omega_*\circ\Phi_f^{-1},\quad \sigma=\sigma_*\circ\Phi_f^{-1}.\tag{7.5}$$

Then we can recover the velocity u in Ω_f by the following div-curl system:

$$\begin{cases} \nabla\times u = P_f^{\mathrm{div}}\omega \ \ \nabla\cdot u = \sigma+\beta & \text{in } \Omega_f, \\ u\cdot N_f = \partial_t f & \text{on } \Gamma_f, \\ u\cdot n_-\big|_{\Gamma_-} = 0, \ \int_{\mathbb{T}^2} u^j \mathrm{d}\overline{x} = \alpha^j, \quad j=1,2 & \text{on} \Gamma_-, \end{cases} \tag{7.6}$$

where P_f^{div} is the operator which projects a vector field in Ω_f to its divergence-free part. More precisely, $P_f^{\mathrm{div}}\omega = \omega - \nabla\psi$ with

$$\begin{cases} \Delta\psi = \nabla\cdot\omega \quad \text{in } \Omega_f, \\ \psi|_{\Gamma_f} = 0 \ \ \partial_3\psi|_{\Gamma_-} = 0. \end{cases}$$

The function β in (7.6) is given by

$$\beta = \beta(t) = \frac{1}{|\Omega_f|}\left\{ \int_{\mathbb{T}^2} \partial_t f \mathrm{d}\overline{x} - \int_{\Omega_f} \sigma \mathrm{d}x \right\}.$$

It is direct to check that $\beta(0)=0$ by the initial data and the fact that $u_I^\nu\cdot n_- = (u_{\mathrm{in}}\cdot n_-)\circ\Phi_{\mathrm{in}}^\nu = 0$. The existence of u follows from Proposition B.1. Denote by

$$D_t = \partial_t + u\cdot\nabla. \tag{7.7}$$

For the iterations of the vorticity W and divergence Q, we solve the following evolution equations in Ω_f:

$$\begin{cases} D_t W = W\cdot\nabla u - W(\nabla\cdot u), \\ W|_{t=0} = \omega_I^\nu, \end{cases} \tag{7.8}$$

and

$$\begin{cases} \epsilon^2 D_t^2 H - \Delta H = \mathrm{tr}(\nabla u)^2, \\ H|_{\Gamma_f} = 0, \quad \partial_3 H|_{\Gamma_-} = 0, \\ H|_{t=0} = h_I^\nu, \ \ D_t H|_{t=0} = -\dfrac{1}{\epsilon^2}\sigma_I^\nu. \end{cases} \tag{7.9}$$

The iteration of the divergence Q can be defined as

$$Q = -\epsilon^2 D_t H. \tag{7.10}$$

To construct the iteration of the free surface F, we consider the following equation:

$$\begin{cases} \overline{\partial}_t^2 F - \nu\Delta F + 2u^j \overline{\partial}_t \overline{\partial}_j F + u^j u^k \overline{\partial}_j \overline{\partial}_k F = -N_f \cdot \nabla h, \\ F|_{t=0} = f_I^\nu, \ \ \partial_t F|_{t=0} = (\partial_t f)_I^\nu, \end{cases} \tag{7.11}$$

where $\overline{\partial}_t, \overline{\partial}_j$ are tangential derivatives along Γ_f defined in (2.4).

As for the iteration of functions $A^j = A^j(t)$ with $j = 1,2$, we use

$$\begin{cases} \partial_t A^j = -\int_{\Gamma_-} u \cdot \nabla u^j \mathrm{d}\overline{x}, \\ A^j(0) = \alpha_I^j. \end{cases} \tag{7.12}$$

With the iterations given by (7.8)-(7.12) in Ω_f, we can define the iteration map

$$\mathcal{F}(f, \omega_*, \sigma_*, \alpha^1, \alpha^2) = (F, W \circ \Phi_f, Q \circ \Phi_f, A^1, A^2) =: (F, W_*, Q_*, A^1, A^2). \tag{7.13}$$

Proposition 7.1. *There exist M_1^ν, M_2^ν, T^ν depending on c_0, M_0 such that the iteration $\mathcal{F}$ is a map from $\mathcal{X}(T^\nu, M_1^\nu, M_2^\nu)$ to itself.*

Proof. We check the conditions in Definitions 7.1. The initial conditions are satisfied automatically. Next, for the velocity u defined in (7.6), Proposition B.1 implies that

$$\|u\|_{H^\kappa(\Omega_f)} \le C(M_1^\nu).$$

Then, for F in (7.11), Proposition 5.1 implies that

$$\|F\|_{H^{\kappa+\frac{1}{2}}(\mathbb{T}^2)} + \|\partial_t F\|_{H^{\kappa-\frac{1}{2}}(\mathbb{T}^2)} \le C(M_0) \mathrm{e}^{T^\nu C(M_1^\nu, M_2^\nu)}.$$

Recall that

$$\Phi_f : y \in \Omega_* \longrightarrow x \in \Omega_f.$$

Denote by

$$U_* = (\nabla_y \Phi_f)^{-1}(u_* - \partial_t \Phi_f), \quad K = (\nabla_y \Phi_f)^{-1}.$$

When pulling (W, H, Q) in (7.8)-(7.10) back to Ω_* by Φ_f, we have

$$(\partial_t + U_* \cdot \nabla_y) W_* = \{\nabla_y u_* K - \mathrm{tr}(\nabla_y u_* K) I\} W_*, \tag{7.14}$$

$$\begin{aligned} &\epsilon^2 (\partial_t + U_* \cdot \nabla_y)^2 H_* - K_i^j K_i^k \partial_{y^j} \partial_{y^k} H_* \\ = &\mathrm{tr}(\nabla_y u_* K)^2 + K_i^j K_i^k K_m^l \partial_{y^l} H_* \frac{\partial^2 \Phi_f^m}{\partial y^j \partial y^k}, \end{aligned} \tag{7.15}$$

Propositions 4.1 and 6.1 ensure that

$$\big\|\big(W_*,\epsilon^{-2}Q_*\big)\big\|_{H^{\kappa-1}(\Omega_*)}\le C(M_0)\mathrm{e}^{T^\nu C(M_1^\nu,M_2^\nu)}.$$

Furthermore, we have from Eqs. (7.8)-(7.11) that

$$\big\|\big(\partial_t W_*,\epsilon^{-2}\partial_t Q_*\big)\big\|_{H^{\kappa-2}(\Omega_*)}+\big\|\partial_t^2 F\big\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}\le C\big(M_1^\nu\big).$$

Obviously, there also hold that

$$\begin{aligned}&|A^j|\le M_0+T^\nu C\big(M_1^\nu\big),\quad |\partial_t A^j|\le C\big(M_1^\nu\big),\\ &\|F-f_*\|_{H^{\kappa-\frac{1}{2}}(\mathbb{T}^2)}\le\int_0^t\|\partial_t F\|_{H^{\kappa-\frac{1}{2}}(\mathbb{T}^2)}.\end{aligned}$$

As the result, we can take $M_1^\nu>2C(M_0)$, and then take $M_2^\nu=C(M_1^\nu)$. By choosing T^ν sufficiently small, we can conclude that the iteration $\mathcal{F}$ maps $\mathcal{X}(T^\nu,M_1^\nu,M_2^\nu)$ to itself. $\square$

7.3 Contraction of iteration map $\mathcal{F}$

We give a sketchy proof that the iteration map $\mathcal{F}$ defined in (7.13) is a contraction when T^ν is sufficiently small. Interested reader can consult [28] for a more detailed discussion.

Let $(f^a,\omega_*^a,\sigma_*^a,\alpha^{a,1},\alpha^{a,2})$ and $(f^b,\omega_*^b,\sigma_*^b,\alpha^{b,1},\alpha^{b,2})$ be two elements in the space $\mathcal{X}(T^\nu,M_1^\nu,M_2^\nu)$, and

$$\big(F^c,W_*^c,Q_*^c,A^{c,1},A^{c,2}\big)=\mathcal{F}\big(f^c,\omega_*^c,\sigma_*^c,\alpha^{c,1},\alpha^{c,2}\big),\quad c=a,b.$$

We denote by $g^d=g^a-g^b$. For example, $f^c=f^a-f^b$, $W_*^d=W_*^a-W_*^b$.

Proposition 7.2. *There exists $T^\nu>0$ depending on c_0,M_0 such that*

$$\begin{aligned}\mathfrak{E}:=&\sup_{0<t<T_\nu}\Big\{\|F^d\|_{H^{\kappa-\frac{1}{2}}(\mathbb{T}^2)}+\big\|\partial_t F^d\big\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}+\big\|\big(W_*^d,\epsilon^{-2}Q_*^d\big)\big\|_{H^{\kappa-2}(\Omega_*)}\\ &\qquad+|A^{d,1}|+|A^{d,2}|\Big\}\\ \le&\frac{1}{2}\sup_{0<t<T_\nu}\Big\{\|f^d\|_{H^{\kappa-\frac{1}{2}}(\mathbb{T}^2)}+\big\|\partial_t f^d\big\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}+\big\|\big(\omega_*^d,\epsilon^{-2}\sigma_*^d\big)\big\|_{H^{\kappa-2}(\Omega_*)}\\ &\qquad+|\alpha^{d,1}|+|\alpha^{d,2}|\Big\}\triangleq\frac{1}{2}\mathfrak{e}.\end{aligned}$$

Proof. By the elliptic estimates, we have

$$\|\Phi_{f^a}-\Phi_{f^b}\|_{H^\kappa(\Omega_*)}\le C\big(M_1^\nu\big)\|f^a-f^b\|_{H^{\kappa-\frac{1}{2}}(\mathbb{T}^2)}\le C\mathfrak{e}.$$

The velocity u^a and u^b given by (7.6) are defined in different domains. To compare them, we introduce

$$u_*^c=u^c\circ\Phi_{f^c},\quad c=a,b.$$

For a vector fields v_* in Ω_*, we define

$$\mathrm{curl}_c v_*=\Big\{\mathrm{curl}\big(v_*\circ\Phi_{f^c}^{-1}\big)\Big\}\circ\Phi_{f^c},\quad \mathrm{div}_c v_*=\Big\{\mathrm{div}\big(v_*\circ\Phi_{f^c}^{-1}\big)\Big\}\circ\Phi_{f^c}.$$

Then, the div-curl system for $u_*^d=u_*^a-u_*^b$ in Ω_* is

$$\begin{cases}\mathrm{curl}_a u_*^d=\omega_*^d+(\mathrm{curl}_b-\mathrm{curl}_a)u_*^b & \text{in }\ \Omega_*,\\ \mathrm{div}_a u_*^d=\sigma_*^d+\beta^d+(\mathrm{div}_b-\mathrm{div}_a)u_*^b & \text{in }\ \Omega_*,\\ u_*^d\cdot N_{f^a}=\partial_t f^d+u_*^b\cdot(N_{f^a}-N_{f^b}) & \text{on }\ \Gamma_*,\\ u_*^d\cdot e_3=0,\ \displaystyle\int_{\Gamma_-}u_*^{d,j}\mathrm{d}\overline{x}=\alpha^{d,j}\quad j=1,2, & \text{on }\ \Gamma_-.\end{cases}$$

An application of Proposition B.1 yields that

$$\big\|u_*^d\big\|_{H^{\kappa-1}(\Omega_*)}\le C\mathfrak{e}.$$

To estimate F^d, since $u^c|_{\Gamma_{f^a}}=u_*^c|_{\Gamma_*}$ $(c=a,b)$, we have from (7.11) that

$$\begin{aligned}&\overline{\partial}_t^2F^d-\nu\Delta F^d+2u^{a,j}\overline{\partial}_t\overline{\partial}_jF^d+u^{a,j}u^{a,k}\overline{\partial}_j\overline{\partial}_kF^d\\ &=-2u^{d,j}\overline{\partial}_t\overline{\partial}_jF^b-\big(u^{a,j}u^{d,k}+u^{d,j}u^{b,k}\big)\overline{\partial}_j\overline{\partial}_kF^b\\ &\quad-N_{f^a}\cdot\nabla h^d-(N_{f^a}-N_{f^b})\cdot\nabla h^b.\end{aligned}$$

We infer from Proposition 5.1 that

$$\|F^d\|_{H^{\kappa-\frac{1}{2}}(\mathbb{T}^2)}+\big\|\partial_tF^d\big\|_{H^{\kappa-\frac{3}{2}}(\mathbb{T}^2)}\le C\big(\mathrm{e}^{CT^\nu}-1\big)\mathfrak{e}.$$

For (W_*^d,H_*^d,Q_*^d), (7.14)-(7.15) yield that

$$\begin{aligned}&\big(\partial_t+U_*^a\cdot\nabla_y\big)W_*^d\\ &=-\big(U_*^a-U_*^b\big)\cdot\nabla_yW_*^b+\big(\nabla_yu_*^aK^a-\mathrm{tr}(\nabla_yu_*^aK^a)I\big)W_*^d\\ &\quad+\Big\{\big(\nabla_yu_*^aK^a-\mathrm{tr}(\nabla_yu_*^aK^a)I\big)-\big(\nabla_yu_*^bK^b-\mathrm{tr}(\nabla_yu_*^bK^b)I\big)\Big\}W_*^b,\end{aligned}$$

$$
\begin{aligned}
&\epsilon^2\big(\partial_t+U_*^a\cdot\nabla_y\big)^2H_*^d-K_i^{a,j}K_i^{a,k}\partial_{y^j}\partial_{y^k}H_*^d\\
=&-\epsilon^2\Big\{\big(\partial_t+U_*^a\cdot\nabla_y\big)\big(U_*^d\cdot\nabla_y\big)+\big(U_*^d\cdot\nabla_y\big)\big(\partial_t+U_*^b\cdot\nabla_y\big)\Big\}H_*^b\\
&+\big(K_i^{a,j}K_i^{a,k}-K_i^{b,j}K_i^{b,k}\big)\partial_{y^j}\partial_{y^k}H_*^b+\Big\{\mathrm{tr}\big(\nabla_y u_*^aK^a\big)^2-\mathrm{tr}\big(\nabla_y u_*^bK^b\big)^2\Big\}\\
&+\left\{K_i^{a,j}K_i^{a,k}K_m^{a,l}\partial_{y^l}H_*^a\frac{\partial^2\Phi_f^{a,m}}{\partial y^j\partial y^k}-K_i^{b,j}K_i^{b,k}K_m^{b,l}\partial_{y^l}H_*^b\frac{\partial^2\Phi_f^{b,m}}{\partial y^j\partial y^k}\right\}.
\end{aligned}
$$

Therefore, similar to Propositions 4.1 and 6.1, we have

$$\big\|\big(W_*^d,\epsilon^{-2}Q_*^d\big)\big\|_{H^{\kappa-2}(\Omega_*)}\le C\big(\mathrm{e}^{CT^\nu}-1\big)\mathfrak{e}.$$

As for $A^{d,j}$, it is direct to verify that

$$\partial_tA^{d,j}=-\int_{\Gamma_-}\big(u^{d,k}\partial_ku^{a,j}+u^{b,k}\partial_ku^{d,j}\big)\mathrm{d}\overline{x},$$

and

$$|A^{d,1}|+|A^{d,2}|\le CT^\nu\mathfrak{e}.$$

As the result, we have

$$\mathfrak{E}\le C\big(\mathrm{e}^{CT^\nu}-1+T^\nu\big)\mathfrak{e}.$$

By choosing T^ν sufficiently small, the proposition is proved. □

7.4 The ν-system

It follows from Propositions 7.1 and 7.2 that the mapping $\mathcal{F}$ has a unique fixed point $(f,\omega_*,\sigma_*,\alpha^1,\alpha^2)$ in $\mathcal{X}(T^\nu,M_1^\nu,M_2^\nu)$ satisfying

$$\overline{\partial}_t^2f-\nu\Delta f+2u^j\overline{\partial}_j\overline{\partial}_tf+u^ju^k\overline{\partial}_j\overline{\partial}_kf=-N_f\cdot\nabla h, \tag{7.16}$$

and

$$\begin{cases}D_t\omega=\omega\cdot\nabla u-\omega\nabla\cdot u & \text{in }\ \Omega_f,\\ \epsilon^2D_t^2h-\Delta h=\mathrm{tr}\,(\nabla u)^2 & \text{in }\ \Omega_f,\\ \sigma=-\epsilon^2D_th & \text{in}\Omega_f,\end{cases} \tag{7.17}$$

where u solves the div-curl system

$$\begin{cases}\nabla\times u=\omega,\ \ \nabla\cdot u=\sigma+\beta & \text{in }\ \Omega_f,\\ u\cdot N_f=\partial_tf & \text{on }\ \Gamma_f,\\ u\cdot n_-=0,\ \displaystyle\int_{\Gamma_-}u^j(\overline{x},-1)\mathrm{d}\overline{x}=\alpha^j & \text{on }\ \Gamma_-\end{cases} \tag{7.18}$$

with

$$\begin{cases} \beta = \dfrac{1}{|\Omega_f|}\left(\displaystyle\int_{\mathbb{T}^2} \partial_t f - \int_{\Omega_f} \sigma\right), \\ \partial_t \alpha^j = -\displaystyle\int_{\Gamma_-} u \cdot \nabla u^j \mathrm{d}\bar{x}. \end{cases} \tag{7.19}$$

We shall prove that $\beta = 0$ and the systems (7.16)-(7.19) can be written as a simpler "ν-system". From the equations (7.17)-(7.18), we have

$$\begin{aligned} &\nabla \times (D_t u + \nabla h) = D_t \omega - \omega \cdot \nabla u + \omega \nabla \cdot u = 0, \\ &\nabla \cdot (D_t u + \nabla h) = D_t \beta - \left(\epsilon^2 D_t^2 h - \Delta h - \mathrm{tr}(\nabla u)^2\right) = \partial_t \beta, \\ &n_- \cdot (D_t u + \nabla h) = D_t (n_- \cdot u) + n_- \cdot \nabla h = 0. \end{aligned}$$

Furthermore, Eq. (7.16) yields that

$$\begin{aligned} &N_f \cdot (D_t u + \nabla h) \\ = &N_f \cdot (\overline{\partial}_t u + u^j \overline{\partial}_j u + \nabla h) \\ = &\overline{\partial}_t (u \cdot N_f) + u^j \overline{\partial}_j (u \cdot N_f) - u \cdot \overline{\partial}_t N_f - u \cdot \left(u^j \overline{\partial}_j N_f\right) + N_f \cdot \nabla h \\ = &\overline{\partial}_t^2 f + 2u^j \overline{\partial}_j \overline{\partial}_t f + u^j u^k \overline{\partial}_j \overline{\partial}_k f + N_f \cdot \nabla h = \nu \Delta f. \end{aligned}$$

That is, the velocity equation of the ν-system has the following form:

$$D_t u + \nabla h = \nu \nabla \varphi, \tag{7.20}$$

where $\varphi \in H^{\kappa+1/2}(\Omega_f)$ solves

$$\begin{cases} \Delta \varphi = \partial_t \beta, \\ N_f \cdot \nabla \varphi|_{\Gamma_f} = \Delta f, \ \ n_- \cdot \nabla \varphi|_{\Gamma_-} = 0. \end{cases}$$

Since $\beta = \beta(t)$ with $\beta(0) = 0$ and

$$|\Omega_f| \partial_t \beta = \int_{\Omega_f} \Delta \varphi = \int_{\mathbb{T}^2} N_f \cdot \nabla \varphi = 0,$$

we have $\beta \equiv 0$. As the result, the systems (7.16)-(7.19) can be written as the following "ν-system":

$$\begin{cases} \partial_t f = u \cdot N_f & \text{on } \Gamma_t, \\ \epsilon^2 D_t h + \nabla \cdot u = 0 & \text{in } \Omega_f, \\ D_t u + \nabla h = \nu \nabla \varphi & \text{in } \Omega_f, \\ h|_{\Gamma_f} = 0, \ \ u^3|_{\Gamma_-} = 0 \end{cases} \tag{7.21}$$

with

$$\begin{cases} \Delta \varphi = 0 \quad \text{in } \Omega_f, \\ N_f \cdot \nabla \varphi|_{\Gamma_f} = \Delta f, \ \ n_- \cdot \nabla \varphi|_{\Gamma_-} = 0. \end{cases} \tag{7.22}$$

7.5 Uniform-in-ν estimates

For the evolution of the surface of the ν-system (7.21)-(7.22), we can derive in the spirit of (3.5)-(3.7) and get that

$$\begin{aligned}D_t^2\overline{\partial}_i f &= \partial_3(h-\nu\varphi)\mathcal{G}\overline{\partial}_i f - \nabla_N\overline{\partial}_i(h-\nu\varphi) - 2\overline{\partial}_i u^j D_t\overline{\partial}_j f\\ &= \nu\{\nabla_N\partial_i\varphi + \partial_i f\nabla_N\partial_3\varphi\} + \partial_3 h\mathcal{G}\overline{\partial}_i f - \nabla_N\overline{\partial}_i h - 2\overline{\partial}_i u^j D_t\overline{\partial}_j f\\ &= \nu\overline{\partial}_i\nabla_N\varphi + \nu\overline{\partial}_{ij}^2 f\partial_j\varphi + \partial_3 h\mathcal{G}\overline{\partial}_i f - \nabla_N\overline{\partial}_i h - 2\overline{\partial}_i u^j D_t\overline{\partial}_j f,\end{aligned}$$

that is,

$$D_t^2\overline{\partial}_i f - \nu\overline{\partial}_i\Delta f - \partial_3 h\mathcal{G}\overline{\partial}_i f = \nu\overline{\partial}_{ij}^2 f\partial_j\varphi - \nabla_N\overline{\partial}_i h - 2\overline{\partial}_i u^j D_t\overline{\partial}_j f. \tag{7.23}$$

The viscosity term $\nu\overline{\partial}_i\Delta f$ can be used to control the term $\nu\overline{\partial}_{ij}^2 f\partial_j\varphi$ on the right-hand side. Therefore, the estimates in Section 5 can be applied to get a uniform-in-ν estimate for f in the ν-system.

Inside the domain Ω_f, since φ is harmonic, the equations of the vorticity $\omega = \nabla\times u$ and the enthalpy h are

$$\begin{cases} D_t\omega = \omega\cdot\nabla u - \omega\nabla\cdot u,\\ \epsilon^2 D_t^2 h - \Delta h = \mathrm{tr}(\nabla u)^2.\end{cases} \tag{7.24}$$

These equations do not depend on ν. We can use the same estimates in Sections 4 and 6 and get the uniform-in-ν estimate for (ω,h) in the ν-system. As for the velocity u itself, we have an extra term $\nu\nabla\varphi$ in (7.21). The elliptic estimates in Lemma B.1 shows that we only need to bound $\|u\|_{L^2(\Omega_f)}$. This lower order energy can be derived directly from (7.21) by the energy method. As the result, we can prove the uniform-in-ν estimates and get a solution to the original problem (1.1)-(1.6) by taking the limit $\nu\to 0$.

8 The incompressible limit

When studying the incompressible limit, we fix a reference domain Ω_* so that we can compare the solutions in different domains. For the solution $(f^\epsilon, u^\epsilon, h^\epsilon)$ to the compressible system (1.1)-(1.5), one can use the harmonic coordinates Φ_{f^ϵ}: $y\in\Omega_*\mapsto x\in\Omega_{f^\epsilon}$ and derive the system of $(f^\epsilon, u_*^\epsilon, h_*^\epsilon) = (f^\epsilon, u^\epsilon\circ\Phi_{f^\epsilon}, h^\epsilon\circ\Phi_{f^\epsilon})$ in Ω_* as

$$
\begin{cases}
(\partial_t + u_*^\epsilon \cdot \nabla_y) f_*^\epsilon = (u_*^\epsilon)^3, \\
(\partial_t + u_*^\epsilon \cdot \nabla_y) u_*^\epsilon + \nabla_y h_*^\epsilon \\
= \nabla_y u_*^\epsilon (\nabla_y \Phi_{f^\epsilon})^{-1} (\partial_t + u_*^\epsilon \cdot \nabla_y)(\Phi_{f^\epsilon} - y) \\
\quad + \nabla_y h_*^\epsilon (\nabla_y \Phi_{f^\epsilon})^{-1} \nabla_y (\Phi_{f^\epsilon} - y), \\
(\partial_t + u_*^\epsilon \cdot \nabla_y) h_*^\epsilon + \nabla_y \cdot u_*^\epsilon \\
= \nabla_y h_*^\epsilon (\nabla_y \Phi_{f^\epsilon})^{-1} (\partial_t + u_*^\epsilon \cdot \nabla_y)(\Phi_{f^\epsilon} - y) \\
\quad + \operatorname{tr} \Big\{ \nabla_y u_*^\epsilon (\nabla_y \Phi_{f^\epsilon})^{-1} \nabla_y (\Phi_{f^\epsilon} - y) \Big\}.
\end{cases} \tag{8.1}
$$

From the uniform bounds of $(f^\epsilon, u^\epsilon, h^\epsilon)$ in the construction, there is a subsequence, where we still use the index ϵ, such that

$$
\big(f^\epsilon, u^\epsilon \circ \Phi_{f^\epsilon}, h^\epsilon \circ \Phi_{f^{\epsilon_k}}\big) \xrightarrow{\text{weak } *} (\widetilde{f}, \widetilde{w}, \widetilde{\eta}) \quad \text{in } L^\infty\Big(0, T; H^\kappa(\mathbb{T}^2) \times \big(H^\kappa(\Omega_*)\big)^4\Big).
$$

Then the uniform bounds of $(\partial_t f^\epsilon, \partial_t u^\epsilon, \partial_t h^\epsilon)$ imply that, after extracting a subsequence,

$$
\big(f^\epsilon, u^\epsilon \circ \Phi_{f^\epsilon}, h^\epsilon \circ \Phi_{f^{\epsilon_k}}\big) \longrightarrow (\widetilde{f}, \widetilde{w}, \widetilde{\eta}) \quad \text{in } C\Big([0,T]; H^s(\mathbb{T}^2) \times \big(H^s(\Omega_*)\big)^4\Big)
$$

for $s < \kappa$. Furthermore, there is another subsequence such that

$$
\big(f^\epsilon, u^\epsilon \circ \Phi_{f^\epsilon}, h^\epsilon \circ \Phi_{f^{\epsilon_k}}\big) \longrightarrow (\widetilde{f}, \widetilde{w}, \widetilde{\eta}) \quad \text{in } C^{\kappa-3}\big([0,T] \times \mathbb{T}^2\big) \times \Big(C^{\kappa-3}\big([0,T] \times \Omega_*\big)\Big)^4.
$$

Then we can pass the limit in the system (8.1) to get that $(\widetilde{f}, \widetilde{u}, \widetilde{p}) = (\widetilde{f}, \widetilde{w} \circ \Phi_{\widetilde{f}}^{-1}, \widetilde{h} \circ \Phi_{\widetilde{f}}^{-1})$ is a solution to the incompressible system (1.10). The uniqueness of the solution to the incompressible system implies that this is the only solution.

Appendix A. Paradifferential operators and commutator estimates

A.1 Paradifferential operators

In this appendix, we shall recall some basic facts on paradifferential operators from [26]. We first introduce the symbols with limited spatial smoothness. Let $W^{k,\infty}(\mathbb{R}^d)$ be the usual Sobolev spaces for $k \in \mathbb{N}$.

Definition A.1. *Given $\mu\in[0,1]$ and $m\in\mathbb{R}$, we denote by $\Gamma^m_\mu(\mathbb{R}^d)$ the space of locally bounded functions $a(x,\xi)$ on $\mathbb{R}^d\times\mathbb{R}^d\backslash\{0\}$, which are C^∞ with respect to ξ for $\xi\neq 0$ such that for all $\alpha\in\mathbb{N}^d$ and $\xi\neq 0$, the function $x\to\partial^\alpha_\xi a(x,\xi)$ belongs to $W^{\mu,\infty}$ and there exists a constant C_α such that*

$$\left\|\partial^\alpha_\xi a(\cdot,\xi)\right\|_{W^{\mu,\infty}}\leq C_\alpha(1+|\xi|)^{m-|\alpha|},\quad \forall|\xi|\geq\frac{1}{2}.$$

The seminorm of the symbol is defined as

$$M^m_\mu(a):=\sup_{|\alpha|\leq\frac{3d}{2}+1+\mu}\ \sup_{|\xi|\geq\frac{1}{2}}\left\|(1+|\xi|)^{-m+|\alpha|}\partial^\alpha_\xi a(\cdot,\xi)\right\|_{W^{\mu,\infty}}.$$

If a is a function independent of ξ, then

$$M^0_\mu(a)=\|a\|_{W^{\mu,\infty}}.$$

Definition A.2. *Given a symbol a, the paradifferential operator T_a is defined by*

$$\widehat{T_a u}(\xi):=(2\pi)^{-d}\int_{\mathbb{R}^d}\chi(\xi-\eta,\eta)\widehat{a}(\xi-\eta,\eta)\psi(\eta)\widehat{u}(\eta)\mathrm{d}\eta, \tag{A.1}$$

where $\widehat{a}$ is the Fourier transform of a with respect to the first variable. $\chi(\xi,\eta)\in C^\infty(\mathbb{R}^d\times\mathbb{R}^d)$ is an admissible cutoff function, that is, there exist $0<\varepsilon_1<\varepsilon_2$ such that

$$\begin{aligned}&\chi(\xi,\eta)=1\quad \textit{for}\quad |\xi|\leq\varepsilon_1|\eta|,\\ &\chi(\xi,\eta)=0\quad \textit{for}\quad |\xi|\geq\varepsilon_2|\eta|,\end{aligned}$$

and

$$\left|\partial^\alpha_\xi\partial^\beta_\eta\chi(\xi,\eta)\right|\leq C_{\alpha,\beta}(1+|\eta|)^{-|\alpha|-|\beta|}\quad \textit{for}\quad (\xi,\eta)\in\mathbb{R}^d\times\mathbb{R}^d.$$

The cutoff function $\psi(\eta)\in C^\infty(\mathbb{R}^d)$ satisfies

$$\begin{aligned}&\psi(\eta)=0\quad \textit{for}\quad |\eta|\leq 1,\\ &\psi(\eta)=1\quad \textit{for}\quad |\eta|\geq 2.\end{aligned}$$

The admissible cutoff function $\chi(\xi,\eta)$ can be chosen as

$$\chi(\xi,\eta)=\sum_{k=0}^\infty\zeta_{k-3}(\xi)\varphi(\eta),$$

where $\zeta(\xi)=1$ for $|\xi|\leq 1.1$, $\zeta(\xi)=0$ for $|\xi|\geq 1.9$, and

$$\begin{cases} \zeta_k(\xi)=\zeta(2^{-k}\xi) & \text{for} \quad k\in\mathbb{Z}, \\ \varphi_0=\zeta, \ \ \varphi_k=\zeta_k-\zeta_{k-1} & \text{for} \quad k\geq 1. \end{cases}$$

We also introduce the Littlewood-Paley operators Δ_k, S_k defined as

$$\begin{aligned} &\Delta_k u=\mathcal{F}^{-1}(\varphi_k\widehat{u}) && \text{for} \quad k\geq 0, \\ &\Delta_k u=0 && \text{for} \quad k<0, \\ &S_k u=\sum_{l\leq k}\Delta_l u && \text{for} \quad k\in\mathbb{Z}. \end{aligned}$$

When the symbol a depends only on the first variable x in $T_a u$, we take $\psi=1$ in (A.1). Then $T_a u$ is just usual Bony's paraproduct defined as

$$T_a u=\sum_{k=0} S_{k-3}a\Delta_k u. \tag{A.2}$$

We have the following Bony's paraproduct decomposition:

$$au=T_a u+T_u a+R(a,u), \tag{A.3}$$

where the remainder term $R(a,u)$ is

$$R(a,u)=\sum_{|k-l|\leq 2}\Delta_k a\Delta_l u.$$

Lemma A.1. *There holds that*

1. *If $s\in\mathbb{R}$ and $\sigma<\frac{d}{2}$, then*

$$\|T_a u\|_{H^s}\lesssim \min\Big\{\|a\|_{L^\infty}\|u\|_{H^s}, \|a\|_{H^\sigma}\|u\|_{H^{s+\frac{d}{2}-\sigma}}, \|a\|_{H^{\frac{d}{2}}}\|u\|_{H^{s+}}\Big\}.$$

2. *If $s>0$ and $s_1,s_2\in\mathbb{R}$ with $s_1+s_2=s+\frac{d}{2}$, then*

$$\|R(a,u)\|_{H^s}\lesssim \|a\|_{H^{s_1}}\|u\|_{H^{s_2}}.$$

3. *If $s>0, s_1\geq s, s_2\geq s$ and $s_1+s_2=s+\frac{d}{2}$, then*

$$\|au\|_{H^s}\lesssim \|a\|_{H^{s_1}}\|u\|_{H^{s_2}}. \tag{A.4}$$

There is also the symbolic calculus of paradifferential operator in Sobolev spaces.

Lemma A.2. *Let $m,m'\in\mathbb{R}$.*

1. *If $a\in\Gamma_0^m(\mathbb{R}^d)$, then for any $s\in\mathbb{R}$,*

$$\|T_a\|_{H^s\to H^{s-m}}\lesssim M_0^m(a).$$

2. *If $a\in\Gamma_\rho^m(\mathbb{R}^d)$ and $b\in\Gamma_\rho^{m'}(\mathbb{R}^d)$ for $\rho>0$, then for any $s\in\mathbb{R}$,*

$$\|T_aT_b-T_{a\sharp b}\|_{H^s\to H^{s-m-m'+\rho}}\lesssim M_\rho^m(a)M_0^{m'}(b)+M_0^m(a)M_\rho^{m'}(b),$$

where

$$a\sharp b=\sum_{|\alpha|<\rho}\partial_\xi^\alpha a(x,\xi)D_x^\alpha b(x,\xi),\quad D_x=-\mathrm{i}\partial_x.$$

3. *If $a\in\Gamma_\rho^m(\mathbb{R}^d)$ for $\rho\in(0,1]$, then for any $s\in\mathbb{R}$,*

$$\big\|(T_a)^*-T_{a^*}\big\|_{H^s\to H^{s-m+\rho}}\lesssim M_\rho^m(a),$$

where $(T_a)^*$ is the adjoint operator of T_a and a^* is the complex conjugate of the symbol a.

A direct corollary of the lemma above is the following commutator estimates.

Lemma A.3. *If $s>1+\frac{d}{2}$, then for $\sigma\le s$,*

$$\big\|[a,\Lambda^\sigma]u\big\|_{L^2}\lesssim\|a\|_{H^s}\|u\|_{H^{\sigma-1}}.\tag{A.5}$$

A.2 Elliptic estimates in a strip

For a strip domain $\Omega_f=\{(\overline{x},x^3):\overline{x}\in\mathbb{T}^2,-1<x^3<f(\overline{x})\}$ with $1+f(\overline{x})\ge c_0>0$, we consider the elliptic boundary value problem

$$\begin{cases}\Delta\Phi=0 & \text{in } \Omega_f,\\ \Phi\big(\overline{x},f(\overline{x})\big)=\phi(\overline{x}) & \text{on } \Gamma_f,\\ \Phi(\overline{x},-1)=0 & \text{on } \Gamma_-.\end{cases}\tag{A.6}$$

Here we have the Dirichlet boundary condition on the bottom. Elliptic theory shows that for $\phi(\overline{x})\in H^{1/2}(\Gamma_f)$, there exists a unique weak solution $\Phi(x)\in H^1(\Omega_f)$ satisfying

$$\|\Phi\|_{H^1(\Omega_f)}\le C\big(c_0,\|f\|_{W^{1,\infty}}\big)\|\phi\|_{H^{\frac12}(\Gamma_f)}.$$

More generally, there holds that

Proposition A.1. *Let $\Phi\in H^1(\Omega_f)$ be a weak solution to* (A.6). *Then for any $s\in[0,\kappa]$, there holds that*

$$\|\Phi\|_{H^{s+\frac12}(\Omega_f)}\le C\big(c_0,\|f\|_{H^\kappa}\big)\|\phi\|_{H^s(\Gamma_f)}. \tag{A.7}$$

To study the modified Dirichlet-Neumann (DN) operator, we follow the method introduced in [2]. The idea is to flatten the strip Ω_f and decouple the elliptic estimates into a forward and a backward parabolic evolution equation. The strip Ω_f can be flattened by a regularized mapping

$$(\overline{x},z)\in S:=\mathbb{T}^2\times(-1,0)\ \longmapsto\ \big(\overline{x},\rho_\delta(\overline{x},z)\big)\in\Omega_f$$

with $\delta>0$ and

$$\rho_\delta(\overline{x},z)=z+(1+z)e^{\delta z|D_{\overline{x}}|}f(\overline{x}). \tag{A.8}$$

It is easy to check that there exists $\delta>0$ depending on c_0 and $\|f\|_{W^{1,\infty}}$ such that

$$\partial_z\rho_\delta(\overline{x},z)\ge\frac{c_0}{2}\quad\text{for}\quad(\overline{x},z)\in S.$$

Denote by

$$\Psi(\overline{x},z)=\Phi\big(\overline{x},\rho_\delta(\overline{x},z)\big).$$

Then $\Psi(\overline{x},z)$ is the solution to the elliptic equation

$$\begin{cases}\partial_z^2\Psi+\alpha\Delta_{\overline{x}}\Psi+\beta\cdot\nabla_{\overline{x}}\partial_z\Psi-\gamma\partial_z\Psi,\\ \Psi(\overline{x},0)=\phi,\ \ \Psi(\overline{x},-1)=0\end{cases} \tag{A.9}$$

with variable coefficients

$$\alpha=\frac{(\partial_z\rho_\delta)^2}{1+|\nabla_{\overline{x}}\rho_\delta|^2},\quad \beta=-2\frac{\partial_z\rho_\delta\nabla_{\overline{x}}\rho_\delta}{1+|\nabla_{\overline{x}}\rho_\delta|^2},$$
$$\gamma=\frac{1}{\partial_z\rho_\delta}\Big(\partial_z^2\rho_\delta+\alpha\Delta_{\overline{x}}\rho_\delta+\beta\cdot\nabla_{\overline{x}}\partial_z\rho_\delta\Big).$$

We introduce the following functional spaces:

$$X^s=L_z^\infty\big((-1,0);H^s(\mathbb{T}^2)\big)\cap L_z^2\big((-1,0);H^{s+\frac12}(\mathbb{T}^2)\big),$$
$$Y^s=L_z^1\big((-1,0);H^s(\mathbb{T}^2)\big)+L_z^2\big((-1,0);H^{s-\frac12}(\mathbb{T}^2)\big).$$

Proposition A.2. *Let $\Psi(\overline{x},z)\in H^1(S)$ be a weak solution of* (A.9). *Assume that $f\in H^\kappa$ for $\kappa>2$. Then for any $s\in[\frac12,\kappa]$, there holds that*

$$\|\nabla_{\overline{x},z}\Psi\|_{X^{s-1}}\le C\big(c_0,\|f\|_{H^\kappa}\big)\|\phi\|_{H^s}. \tag{A.10}$$

To prove the proposition, we first paralinearize the elliptic equation (A.9) as

$$\partial_z^2\Psi+T_\alpha\Delta_{\overline{x}}\Psi+T_\beta\cdot\nabla_{\overline{x}}\partial_z\Psi=F_1+F_2 \tag{A.11}$$

with

$$F_1=\gamma\partial_z\Psi,\quad F_2=(T_\alpha-\alpha)\Delta_{\overline{x}}\Psi+(T_\beta-\beta)\cdot\nabla_{\overline{x}}\partial_z\Psi.$$

Then Eq. (A.11) can be decoupled into a forward and a backward parabolic evolution equation

$$(\partial_z-T_a)(\partial_z-T_A)\Psi=F_1+F_2+F_3:=F \tag{A.12}$$

with the symbols and F_3 as

$$\begin{aligned}
a&=\frac{1}{2}\left(-\mathrm{i}\beta\cdot\xi-\sqrt{4\alpha|\xi|^2-(\beta\cdot\xi)^2}\right),\\
A&=\frac{1}{2}\left(-\mathrm{i}\beta\cdot\xi+\sqrt{4\alpha|\xi|^2-(\beta\cdot\xi)^2}\right),\\
F&=(T_aT_A-T_\alpha\Delta_{\overline{x}})\Psi-(T_a+T_A+T_\beta\cdot\nabla_{\overline{x}})\partial_z\Psi-T_{\partial_zA}\Psi.
\end{aligned}$$

Denote by $\Gamma_r^m(\mathbb{T}^2\times(-1,0))$ the space of symbols $a(\overline{x},\xi;z)$ satisfying

$$M_r^m(a)=\sup_{z\in(-1,0)}\sup_{|\alpha|\le r+2}\sup_{|\xi|\ge\frac{1}{2}}\left\|\langle\xi\rangle^{|\alpha|-m}\partial_\xi^\alpha a(\cdot,\xi;z)\right\|_{W^{r,\infty}}<\infty.$$

It is direct to verify that if $f\in H^\kappa(\mathbb{T}^2)$ for $\kappa>\frac{3}{2}$, then $a,A\in M_\delta^1$ for some $\delta>0$ and

$$M_\delta^1(a)+M_\delta^1(A)\le C\big(c_0,\|f\|_{H^\kappa}\big).$$

Here is a lemma about the parabolic evolution.

Lemma A.4. *Let $a\in\Gamma_\delta^1((-1,0)\times\mathbb{T}^2)$ for some $\delta>0$ be elliptic in the sense that, there exists $c_1>0$ such that*

$$\Re a(x,\xi;z)\ge c_1|\xi|,\quad\forall(x,\xi,z)\in\mathbb{T}^2\times\mathbb{R}^2\times(-1,0).$$

Consider the parabolic equation

$$\partial_zw+T_aw=g,\quad w|_{z=z_0}=w_0.$$

If $w_0\in H^s$ and $g\in Y^s$ for $s\in\mathbb{T}$, then there exists a unique solution $w\in X^s$ such that

$$\|w\|_{X^s}\le\big(c_1,M_\delta^1\big)\big(\|w_0\|_{H^s}+\|g\|_{Y^s}\big).$$

Fix $\delta>0$ with $\delta<\max(\frac{1}{2},\kappa-2)$. The symbol estimates in Lemma A.2 yield the following [2]:

Lemma A.5. *For all $s\in[-\frac{1}{2},\kappa-1-\delta]$, there holds that*

$$\begin{aligned}
&\|F_1\|_{Y^{s+\delta}}\le C\big(c_0,\|f\|_{H^\kappa}\big)\|\partial_z\Psi\|_{X^s},\\
&\|F_2\|_{Y^{s+\delta}}\le C\big(c_0,\|f\|_{H^\kappa}\big)\|\nabla_{\overline{x},z}\Psi\|_{X^s},\\
&\|F_3\|_{L^2_z((-1,0);H^{s-1-\delta})}\le C\big(c_0,\|f\|_{H^\kappa}\big)\|\partial_z\Psi\|_{L^2_z((-1,0);H^s)}.
\end{aligned}$$

Now we can prove Proposition A.2.

*Proof of Proposition A.*2. We shall prove by induction. The case $s=\frac{1}{2}$ is easy to verify. Assume that the statement is true for some $s\in(\frac{1}{2},\kappa-1-\delta]$. We just need to prove that

$$\|\nabla_{\overline{x},z}\Psi\|_{X^{s-1+\delta}}\le C\big(c_0,\|f\|_{H^\kappa}\big)\|\phi\|_{H^{s+\delta}}. \tag{A.13}$$

Set $W=(\partial_z-T_a)\Psi$, which is the solution to the forward parabolic evolution equation

$$\partial_z W-T_aW=F,\quad W(-1)=(\partial_z-T_a)\Psi\big|_{z=-1}. \tag{A.14}$$

Using a localization argument as in [1, Lemma 2.8], we can prove that

$$\|\Psi(\overline{x},-1)\|_{H^\kappa}\le C(c_0)\|\phi\|_{H^{\frac{1}{2}}}. \tag{A.15}$$

By the induction assumption and (A.15), it follows from Lemmas A.4 and A.5 that

$$\begin{aligned}
\|W\|_{X^{s-1+\delta}}&\le C\big(c_0,\|f\|_{H^\kappa}\big)\big(\|W(-1)\|_{H^{s+\delta}}+\|F\|_{Y^{s+\delta}}\big)\\
&\le C\big(c_0,\|f\|_{H^\kappa}\big)\big(\|(\partial_z-T_a)\Psi\big|_{z=-1}\|_{H^{s+\delta}}+\|\nabla_{\overline{x},z}\Psi\|_{X^s}\big)\\
&\le C\big(c_0,\|f\|_{H^\kappa}\big)\|\phi\|_{H^s}.
\end{aligned}\tag{A.16}$$

For the backward parabolic evolution equation

$$\partial_z\Psi-T_A\Psi=W,\quad \Psi|_{z=0}=\phi, \tag{A.17}$$

It follows from Lemma A.4 and (A.16) that

$$\begin{aligned}
\|\Psi\|_{X^{s+\delta}}&\le C\big(c_0,\|f\|_{H^\kappa}\big)\big(\|\phi\|_{H^{s+\delta}}+\|W\|_{L^2_z((-1,0);H^{s-\frac{1}{2}+\delta})}\big)\\
&\le C\big(c_0,\|f\|_{H^\kappa}\big)\|\phi\|_{H^{s+\delta}}.
\end{aligned}$$

Using $\partial_z\Psi=T_A\Psi+W$, we get

$$\|\partial_z\Psi\|_{X^{s-1+\delta}}\le C\big(c_0,\|f\|_{H^\kappa}\big)\|\phi\|_{H^{s+\delta}}.$$

The proposition is proved. $\square$

A.3 The DN operator

Recall that the DN operator is defined as

$$\mathcal{G}\phi = N\cdot\nabla\mathcal{H}\phi\big|_{\Gamma_t}.$$

Noticing that a Dirichlet boundary condition is posed instead of the Neumann boundary condition on the bottom in the definition of the harmonic extension $\mathcal{H}$ in (2.2). From the discussion of the elliptic problem in a strip in the previous subsection, we shall prove that this modification does not change the leading order terms of the DN operator, and hence keeps all regularity properties of the standard DN operator.

In terms of Ψ, the DN operator can be written as

$$\mathcal{G}\phi = \left(\frac{1+|\nabla_{\overline{x}}\rho_\delta|^2}{\partial_z\rho_\delta}\partial_z\Psi - \nabla_{\overline{x}}\rho_\delta\cdot\nabla_{\overline{x}}\Psi\right)\Big|_{z=0}.$$

Denote the above coefficients by

$$\begin{aligned}
&\zeta_1(\overline{x}) = \frac{1+|\nabla_{\overline{x}}\rho_\delta|^2}{\partial_z\rho_\delta}\Big|_{z=0} = \frac{1+|\nabla_{\overline{x}}f|^2}{\partial_z\rho_\delta|_{z=0}},\\
&\zeta_2(\overline{x}) = \nabla_{\overline{x}}\rho_\delta\big|_{z=0} = \nabla_{\overline{x}}f.
\end{aligned}$$

It is direct to check that for $\kappa>2$,

$$\|\zeta_1-1\|_{H^{\kappa-1}} + \|\zeta_2\|_{H^{\kappa-1}} \le C\big(c_0,\|f\|_{H^\kappa}\big). \tag{A.18}$$

Using Bony's decomposition, the DN operator can be decomposed as

$$\begin{aligned}
\mathcal{G}\phi = \Big\{&\partial_z\Psi + T_{\zeta_1-1}\partial_z\Psi + T_{\partial_z\Psi}(\zeta_1-1) + R(\zeta_1-1,\partial_z\Psi)\\
&-T_{\mathrm{i}\zeta_2\cdot\xi}\Psi - T_{\nabla_{\overline{x}}\Psi}\cdot\zeta_2 - R(\zeta_2,\nabla_{\overline{x}}\Psi)\Big\}\Big|_{z=0}.
\end{aligned}$$

Noticing (A.16), we can replace the normal derivative $\partial_z\Psi$ by the tangential derivative $T_A\Psi$ by (A.17) and have the following decomposition of the DN operator:

$$\mathcal{G}\phi = T_\lambda\phi + R_f\phi, \tag{A.19}$$

where the symbol $\lambda(x,\xi)$ of the leading order term is given by

$$\lambda(x,\xi) = \sqrt{(1+|\nabla f|^2)|\xi|^2 - (\nabla f\cdot\xi)^2}.$$

Obviously, $\lambda\in\Gamma^1_\delta(\mathbb{R}^2)$ with the bound

$$M^1_\delta(\lambda)\le C(\|f\|_{H^\kappa}),\quad \forall\delta\in(0,\kappa-2). \tag{A.20}$$

The remain term R_f is given by

$$R_f=R_{1,f}+R_{2,f}+R_{3,f} \tag{A.21}$$

with

$$\begin{aligned}&R_{1,f}\phi=(T_{\zeta_1}T_A-T_{\zeta_1 A})\Psi\big|_{z=0},\quad R_{2,f}\phi=-T_{\zeta_1}(\partial_z-T_A)\Psi\big|_{z=0},\\&R_{3,f}\phi=\Big\{S_2(\partial_z\Psi)+T_{\partial_z\Psi}(\zeta_1-1)+R(\zeta_1-1,\partial_z\Psi)-T_{\nabla_{\overline{x}}\Psi}\cdot\zeta_2-R(\nabla_{\overline{x}}\Psi,\zeta_2)\Big\}\Big|_{z=0}.\end{aligned}$$

Let us recall the following results from [28].

Lemma A.6. *If $\kappa>3$, then there holds that*

$$\|R_fg\|_{H^s}\le C(\|f\|_{H^\kappa})\|g\|_{H^s},\quad \forall s\in[1/2,\kappa-1]. \tag{A.22}$$

Lemma A.7. *If $\kappa>3$, then there holds that*

$$\|\mathcal{G}g\|_{H^{s-1}}\le C(\|f\|_{H^\kappa})\|g\|_{H^s},\quad \forall s\in[1/2,\kappa]. \tag{A.23}$$

In a similar fashion, we can decompose $\mathcal{G}^{1/2}$ as

$$\mathcal{G}^{\frac12}=T_{\sqrt{\lambda}}+\widetilde{R}_f, \tag{A.24}$$

where the remainder term $\widetilde{R}_f$ satisfies that, if $\kappa>3$,

$$\|\widetilde{R}_f\|_{H^{\sigma+\frac12}\to H^\sigma}\le C(c_0,\|f\|_{H^\kappa}),\quad \forall\sigma\in[1/2,\kappa-1]. \tag{A.25}$$

We have the following commutator estimate.

Lemma A.8. *If $s>\frac52$, then there holds that*

$$\big\|[\mathcal{G}^{\frac12},a]\big\|_{H^\sigma\to H^{\sigma-\frac12}}\le C\big(\|f\|_{H^{s+\frac12}}\big)\|a\|_{H^{\sigma+1}},\quad \forall\sigma\in[-1/2,s-1/2]. \tag{A.26}$$

In this paper, we need to estimate the commutator $[\mathcal{G}^{1/2},D_t]$. To this end, we need an auxiliary result.

Lemma A.9 (cf. [2, Lemma 2.15]). *Consider a symbol $p=p(t,x,\xi)$ which is homogeneous of order m. There holds that*

$$\big\|[T_p,\partial_t+T_u\cdot\nabla]\big\|_{L^2}\lesssim\Big\{M^m_0(p)\|u\|_{C^{1+}_\star}+M^m_0(D_tp)\Big\}\|u\|_{H^m}.$$

Then the decomposition (A.24) yields the following result.

Lemma A.10. *There holds that*

$$\big\|[\mathcal{G}^{\frac12},D_t]g\big\|_{L^2}\lesssim\Big(\|f\|_{W^{\frac32,\infty}}\|u\|_{H^4}+\|D_tf\|_{W^{\frac32,\infty}}\Big)\|g\|_{H^{\frac12}}. \tag{A.27}$$

Appendix B. The div-curl system

In this appendix, we give the estimates for the div-curl system

$$\begin{cases} \nabla\times u=\omega, \ \nabla\cdot u=\sigma, & \text{in } \Omega_f, \\ u\cdot N_f=\theta, & \text{on } \Gamma_f, \\ u\cdot n_-=0, \ \int_{\mathbb{T}^2} u^j \mathrm{d}\overline{x}=\alpha^j, \ j=1,2 & \text{on } \Gamma_-. \end{cases} \tag{B.1}$$

Firstly, let us recall the existence result in [28] (see also [5,27]).

Proposition B.1. *Assume that $s\in[2,\kappa-1]$ is an integer. Given $(\omega,\sigma)\in H^{s-1}(\Omega_f)$ and $\theta\in H^{s-1/2}(\mathbb{T}^2)$ with the compatibility conditions*

$$\int_{\Omega_f}\sigma \mathrm{d}x=\int_{\mathbb{T}^2}\theta \mathrm{d}\overline{x}, \quad \nabla\cdot\omega=0, in\ \Omega_f, \quad \int_{\Gamma_-}\omega^3 \mathrm{d}\overline{x}=0,$$

there exists a unique $u\in H^\sigma(\Omega_f)$ to the div-curl system (B.1) *such that*

$$\|u\|_{H^s(\Omega_f)}\le C(L)\Big\{\|(\omega,\sigma)\|_{H^{s-1}(\Omega_f)}+\|\theta\|_{H^{s-\frac{1}{2}}(\mathbb{T}^2)}+|\alpha^1|+|\alpha^2|\Big\} \tag{B.2}$$

with $L=\|f\|_{H^{\kappa-1/2}}$.

The regularity of the solution of the div-curl system can be increased if we use the tangential derivatives for the boundary condition on the surface Γ_f.

Lemma B.1. *For a vector $u\in H^\kappa(\Omega_t)$, there holds that*

$$\begin{aligned} \|u\|_{H^\kappa(\Omega_t)}\lesssim C(L)\Big\{&\|\nabla\times u\|_{H^{\kappa-1}(\Omega_t)}+\|\nabla\cdot u\|_{H^{\kappa-1}(\Omega_t)} \\ &+\sum_{i=1,2}\|\overline{\partial}_i u\cdot N\|_{H^{\kappa-\frac{3}{2}}(\Gamma_t)}+\|u\|_{H^{\kappa-1}(\Omega_t)}\Big\}, \end{aligned} \tag{B.3}$$

where $L=\|f\|_{H^{\kappa-1/2}}$.

Proof. Denote the right-hand side of (B.3) by RHS.

Step 1. Since

$$\Delta u=\nabla(\nabla\cdot u)-\nabla\times(\nabla\times u),$$

standard elliptic theory shows that it suffices to prove that

$$\|N\cdot\nabla u\|_{H^{\kappa-\frac{3}{2}}(\Gamma_t)}\le \text{RHS}. \tag{B.4}$$

Step 2. Set

$$w=\nabla u^{\top}N=(\partial_1 u\cdot N,\partial_2 u\cdot N,\partial_2 u\cdot N)^{\top}.$$

To prove (B.4), noticing that

$$N\cdot\nabla u-w=(\nabla u-\nabla u^{\top})N=(\nabla\times u)\times N,$$

and

$$\|N\cdot\nabla u-w\|_{H^{\kappa-\frac{3}{2}}(\Gamma_t)}\le C(L)\|\nabla\times u\|_{H^{\kappa-1}},$$

we just need to prove that

$$\|w\|_{H^{\kappa-\frac{3}{2}}(\Gamma_t)}\le \text{RHS}. \tag{B.5}$$

Step 3. Set

$$\tau_1=(1,0,\partial_1 f)^{\top},\quad \tau_2=(0,1,\partial_2 f)^{\top}.$$

Then $\operatorname{span}\{\tau_1,\tau_2,N\}=\mathbb{R}^3$ and $\tau_i\cdot\nabla=\overline{\partial}_i$ on Γ_t. Consider the projections of $w=\nabla u^{\top}N$ over directions τ_i, $i=1,2$,

$$w\cdot\tau_i=(\partial_i u+\partial_i f\partial_3 u)\cdot N=\overline{\partial}_i u\cdot N.$$

Therefore, to prove (B.5), one just needs to prove that

$$\|w\cdot N\|_{H^{\kappa-\frac{3}{2}}(\Gamma_t)}=\|N\otimes N:\nabla u\|_{H^{\kappa-\frac{3}{2}}(\Gamma_t)}\le \text{RHS}. \tag{B.6}$$

Using the property of the DN operator $\mathcal{G}=N\cdot\nabla\mathcal{H}$, it suffices to prove that

$$\|\mathcal{G}(w\cdot N)\|_{H^{\kappa-\frac{5}{2}}(\Gamma_t)}=\|\mathcal{G}(N\otimes N:\nabla u)\|_{H^{\kappa-\frac{5}{2}}(\Gamma_t)}\le \text{RHS}. \tag{B.7}$$

Step 4. Denote the harmonic extension of N from Γ_t to Ω_t by $N_{\mathcal{H}}$. A direct computation shows that

$$(N\otimes N:\nabla^2)u\cdot N=N\cdot\nabla(N_{\mathcal{H}}\otimes N_{\mathcal{H}}:\nabla u)-N\cdot\nabla(N_{\mathcal{H}}\otimes N_{\mathcal{H}}):\nabla u.$$

As the result,

$$\begin{aligned}\mathcal{G}(w\cdot N)&=\mathcal{G}(N\otimes N:\nabla u)\\&=\underbrace{(N\otimes N:\nabla^2)u\cdot N}_{:=\mathrm{I}}+\underbrace{N\cdot\nabla(N_{\mathcal{H}}\otimes N_{\mathcal{H}}):\nabla u}_{:=\mathrm{II}}\\&\quad+\underbrace{N\cdot\nabla\Big\{\mathcal{H}(N\otimes N:\nabla u)-N_{\mathcal{H}}\otimes N_{\mathcal{H}}:\nabla u\Big\}}_{:=\mathrm{III}}.\end{aligned} \tag{B.8}$$

For the term II in (B.8),

$$\|\mathrm{II}\|_{H^{\kappa-\frac{5}{2}}(\Gamma_t)} \leq C(L)\|u\|_{H^{\kappa-1}(\Omega_t)}. \tag{B.9}$$

For the term III in (B.8), set

$$T = \mathcal{H}(N\otimes N:\nabla u) - N_{\mathcal{H}}\otimes N_{\mathcal{H}}:\nabla u.$$

One has $T|_{\Gamma_t} = T|_{\Gamma_-} = 0$ and

$$\|\Delta T\|_{H^{\kappa-3}(\Omega_t)} \leq C(L)\Big\{\|\nabla\times u\|_{H^{\kappa-1}(\Omega_t)} + \|\nabla\cdot u\|_{H^{\kappa-1}(\Omega_t)}\Big\},$$

which implies that

$$\|\mathrm{III}\|_{H^{\kappa-\frac{5}{2}}(\Omega_t)} \leq C(L)\Big\{\|\nabla\times u\|_{H^{\kappa-1}(\Omega_t)} + \|\nabla\cdot u\|_{H^{\kappa-1}(\Omega_t)}\Big\}. \tag{B.10}$$

Step 5. To cope with the term I in (B.8), we use the following relation:

$$\begin{aligned} N\otimes N = &(1+|\nabla f|^2)I - (1+|\partial_2 f|^2)\tau_1\otimes\tau_1 - (1+|\partial_1 f|^2)\tau_2\otimes\tau_2 \\ &+\partial_1 f\partial_2 f(\tau_1\otimes\tau_2 + \tau_2\otimes\tau_1). \end{aligned}$$

Since

$$\begin{aligned} (\tau_i\otimes\tau_j:\nabla^2)u\cdot N &= \overline{\partial}_i\overline{\partial}_j u\cdot N - \partial_i\partial_j f\partial_3 u\cdot N \\ &= \overline{\partial}_i(\overline{\partial}_j u\cdot N) - \overline{\partial}_j\cdot\overline{\partial}_i N - \partial_i\partial_j f\partial_3 u\cdot N, \end{aligned}$$

we have

$$\begin{aligned} &\big\|(\tau_i\otimes\tau_j:\nabla^2)u\cdot N\big\|_{H^{\kappa-\frac{5}{2}}(\Gamma_t)} \\ \leq &C(L)\Big\{\sum_{k=1,2}\|\overline{\partial}_k u\cdot N\|_{H^{\kappa-\frac{3}{2}}(\Gamma_t)} + \|u\|_{H^{\kappa-1}(\Omega_t)}\Big\}, \end{aligned}$$

and

$$\begin{aligned} &\big\|\big((1+|\nabla f|^2)I:\nabla^2\big)u\cdot N\big\|_{H^{\kappa-\frac{5}{2}}(\Gamma_t)} \\ =&\big\|(1+|\nabla f|^2)\Delta u\cdot N\big\|_{H^{\kappa-\frac{5}{2}}(\Gamma_t)} \\ \leq &C(L)\Big\{\|\nabla\times u\|_{H^{\kappa-1}(\Omega_t)} + \|\nabla\cdot u\|_{H^{\kappa-1}(\Omega_t)}\Big\}. \end{aligned}$$

Therefore

$$\|\mathrm{I}\|_{H^{\kappa-\frac{5}{2}}(\Gamma_t)} \leq \mathrm{RHS}. \tag{B.11}$$

Combining the estimates (B.4)-(B.7) and (B.9)-(B.11) finishes the proof. $\square$

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