3D Hyperbolic Navier-Stokes Equations in a Thin Strip: Global Well-Posedness and Hydrostatic Limit in Gevrey Space

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Abstract. We consider a hyperbolic version of three-dimensional anisotropic Navier-Stokes equations in a thin strip and its hydrostatic limit that is a hyperbolic Prandtl type equations. We prove the global-in-time existence and uniqueness for the two systems and the hydrostatic limit when the initial data belong to the Gevrey function space with index 2. The proof is based on a direct energy method by observing the damping effect in the systems.

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Key words: 3D hydrostatic Navier-Stokes equations, global well-posedness, Gevrey class, hydrostatic limit.

1   Introduction and the main result

There have been extensive studies on the well-posedness of the Prandtl type equations, while most of them are concerned with the local-in-time existence and uniqueness. Compared with the local theory, the global in time property is far from being well investigated. Here, we mention Xin-Zhang's work [51] on global

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weak solutions and some recent papers [1, 23, 36, 41–43, 50] on global analytic or Gevrey solutions. Note the above results are obtained mainly in the two-dimensional setting so that the global well-posedness of the three-dimensional case remains open.

In this paper, we aim to establish global well-posedness theories for some Prandtl type equations in the three-dimensional (3D) setting. Precisely, we will investigate the global-in-time existence and uniqueness of the hyperbolic version of 3D anisotropic Navier-Stokes equations and 3D hydrostatic Navier-Stokes equations. The proof relies on an observation that the vertical diffusion leads to a damping effect and the argument is a direct energy method. Note that this argument does not apply to the classical Prandtl equation because of the lack of Poincaré inequality in the half-space.

The system of hydrostatic Navier-Stokes equations plays an important role in the atmospheric and oceanic sciences and it describes the large scale motion of geophysical flow as a limit of Navier-Stokes equations in a thin domain where the vertical scale is significantly smaller than the horizontal one. By a proper rescaling (cf. [14, 43, 46] for instance and references therein), the 3D anisotropic Navier-Stokes equations in a thin domain read

$$\begin{cases} \left(\partial_t + u^\varepsilon \cdot \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \Delta_x - \partial_y^2\right) u^\varepsilon + \partial_x p^\varepsilon = 0, & (x,y) \in \mathbb{R}^2 \times]0,1[, \\ \varepsilon^2 \left(\partial_t + u^\varepsilon \cdot \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \Delta_x - \partial_y^2\right) v^\varepsilon + \partial_y p^\varepsilon = 0, & (x,y) \in \mathbb{R}^2 \times]0,1[, \\ \partial_x \cdot u^\varepsilon + \partial_y v^\varepsilon = 0, & (x,y) \in \mathbb{R}^2 \times]0,1[, \end{cases} \quad (1.1)$$

where $u^\varepsilon, v^\varepsilon$ stand for tangential and normal components of the velocity field respectively, and the viscosity coefficient is denoted by ε^2. In this paper, the above system is considered with the following no-slip Dirichlet boundary condition:

$$u^\varepsilon|_{y=0,1} = 0, \quad v^\varepsilon|_{y=0,1} = 0.$$

By letting $\varepsilon \to 0$, the first order approximation yields the following hydrostatic Navier-Stokes equations:

$$\begin{cases} \left(\partial_t + u \cdot \partial_x + v \partial_y - \partial_y^2\right) u + \partial_x p = 0, & (x,y) \in \mathbb{R}^2 \times]0,1[, \\ \partial_y p = 0, & (x,y) \in \mathbb{R}^2 \times]0,1[, \\ \partial_x \cdot u + \partial_y v = 0, & (x,y) \in \mathbb{R}^2 \times]0,1[, \\ u|_{y=0,1} = 0, \ v|_{y=0,1} = 0, & x \in \mathbb{R}^2, \\ u|_{t=0} = u_0^H, & (x,y) \in \mathbb{R}^2 \times]0,1[. \end{cases} \quad (1.2)$$

Here, v is a scalar function and $u = (u_1, u_2)$ is vector-valued, standing for the normal and the tangential velocity fields respectively. Compared with the Navier-Stokes equations, there is no time evolution equation for the normal velocity v

and the loss of tangential derivative property occurs in the non-local term v. This is the main degeneracy feature of the Prandtl type equations. Note that the classical Prandtl equations are considered in the half-space

$$\begin{cases} \left(\partial_t + u \cdot \partial_x + v\partial_y - \partial_y^2\right)u + \partial_x p = 0, & (x,y) \in \mathbb{R}^2 \times]0,+\infty[, \\ \partial_x \cdot u + \partial_y v = 0, & (x,y) \in \mathbb{R}^2 \times]0,+\infty[, \\ u|_{y=0} = 0, \; v|_{y=0} = 0, \; \lim_{y \to +\infty} u = U, & x \in \mathbb{R}^2, \\ u|_{t=0} = u_0^P, & (x,y) \in \mathbb{R}^2 \times]0,+\infty[, \end{cases} \tag{1.3}$$

where p and U are given by the trace of the Euler flow on the boundary.

Other Prandtl type equations include hydrostatic Euler equations and MHD boundary layer system. The former is an inviscid form of (1.2) and the latter is a system of Prandtl type equations on velocity and magnetic fields. For Prandtl type equations without structural assumption, there are results showing that either analyticity or Gevrey regularity is sufficient for the well-posedness, cf. [10, 21, 24, 30, 43, 48] and references therein. In particular, the Gevrey 2 function space is optimal for classical Prandtl equation [12], while the optimal index for MHD boundary layer system remains unknown. On the other hand, the analyticity is necessary for the well-posedness of the hydrostatic Navier-Stokes equations (1.2), cf. [46].

Recently, Aarach [1] and Paicu-Zhang [42] studied the hyperbolic version of 2D hydrostatic Navier-Stokes equations and established global solutions in analytic and Gevrey class 2, respectively. This shows the hyperbolic feature yields some stabilizing effect. Note that the hyperbolic version of the hydrostatic Navier-Stokes equations can be derived as a hydrostatic limit of the hyperbolic Navier-Stokes equations that was proposed by Cattaneo [5] to avoid the non-physical property of infinite propagation speed. There have been many results on the hyperbolic Navier-Stokes equations, cf. [2, 4, 8, 40, 44, 45]. By performing proper change of scales as in [42] we have the following hyperbolic version of the anisotropic Navier-Stokes equations (1.1):

$$\begin{cases} \left(\partial_t^2 + \partial_t + u^\varepsilon \cdot \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \Delta_x - \partial_y^2\right)u^\varepsilon + \partial_x p^\varepsilon = 0, & (x,y) \in \mathbb{R}^2 \times]0,1[, \\ \varepsilon^2 \left(\partial_t^2 + \partial_t + u^\varepsilon \cdot \partial_x + v^\varepsilon \partial_y - \varepsilon^2 \Delta_x - \partial_y^2\right)v^\varepsilon + \partial_y p^\varepsilon = 0, & (x,y) \in \mathbb{R}^2 \times]0,1[, \\ \partial_x \cdot u^\varepsilon + \partial_y v^\varepsilon = 0, & (x,y) \in \mathbb{R}^2 \times]0,1[, \\ u^\varepsilon|_{y=0,1} = 0, \; v^\varepsilon|_{y=0,1} = 0, & x \in \mathbb{R}^2, \\ (u^\varepsilon, v^\varepsilon)|_{t=0} = (u_0^\varepsilon, v_0^\varepsilon), \; (\partial_t u, \partial_t v)|_{t=0} = (u_1^\varepsilon, v_1^\varepsilon), & (x,y) \in \mathbb{R}^2 \times]0,1[. \end{cases} \tag{1.4}$$

Letting $\varepsilon \to 0$ in the above system (1.4) gives

$$\left(\partial_t^2 + \partial_t + u \cdot \partial_x + v\partial_y - \partial_y^2\right)u + \partial_x p = 0, \quad (x,y) \in \mathbb{R}^2 \times]0,1[, \tag{1.5a}$$

$$\partial_y p = 0, \qquad\qquad (x,y) \in \mathbb{R}^2 \times]0,1[, \qquad\qquad (1.5b)$$

$$\partial_x \cdot u + \partial_y v = 0, \qquad\qquad (x,y) \in \mathbb{R}^2 \times]0,1[, \qquad\qquad (1.5c)$$

$$u|_{y=0,1} = 0, \ \ v|_{y=0,1} = 0, \qquad x \in \mathbb{R}^2, \qquad\qquad (1.5d)$$

$$u|_{t=0} = u_0, \ \ \partial_t u|_{t=0} = u_1, \quad (x,y) \in \mathbb{R}^2 \times]0,1[, \qquad (1.5e)$$

which is a hyperbolic version of the hydrostatic Navier-Stokes equations (1.2). For this system, according to the recent work of Paicu-Zhang [42], we have well-posedness in Gevrey function space rather than analytic space. This is different from its parabolic version because analyticity is necessary for well-posedness of the hydrostatic Navier-Stokes equations without any structural assumption.

Before stating the main result on the global well-posedness for the hyperbolic version of the hydrostatic Navier-Stokes system (1.5), we briefly review some previous related works on the Prandtl type equations as follows.

1.1 Classical Prandtl equation

The well-posedness theory of Prandtl equation (1.3) has been well investigated, cf. [3, 6, 10–13, 18–20, 25, 26, 29, 32, 33, 35, 51–54] and the references therein. For the 2D case, under Oleinik's monotonicity condition the well-posedness theory in Sobolev space was justified in the pioneer work by Oleinik [39]. This classical result was revisited in two independent work of Alexandre *et al.* [3] and Masmoudi and Wong [38] by using energy method. However, the Sobolev well-posedness of 3D Prandtl equation under suitable structure condition remains unsolved despite some attempts like Liu *et al.* [33]. If Oleinik's monotonicity condition is violated, the ill-posedness and the related instability phenomena were well investigated, cf. [7, 9, 11, 12, 15–17, 32] and the references therein. Without any structural assumption, it is now well-understood that the Prandtl equation is well-posed in Gevrey class with optimal Gevrey index less or equal to 2 by the instability analysis of Dietert and Gérard-Varet [10] and the work on well-posedness by Dietert and Gérard-Varet [10] and Li *et al.* [24]. This generalizes the classical result of Sammartino and Caflisch [48] in the analytic framework. Similar well-posedness properties of hyperbolic Prandtl equations in Gevrey class were proven in [27].

On the other hand, in the fully nonlinear regime, Prandtl type system can be derived from the MHD system. In this regime, the tangential magnetic field has stabilizing effort as shown in the 2D case by Liu *et al.* [34] (see also [28, 31] for the further generalization), where the Sobolev well-posedness theory was established without Oleinik's monotonicity condition on the velocity field provided the tangential magnetic field dominates. Without any structural assumption, the

Gevrey well-posedness was studied in [30] with Gevrey index less or equal to $\frac{3}{2}$ that is not known to be optimal.

As for global-in-time existence of the classical Prandtl equation, there is an early work on weak solution by Xin and Zhang [51], and work on analytic solution by Paicu and Zhang [41], cf. also some other related work [19, 52, 54]. Recently, in [50] the authors also proved the global well-posedness property in Gevrey class 2. On the other hand, the global analytic solution to MHD boundary layer system was obtained by Liu and Zhang [36] and Li and Xie [23]. Note that all these global-in-time existence results are in 2D setting and some suitable structural condition on the initial data is required. Hence, the global property of these systems in 3D setting remains unknown.

1.2   Hydrostatic Navier-Stokes equations and related models

Compared with Prandtl equation, the hydrostatic Navier-Stokes equations (1.2) is less being well understood. In fact, the Sobolev well-posedness of the hydrostatic Navier-Stokes equations is still unclear. Under the convex assumption, only the Gevrey well-posedness has been obtained, cf. the recent work by Gérard-Varet *et al.* [14] with Gevrey index up to $\frac{9}{8}$ that seems not to be optimal. On the other hand, Masmoudi and Wong [37] proved that the convex condition is sufficient for the Sobolev well-posedness of hydrostatic Euler equations which is the inviscid form of hydrostatic Navier-Stokes equations. And Renardy [47] obtained the classical solutions to hydrostatic MHD equations provided the horizontal component of the magnetic field is not degenerate.

Furthermore, the global well-posedness property of the hydrostatic Navier-Stokes equations (1.2) was investigated by Paicu *et al.* [43] in analytic function space. In addition, the global well-posedness theory of the hyperbolic version of 2D hydrostatic Navier-Stokes equations (1.5) was established recently by Aarach [1] and Paicu and Zhang [42] in analytic and Gevrey function spaces respectively.

1.3   Statement of the main results

In this paper, we study the global Gevrey well-posedness of the hyperbolic version (1.5) for 3D hydrostatic Navier-Stokes system. For this, we first list some notations to be used.

Notation. In the following, we will use $\|\cdot\|_{L^2}$ and $(\cdot,\cdot)_{L^2}$ to denote the norm and inner product of $L^2 = L^2(\mathbb{R}^2 \times [0,1])$ and use the notation $\|\cdot\|_{L^2_x}$ and $\|\cdot,\cdot\|_{L^2_x}$ when the variable x is specified. Similar notation will be used for L^∞. In addition, we

use

$$L_x^p\left(L_y^q\right) = L^p\left(\mathbb{R}^2; L^q([0,1])\right)$$

for the classical Sobolev space. For a vector-valued function $A = (A_1, A_2, \ldots, A_n)$, we used the convention that

$$\|A\|_{L^2}^2 = \sum_{1 \le j \le n} \|A_j\|_{L^2}^2.$$

In the following discussion, we only require the Gevrey regularity in the tangential variable $x \in \mathbb{R}^2$. Precisely, the Gevrey function spaces are defined as follows.

Definition 1.1. *The space X_ρ of (partial) Gevrey functions consists of all smooth (scalar or vector-valued) functions $h(t, x)$ such that the norm $|h(t)|_{X_{\rho(t)}} < +\infty$, where*

$$|h|_{X_\rho}^2 = \sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho,m}^2 \left(\left\| \partial_t \partial_{x_j}^m h \right\|_{L^2}^2 + \left\| \partial_y \partial_{x_j}^m h \right\|_{L^2}^2 + \left\| \partial_{x_j}^m h \right\|_{L^2}^2 \right)$$

with

$$L_{\rho,m} = \frac{\rho^{m+1}(m+1)^7}{(m!)^2}, \quad m \ge 0, \quad \rho > 0. \tag{1.6}$$

In the following discussion, ρ depends on time but we only write it as ρ for simplicity of notations. On the other hand, if h is independent of t, then we use the notation

$$|h|_{X_{\rho*}}^2 = \sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho*,m}^2 \left(\left\| \partial_y \partial_{x_j}^m h \right\|_{L^2}^2 + \left\| \partial_{x_j}^m h \right\|_{L^2}^2 \right)$$

with ρ_* being a real number.

Remark 1.1. The norm $|h|_{X_\rho}$ defined above is equivalent to the standard Gevrey norm

$$\|h\|_\rho^2 = \sum_{\alpha \ge 0} L_{\rho,|\alpha|}^2 \left(\left\| \partial_t \partial_x^\alpha h \right\|_{L^2}^2 + \left\| \partial_y \partial_x^\alpha h \right\|_{L^2}^2 + \left\| \partial_x^\alpha h \right\|_{L^2}^2 \right),$$

in the sense that

$$\frac{1}{2} \|h\|_{\rho/2}^2 \le |h|_{X_\rho}^2 \le \|h\|_\rho^2,$$

where the last inequality is trivial and the first inequality follows from the fact that

$$\forall \alpha \in \mathbb{Z}_+^2, \quad \left\| \partial_x^\alpha u \right\|_{L^2}^2 \le \left\| \partial_{x_1}^{|\alpha|} u \right\|_{L^2}^2 + \left\| \partial_{x_2}^{|\alpha|} u \right\|_{L^2}^2. \tag{1.7}$$

Note the initial data u_0, u_1 of (1.5) satisfy the following compatibility condition:

$$\forall x \in \mathbb{R}^2, \quad u_0|_{y=0,1} = u_1|_{y=0,1} = 0, \quad \int_0^1 \partial_x \cdot u_0(x,y) dy = \int_0^1 \partial_x \cdot u_1(x,y) dy = 0. \quad (1.8)$$

The main results of this paper can now be stated as follows.

Theorem 1.1 (Global well-posedness of system (1.5)). *Let $(X_\rho, \|\cdot\|_{X_\rho})$ be given in Definition 1.1. If the initial data $u_0, u_1 \in X_{2\rho_0}$ for some $\rho_0 > 0$ satisfying the compatibility (1.8) and*

$$|u_0|_{X_{2\rho_0}} + |u_1|_{X_{2\rho_0}} \leq \varepsilon_0$$

for some small $\varepsilon_0 > 0$, then the hyperbolic version of 3D hydrostatic Navier-Stokes equations (1.5) admit a unique global-in-time solution $u \in C([0, +\infty[, X_\rho)$ provided ε_0 is sufficiently small. Moreover,

$$\forall t \geq 0, \quad |u(t)|_{X_{\rho(t)}} \leq 4\varepsilon_0 e^{-\frac{t}{32}},$$

where and throughout the paper

$$\rho(t) = \frac{\rho_0}{2} + \frac{\rho_0}{2} e^{-at}, \quad a = \frac{1}{96}. \quad (1.9)$$

If in addition $\partial_y u_0 \in X_{2\rho_0}$ with $|\partial_y u_0|_{X_{2\rho_0}} \leq \varepsilon_0$, then $\partial_t u \in C([0, +\infty[, X_\rho)$ and $\partial_y u \in C([0, +\infty[, X_{\rho(t)/2})$. Moreover, there exists a constant C, depending only on the Sobolev embedding constant, such that

$$\forall t \geq 0, \quad |\partial_t u(t)|_{X_{\rho(t)}} + |\partial_y u(t)|_{X_{\rho(t)/2}} \leq C\varepsilon_0 e^{-\frac{t}{32}}.$$

Theorem 1.2 (Global well-posedness of system (1.4)). *Suppose the initial data in (1.4) satisfy that $(u_j^\varepsilon, \varepsilon v_j^\varepsilon) \in X_{2\rho_0}, j = 0,1$, for some $\rho_0 > 0$, compatible to the boundary conditions in (1.4). Then the anisotropic hyperbolic Navier-Stokes equations (1.4) admit a unique global-in-time solution $(u^\varepsilon, \varepsilon v^\varepsilon) \in C([0, +\infty[, X_\rho)$, provided*

$$\left|(u_0^\varepsilon, \varepsilon v_0^\varepsilon)\right|_{X_{2\rho_0}} + \left|(u_1^\varepsilon, \varepsilon v_1^\varepsilon)\right|_{X_{2\rho_0}} \leq \delta_0$$

with δ_0 sufficiently small. Moreover,

$$\forall t \geq 0, \quad \left|(u^\varepsilon(t), \varepsilon v^\varepsilon(t))\right|_{X_{\rho(t)}} \leq 4\delta_0 e^{-\frac{t}{32}},$$

where ρ is defined by (1.9).

Theorem 1.3 (Hydrostatic limit). *Suppose all the assumptions in Theorems* 1.1 *and* 1.2 *hold, that is, the initial data of* (1.5) *and* (1.4) *satisfy*

$$|u_0|_{X_{2\rho_0}} + |u_1|_{X_{2\rho_0}} + |\partial_y u_0|_{X_{2\rho_0}} \leq \varepsilon_0,$$

and

$$\left|(u_0^\varepsilon, \varepsilon v_0^\varepsilon)\right|_{X_{2\rho_0}} + \left|(u_1^\varepsilon, \varepsilon v_1^\varepsilon)\right|_{X_{2\rho_0}} \leq \delta_0$$

for some small constants ε_0 and δ_0. Let u and $(u^\varepsilon, v^\varepsilon)$ be given in Theorems 1.1 *and* 1.2 *that solve* (1.5) *and* (1.4), *respectively. Then there exists a constant C, depending only on the constants $\varepsilon_0, \delta_0, \rho_0$ and the Sobolev embedding constant but independent of ε, such that*

$$\sup_{t \geq 0} |u^\varepsilon(t) - u(t)|_{X_{\rho(t)/2}} \leq C \left(\left|u_0^\varepsilon - u_0\right|_{X_{2\rho_0}} + \left|u_1^\varepsilon - u_1\right|_{X_{2\rho_0}} + \varepsilon \right).$$

Remark 1.2. The following analysis implies that the global well-posedness property holds for Gevrey function space with the Gevrey index less or equal to 2.

Remark 1.3. The proof given in this paper is based on a direct energy method that is substantially different from the elegant and subtle arguments used in [1,42] that involve the Littlewood-Paley decomposition.

2   Global well-posedness of hydrostatic system

In this section, we will prove Theorem 1.1 that is based on the proof of a priori estimates so that the existence and uniqueness follow by a standard argument. In fact, a self-contained proof consists of two parts. The first part is about the construction of approximate solutions that follows from the standard parabolic and hyperbolic theories. And then the uniform estimate on approximate solutions can be derived following a priori estimates. Hence, for brevity we only present the proof of a priori estimates for Gevrey solutions. Precisely, we will prove the following theorem.

Theorem 2.1 (A priori estimate). *If $u \in C([0, +\infty[, X_\rho)$ solves the hyperbolic version* (1.5) *of the hydrostatic Navier-Stokes equations with the initial data satisfying*

$$|u_0|_{X_{2\rho_0}} + |u_1|_{X_{2\rho_0}} \leq \epsilon_0 \tag{2.1}$$

for some small $\epsilon_0 > 0$, then

$$\forall t \geq 0, \quad |u(t)|_{X_{\rho(t)}} \leq 4\varepsilon_0 e^{-\frac{t}{32}}, \tag{2.2}$$

where the function ρ is defined by (1.9). Moreover, suppose

$$\left|\partial_y u_0\right|_{X_{2\rho_0}} \leq \varepsilon_0,$$

and $\partial_t u \in C([0,+\infty[, X_\rho), \partial_y u \in C([0,+\infty[, X_{\rho/2})$. Then there exists a constant C depending only on ρ_0 and the Sobolev embedding constant such that

$$\forall t \geq 0, \quad \left|\partial_t u(t)\right|_{X_{\rho(t)}} \leq C\varepsilon_0 e^{-\frac{t}{32}}, \tag{2.3}$$

$$\forall t \geq 0, \quad \left|\partial_y u(t)\right|_{X_{\rho(t)/2}} \leq C\varepsilon_0 e^{-\frac{t}{32}}. \tag{2.4}$$

We will use a bootstrap principle to prove the above a priori estimate. To do so, we first recall an abstract version of the bootstrap principle given in [49].

Proposition 2.1 ([49, Proposition 1.21]). *Let I be a time interval, and for each $t \in I$ we have two statements, a "hypothesis" $\mathbf{H}(t)$ and a "conclusion" $\mathbf{C}(t)$. Suppose we can verify the following four statements:*

(i) If $\mathbf{H}(t)$ is true for some time $t \in I$ then $\mathbf{C}(t)$ is also true for the time t.

(ii) If $\mathbf{C}(t)$ is true for some $t \in I$ then $\mathbf{H}(t')$ holds for all t' in a neighborhood of t.

(iii) If $t_1, t_2, \ldots$ is a sequence of times in I which converges to another time $t \in I$ and $\mathbf{C}(t_n)$ is true for all t_n, then $\mathbf{C}(t)$ is true.

(iv) $\mathbf{H}(t)$ is true for at least one time $t \in I$.

Then $\mathbf{C}(t)$ is true for all $t \in I$.

The rest of the section is to apply this bootstrap principle to obtain the a priori estimate in Theorem 2.1. The Gevrey class enables us to overcome the loss of tangential derivatives by shrinking the radius ρ. For this, we can either apply the abstract Cauchy-Kowalewski theorem (cf. [24, 29, 48] for instance) or use an auxiliary norm $|\cdot|_{Y_\rho}$ as in [22]. Here we will make use of the latter approach. Define

$$|h|_{Y_\rho}^2 = \sum_{j=1}^{2} \sum_{m=0}^{+\infty} L_{\rho,m}^2 \left[\frac{m+1}{\rho} \left(\left\|\partial_t \partial_{x_j}^m h\right\|_{L^2}^2 + \left\|\partial_y \partial_{x_j}^m h\right\|_{L^2}^2 \right) + \frac{(m+1)^3}{\rho^3} \left\|\partial_{x_j}^m h\right\|_{L^2}^2 \right] \tag{2.5}$$

for (scalar or vector-valued) functions h, where $L_{\rho,m}$ is given by (1.6).

2.1　Proof of Theorem 2.1: The first assertion

In this part we will present in details the proof of the first assertion (2.2). To simplify the notation, we assume without loss of generality that $\rho_0 \le 1$. In the following discussion, we will omit the time dependence of ρ in the notation, and denote by ρ' and ρ'' the first and the second order time derivatives respectively. Note that

$$\forall t \ge 0, \quad \frac{\rho_0}{2} \le \rho(t) \le \rho_0, \quad \rho'(t) \le \rho'^3(t) \le 0. \tag{2.6}$$

For each $t \in [0,+\infty[$, let $\mathbf{H}(t)$ be the statement

$$\forall s \in [0,t], \quad |u(s)|_{X_{\rho(s)}} \le 8\varepsilon_0 e^{-\frac{s}{32}}, \tag{2.7}$$

and let $\mathbf{C}(t)$ be the statement

$$\forall s \in [0,t], \quad |u(s)|_{X_{\rho(s)}} \le 4\varepsilon_0 e^{-\frac{s}{32}}. \tag{2.8}$$

Statements (ii)-(iv) in Proposition 2.1 follow from the continuity of $t \mapsto |u(t)|_{X_{\rho(t)}}$ and the condition (2.1). Then by Proposition 2.1, $\mathbf{C}(t)$ holds for all $t \in [0,+\infty[$ if we can show the following statement:

$$\mathbf{H}(t) \text{ is true for some time } t \in [0,\infty[\implies$$
$$\mathbf{C}(t) \text{ is also true for the same time } t. \tag{2.9}$$

We now turn to prove (2.9). In the following argument, we assume (2.7) holds with some fixed time t. Applying $\partial_{x_j}^m, j = 1,2$ to Eq. (1.5a) gives

$$(\partial_t^2 + \partial_t - \partial_y^2)\partial_{x_j}^m u$$
$$= -\partial_x \partial_{x_j}^m p - \sum_{k=0}^{m} \binom{m}{k} \left[(\partial_{x_j}^k u \cdot \partial_x)\partial_{x_j}^{m-k} u + (\partial_{x_j}^k v)\partial_{x_j}^{m-k}\partial_y u \right] \overset{def}{=} H_m. \tag{2.10}$$

Taking the L^2 inner product with $\partial_{x_j}^m u$ and $\partial_t \partial_{x_j}^m u$, respectively, on both sides of (2.10) and observing $u|_{y=0,1} = 0$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left[\frac{d}{dt}\|\partial_{x_j}^m u\|_{L^2}^2 + \|\partial_{x_j}^m u\|_{L^2}^2\right] + \|\partial_y \partial_{x_j}^m u\|_{L^2}^2 = \left(H_m, \partial_{x_j}^m u\right)_{L^2} + \|\partial_t \partial_{x_j}^m u\|_{L^2}^2, \tag{2.11}$$

$$\frac{d}{dt}\left(\|\partial_t \partial_{x_j}^m u\|_{L^2}^2 + \|\partial_y \partial_{x_j}^m u\|_{L^2}^2\right) + 2\|\partial_t \partial_{x_j}^m u\|_{L^2}^2 = 2\left(H_m, \partial_t \partial_{x_j}^m u\right)_{L^2}. \tag{2.12}$$

This yields by taking the summation of (2.11) and (2.12)

$$\frac{1}{2}\frac{d}{dt}\left[2\|\partial_t\partial_{x_j}^m u\|_{L^2}^2+2\|\partial_y\partial_{x_j}^m u\|_{L^2}^2+\frac{d}{dt}\|\partial_{x_j}^m u\|_{L^2}^2+\|\partial_{x_j}^m u\|_{L^2}^2\right]$$
$$+\|\partial_t\partial_{x_j}^m u\|_{L^2}^2+\|\partial_y\partial_{x_j}^m u\|_{L^2}^2$$
$$=\left(H_m,\partial_{x_j}^m u+2\partial_t\partial_{x_j}^m u\right)_{L^2}.$$

The above equality and the Poincaré inequality in the interval $[0,1]$

$$\frac{1}{4}\|\partial_{x_j}^m u\|_{L^2}^2\le\|\partial_y\partial_{x_j}^m u\|_{L^2}^2$$

because $u|_{y=0,1}=0$ give

$$\frac{1}{2}\frac{d}{dt}\left[2\|\partial_t\partial_{x_j}^m u\|_{L^2}^2+2\|\partial_y\partial_{x_j}^m u\|_{L^2}^2+\frac{d}{dt}\|\partial_{x_j}^m u\|_{L^2}^2+\|\partial_{x_j}^m u\|_{L^2}^2\right]$$
$$+\frac{1}{8}\|\partial_{x_j}^m u\|_{L^2}^2+\|\partial_t\partial_{x_j}^m u\|_{L^2}^2+\frac{1}{2}\|\partial_y\partial_{x_j}^m u\|_{L^2}^2$$
$$\le\left(H_m,\partial_{x_j}^m u+2\partial_t\partial_{x_j}^m u\right)_{L^2}.$$

Thus, by multiplying the above inequality by $L_{\rho,m}^2$ and observing

$$\frac{d}{dt}L_{\rho,m}^2=2\rho'\frac{m+1}{\rho}L_{\rho,m}^2,$$

taking summation over m gives

$$\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left(2\|\partial_t\partial_{x_j}^m u\|_{L^2}^2+2\|\partial_y\partial_{x_j}^m u\|_{L^2}^2+\frac{d}{dt}\|\partial_{x_j}^m u\|_{L^2}^2+\|\partial_{x_j}^m u\|_{L^2}^2\right)$$
$$+\frac{1}{8}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left(\|\partial_{x_j}^m u\|_{L^2}^2+\|\partial_t\partial_{x_j}^m u\|_{L^2}^2+\|\partial_y\partial_{x_j}^m u\|_{L^2}^2\right)$$
$$\le\sum_{m=0}^{+\infty}\rho'\frac{m+1}{\rho}L_{\rho,m}^2\left(2\|\partial_t\partial_{x_j}^m u\|_{L^2}^2+2\|\partial_y\partial_{x_j}^m u\|_{L^2}^2+\frac{d}{dt}\|\partial_{x_j}^m u\|_{L^2}^2+\|\partial_{x_j}^m u\|_{L^2}^2\right)$$
$$+\sum_{m=0}^{+\infty}L_{\rho,m}^2\left(H_m,\partial_{x_j}^m u+2\partial_t\partial_{x_j}^m u\right)_{L^2}$$
$$\le\sum_{m=0}^{+\infty}\rho'\frac{m+1}{\rho}L_{\rho,m}^2\left(\|\partial_t\partial_{x_j}^m u\|_{L^2}^2+\|\partial_y\partial_{x_j}^m u\|_{L^2}^2\right)+\sum_{m=0}^{+\infty}\rho'\frac{m+1}{\rho}L_{\rho,m}^2\|\partial_t\partial_{x_j}^m u\|_{L^2}^2$$

$$+\sum_{m=0}^{+\infty}\rho'\frac{m+1}{\rho}L_{\rho,m}^2\frac{d}{dt}\left\|\partial_{x_j}^m u\right\|_{L^2}^2+\sum_{m=0}^{+\infty}L_{\rho,m}^2\left|\left(H_m,\partial_{x_j}^m u\right)_{L^2}\right|$$

$$+2\sum_{m=0}^{+\infty}L_{\rho,m}^2\left|\left(H_m,\partial_t\partial_{x_j}^m u\right)_{L^2}\right|, \tag{2.13}$$

where the last inequality holds because of $\rho'\leq 0$. Moreover, for the second summation term on the right hand side of (2.13), by noting $\rho'\leq 0$, we have

$$\rho'\frac{m+1}{\rho}L_{\rho,m}^2\left\|\partial_t\partial_{x_j}^m u\right\|_{L^2}^2$$

$$=\rho'\frac{m+1}{\rho}\left\|\partial_t\left(L_{\rho,m}\partial_{x_j}^m u\right)\right\|_{L^2}^2+\rho'^3\frac{(m+1)^3}{\rho^3}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2$$

$$-2\rho'^2\frac{(m+1)^2}{\rho^2}\left(\partial_t\left(L_{\rho,m}\partial_{x_j}^m u\right),L_{\rho,m}\partial_{x_j}^m u\right)_{L^2}$$

$$\leq\rho'^3\frac{(m+1)^3}{\rho^3}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2-\rho'^2\frac{(m+1)^2}{\rho^2}\frac{d}{dt}\left\|L_{\rho,m}\partial_{x_j}^m u\right\|_{L^2}^2.$$

Here, the last term can be written as

$$-\rho'^2\frac{(m+1)^2}{\rho^2}\frac{d}{dt}\left\|L_{\rho,m}\partial_{x_j}^m u\right\|_{L^2}^2$$

$$=-\frac{d}{dt}\left(\rho'^2\frac{(m+1)^2}{\rho^2}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2\right)$$

$$+2\rho'\left(\rho''-\frac{\rho'^2}{\rho}\right)\frac{(m+1)^2}{\rho^2}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2$$

$$\leq-\frac{d}{dt}\left(\rho'^2\frac{(m+1)^2}{\rho^2}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2\right),$$

where the last inequality holds because

$$\rho''-\frac{\rho'^2}{\rho}=\frac{\rho_0 a^2 e^{-at}}{2(1+e^{-at})}\geq 0. \tag{2.14}$$

Combining the above inequalities yields

$$\rho'\frac{m+1}{\rho}L_{\rho,m}^2\left\|\partial_t\partial_{x_j}^m u\right\|_{L^2}^2$$

$$\leq\rho'^3\frac{(m+1)^3}{\rho^3}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2-\frac{d}{dt}\left(\rho'^2\frac{(m+1)^2}{\rho^2}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2\right).$$

For the third summation term on the right-hand side of (2.13), direct computation gives

$$
\begin{aligned}
&\rho'\frac{m+1}{\rho}L_{\rho,m}^2\frac{d}{dt}\big\|\partial_{x_j}^m u\big\|_{L^2}^2 \\
&=\frac{d}{dt}\left(\rho'\frac{m+1}{\rho}L_{\rho,m}^2\big\|\partial_{x_j}^m u\big\|_{L^2}^2\right)-\left(\rho''-\frac{\rho'^2}{\rho}\right)\frac{m+1}{\rho}L_{\rho,m}^2\big\|\partial_{x_j}^m u\big\|_{L^2}^2 \\
&\quad-2\rho'^2\frac{(m+1)^2}{\rho^2}L_{\rho,m}^2\big\|\partial_{x_j}^m u\big\|_{L^2}^2 \\
&\leq\frac{d}{dt}\left(\rho'\frac{m+1}{\rho}L_{\rho,m}^2\big\|\partial_{x_j}^m u\big\|_{L^2}^2\right)-2\rho'^2\frac{(m+1)^2}{\rho^2}L_{\rho,m}^2\big\|\partial_{x_j}^m u\big\|_{L^2}^2,
\end{aligned}
$$

where (2.14) is also used in the last inequality. Now we substitute the above two estimates into (2.13) to have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left(2\big\|\partial_t\partial_{x_j}^m u\big\|_{L^2}^2+2\big\|\partial_y\partial_{x_j}^m u\big\|_{L^2}^2+\frac{d}{dt}\big\|\partial_{x_j}^m u\big\|_{L^2}^2+\big\|\partial_{x_j}^m u\big\|_{L^2}^2\right) \\
&\quad+\frac{d}{dt}\sum_{m=0}^{+\infty}L_{\rho,m}^2\big\|\partial_{x_j}^m u\big\|_{L^2}^2\left(\rho'^2\frac{(m+1)^2}{\rho^2}-\rho'\frac{m+1}{\rho}\right) \\
&\quad+\frac{1}{8}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left(\big\|\partial_{x_j}^m u\big\|_{L^2}^2+\big\|\partial_t\partial_{x_j}^m u\big\|_{L^2}^2+\big\|\partial_y\partial_{x_j}^m u\big\|_{L^2}^2\right) \\
&\quad+2\sum_{m=0}^{+\infty}\rho'^2\frac{(m+1)^2}{\rho^2}L_{\rho,m}^2\big\|\partial_{x_j}^m u\big\|_{L^2}^2 \\
&\leq\rho'\sum_{m=0}^{+\infty}\frac{m+1}{\rho}L_{\rho,m}^2\left(\big\|\partial_t\partial_{x_j}^m u\big\|_{L^2}^2+\big\|\partial_y\partial_{x_j}^m u\big\|_{L^2}^2\right)+\rho'^3\sum_{m=0}^{+\infty}\frac{(m+1)^3}{\rho^3}L_{\rho,m}^2\big\|\partial_{x_j}^m u\big\|_{L^2}^2 \\
&\quad+\sum_{m=0}^{+\infty}L_{\rho,m}^2\big|\big(H_m,\partial_{x_j}^m u\big)_{L^2}\big|+2\sum_{m=0}^{+\infty}L_{\rho,m}^2\big|\big(H_m,\partial_t\partial_{x_j}^m u\big)_{L^2}\big| \\
&\leq\rho'^3\sum_{m=0}^{+\infty}L_{\rho,m}^2\left[\frac{m+1}{\rho}\left(\big\|\partial_t\partial_{x_j}^m u\big\|_{L^2}^2+\big\|\partial_y\partial_{x_j}^m u\big\|_{L^2}^2\right)+\frac{(m+1)^3}{\rho^3}\big\|\partial_{x_j}^m u\big\|_{L^2}^2\right] \\
&\quad+\sum_{m=0}^{+\infty}L_{\rho,m}^2\big|\big(H_m,\partial_{x_j}^m u\big)_{L^2}\big|+2\sum_{m=0}^{+\infty}L_{\rho,m}^2\big|\big(H_m,\partial_t\partial_{x_j}^m u\big)_{L^2}\big|,
\end{aligned}
\tag{2.15}
$$

where (2.6) is used in the last inequality. We claim that there exists a constant C depending only on the Sobolev embedding constant such that

$$\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^{2}\big|\big(H_{m},\partial_{x_{j}}^{m}u\big)_{L^{2}}\big|+2\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^{2}\big|\big(H_{m},\partial_{t}\partial_{x_{j}}^{m}u\big)_{L^{2}}\big|$$

$$\leq C\rho^{-2}|u|_{X_{\rho}}|u|_{Y_{\rho}}^{2} \tag{2.16}$$

with $|u|_{Y_{\rho}}$ defined by (2.5) and $|u|_{X_{\rho}}$ in Definition 1.1.

For brevity of presentation, the proof of the statement (2.16) is postponed to the end of this section. We combine (2.16) with the fact that

$$\rho'^{3}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^{2}\left[\frac{m+1}{\rho}\left(\big\|\partial_{t}\partial_{x_{j}}^{m}u\big\|_{L^{2}}^{2}+\big\|\partial_{y}\partial_{x_{j}}^{m}u\big\|_{L^{2}}^{2}\right)+\frac{(m+1)^{3}}{\rho^{3}}\big\|\partial_{x_{j}}^{m}u\big\|_{L^{2}}^{2}\right]=\rho'^{3}|u|_{Y_{\rho}}^{2}$$

to conclude for any $s\in[0,t]$ with the same time t given in (2.7), the following estimate holds:

$$\rho'(s)^{3}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho(s),m}^{2}$$

$$\times\left[\frac{m+1}{\rho(s)}\left(\big\|\partial_{t}\partial_{x_{j}}^{m}u(s)\big\|_{L^{2}}^{2}+\big\|\partial_{y}\partial_{x_{j}}^{m}u(s)\big\|_{L^{2}}^{2}\right)+\frac{(m+1)^{3}}{\rho(s)^{3}}\big\|\partial_{x_{j}}^{m}u(s)\big\|_{L^{2}}^{2}\right]$$

$$+\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho(s),m}^{2}\big|\big(H_{m}(s),\partial_{x_{j}}^{m}u(s)\big)_{L^{2}}\big|$$

$$+2\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^{2}\big|\big(H_{m}(s),\partial_{t}\partial_{x_{j}}^{m}u(s)\big)_{L^{2}}\big|$$

$$\leq\left(\rho'(s)^{3}+C\rho(s)^{-2}|u(s)|_{X_{\rho(s)}}\right)|u(s)|_{Y_{\rho(s)}}^{2}$$

$$\leq-\left(\frac{\rho_{0}^{3}a^{3}}{8}-32\varepsilon_{0}C\rho_{0}^{-2}\right)e^{-\frac{s}{32}}|u(s)|_{Y_{\rho(s)}}^{2}\leq0,$$

where the last inequality follows from the condition (2.7) and the property (2.6) by choosing ε_{0} in (2.7) to be sufficiently small. As the result, we take summation on both sides of (2.15) for $j=1,2$ to obtain

$$\frac{1}{2}\frac{d}{dt}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^{2}\left(2\big\|\partial_{t}\partial_{x_{j}}^{m}u\big\|_{L^{2}}^{2}+2\big\|\partial_{y}\partial_{x_{j}}^{m}u\big\|_{L^{2}}^{2}+\frac{d}{dt}\big\|\partial_{x_{j}}^{m}u\big\|_{L^{2}}^{2}+\big\|\partial_{x_{j}}^{m}u\big\|_{L^{2}}^{2}\right)$$

$$+\frac{d}{dt}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^{2}\big\|\partial_{x_{j}}^{m}u\big\|_{L^{2}}^{2}\left(\rho'^{2}\frac{(m+1)^{2}}{\rho^{2}}-\rho'\frac{m+1}{\rho}\right)$$

$$+\frac{1}{8}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left(\left\|\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_t\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_y\partial_{x_j}^m u\right\|_{L^2}^2\right)$$

$$+2\sum_{j=1}^{2}\sum_{m=0}^{+\infty}\rho'^2\frac{(m+1)^2}{\rho^2}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2\leq 0.$$

Thus, multiplying the both sides by $e^{\frac{t}{16}}$ gives

$$\frac{1}{2}\frac{d}{dt}e^{\frac{t}{16}}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left(2\left\|\partial_t\partial_{x_j}^m u\right\|_{L^2}^2+2\left\|\partial_y\partial_{x_j}^m u\right\|_{L^2}^2+\frac{d}{dt}\left\|\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_{x_j}^m u\right\|_{L^2}^2\right)$$

$$+\frac{d}{dt}e^{\frac{t}{16}}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2\left(\rho'^2\frac{(m+1)^2}{\rho^2}-\rho'\frac{m+1}{\rho}\right)+\mathcal{A}\leq 0 \qquad (2.17)$$

with

$$\mathcal{A}=-\frac{1}{32}e^{\frac{t}{16}}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left(2\left\|\partial_t\partial_{x_j}^m u\right\|_{L^2}^2+2\left\|\partial_y\partial_{x_j}^m u\right\|_{L^2}^2+\frac{d}{dt}\left\|\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_{x_j}^m u\right\|_{L^2}^2\right)$$

$$-\frac{1}{16}e^{\frac{t}{16}}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2\left(\rho'^2\frac{(m+1)^2}{\rho^2}-\rho'\frac{m+1}{\rho}\right)$$

$$+\frac{1}{8}e^{\frac{t}{16}}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}L_{\rho,m}^2\left(\left\|\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_t\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_y\partial_{x_j}^m u\right\|_{L^2}^2\right)$$

$$+2e^{\frac{t}{16}}\sum_{j=1}^{2}\sum_{m=0}^{+\infty}\rho'^2\frac{(m+1)^2}{\rho^2}L_{\rho,m}^2\left\|\partial_{x_j}^m u\right\|_{L^2}^2.$$

By noting that
$$\frac{2|\rho'|(m+1)}{\rho}\leq 1+\frac{\rho'^2(m+1)^2}{\rho^2}$$

and

$$\frac{1}{4}\left(\left\|\partial_t\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_y\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_{x_j}^m u\right\|_{L^2}^2\right)$$

$$\leq 2\left\|\partial_t\partial_{x_j}^m u\right\|_{L^2}^2+2\left\|\partial_y\partial_{x_j}^m u\right\|_{L^2}^2+\frac{d}{dt}\left\|\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_{x_j}^m u\right\|_{L^2}^2$$

$$\leq 3\left(\left\|\partial_t\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_y\partial_{x_j}^m u\right\|_{L^2}^2+\left\|\partial_{x_j}^m u\right\|_{L^2}^2\right), \qquad (2.18)$$

we have $\mathcal{A}\geq 0$.

We now integrate both sides of (2.17) over $[0,s]$ for all $s \leq t$. By the above inequality, (2.18) and the fact that $\rho' \leq 0$, we obtain

$$
e^{\frac{s}{16}} \sum_{j=1}^{2} \sum_{m=0}^{+\infty} L^2_{\rho(s),m} \left(\left\| \partial_{x_j}^m u(s) \right\|_{L^2}^2 + \left\| \partial_t \partial_{x_j}^m u(s) \right\|_{L^2}^2 + \left\| \partial_y \partial_{x_j}^m u(s) \right\|_{L^2}^2 \right)
$$

$$
\leq 12 \left(|u_0|_{X_{\rho_0}}^2 + |u_1|_{X_{\rho_0}}^2 \right) + 8 \sum_{j=1}^{2} \sum_{m=0}^{+\infty} L^2_{\rho_0,m} \left\| \partial_{x_j}^m u_0 \right\|_{L^2}^2 \left(\rho'(0)^2 \frac{(m+1)^2}{\rho_0^2} - \rho'(0) \frac{m+1}{\rho_0} \right)
$$

$$
\leq 16 \left(|u_0|_{X_{2\rho_0}}^2 + |u_1|_{X_{2\rho_0}}^2 \right), \tag{2.19}
$$

that is, by Definition 1.1 and (2.1),

$$
\forall s \in [0,t], \quad |u(s)|_{X_{\rho(s)}} \leq 4e^{-\frac{s}{32}} \left(|u_0|_{X_{2\rho_0}} + |u_1|_{X_{2\rho_0}} \right) \leq 4\varepsilon_0 e^{-\frac{s}{32}}.
$$

Then we have (2.9). Hence, $\mathbf{C}(t)$ in (2.8) holds for all $t \geq 0$ by Proposition 2.1 so that (2.2) holds. We have proven the first assertion (2.2) in Theorem 2.1.

Now it remains to prove (2.16) as follows.

Proof. Recall H_m given by (2.10). We first estimate the terms involving the pressure function. Firstly, note that the following compatibility condition for the solution u to (1.5) holds:

$$
\forall x \in \mathbb{R}^2, \quad \int_0^1 \partial_x \cdot u(x,y) dy = 0. \tag{2.20}
$$

Then integrating by parts gives

$$
\left| \left(\partial_x \partial_{x_j}^m p, \, \partial_{x_j}^m u \right)_{L^2} \right| = \left| \left(\partial_{x_j}^m p, \, \partial_{x_j}^m \partial_x \cdot u \right)_{L^2} \right| = \left| \left(\partial_{x_j}^m p, \, \partial_y \partial_{x_j}^m v \right)_{L^2} \right| = 0, \tag{2.21}
$$

where the last equality follows from integration by parts and the fact that $\partial_y p = 0$. The above equality also holds when $\partial_{x_j}^m u$ is replaced by $\partial_t \partial_{x_j}^m u$. That is,

$$
\left| \left(\partial_x \partial_{x_j}^m p, \partial_{x_j}^m u \right)_{L^2} \right| + \left| \left(\partial_x \partial_{x_j}^m p, \partial_t \partial_{x_j}^m u \right)_{L^2} \right| = 0. \tag{2.22}
$$

It remains to estimate the other terms in H_m and show that

$$
\sum_{m=0}^{+\infty} L^2_{\rho,m} \left| \left(\sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \left[(\partial_{x_j}^k u \cdot \partial_x) \partial_{x_j}^{m-k} u + (\partial_{x_j}^k v) \partial_{x_j}^{m-k} \partial_y u \right], \partial_t \partial_{x_j}^m u \right)_{L^2} \right|
$$

$$
+ \sum_{m=0}^{+\infty} L^2_{\rho,m} \left| \left(\sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \left[(\partial_{x_j}^k u \cdot \partial_x) \partial_{x_j}^{m-k} u + (\partial_{x_j}^k v) \partial_{x_j}^{m-k} \partial_y u \right], \partial_{x_j}^m u \right)_{L^2} \right|
$$

$$
\leq C |u|_{X_\rho} |u|_{Y_\rho}^2. \tag{2.23}
$$

To prove (2.23), we write

$$\sum_{m=0}^{+\infty} L_{\rho,m}^2 \left| \left(\sum_{k=0}^{m} \frac{m!}{k!(m-k)!} \left[(\partial_{x_j}^k u \cdot \partial_x) \partial_{x_j}^{m-k} u + (\partial_{x_j}^k v) \partial_{x_j}^{m-k} \partial_y u \right], \partial_t \partial_{x_j}^m u \right)_{L^2} \right|$$
$$\leq S_1 + S_2, \tag{2.24}$$

where

$$\begin{cases} S_1 = \sum_{m=0}^{+\infty} \sum_{k=0}^{m} L_{\rho,m}^2 \dfrac{m!}{k!(m-k)!} \left\| (\partial_{x_j}^k u \cdot \partial_x) \partial_{x_j}^{m-k} u \right\|_{L^2} \left\| \partial_t \partial_{x_j}^m u \right\|_{L^2}, \\[4mm] S_2 = \sum_{m=0}^{+\infty} \sum_{k=0}^{m} L_{\rho,m}^2 \dfrac{m!}{k!(m-k)!} \left\| (\partial_{x_j}^k v) \partial_y \partial_{x_j}^{m-k} u \right\|_{L^2} \left\| \partial_t \partial_{x_j}^m u \right\|_{L^2}. \end{cases} \tag{2.25}$$

We first estimate S_1 as follows.

$$S_1 \leq \sum_{m=0}^{+\infty} \sum_{k=0}^{[\frac{m}{2}]} L_{\rho,m}^2 \frac{m!}{k!(m-k)!} \left\| \partial_{x_j}^k u \right\|_{L^\infty} \left\| \partial_x \partial_{x_j}^{m-k} u \right\|_{L^2} \left\| \partial_t \partial_{x_j}^m u \right\|_{L^2}$$
$$+ \sum_{m=0}^{+\infty} \sum_{k=[\frac{m}{2}]+1}^{m} L_{\rho,m}^2 \frac{m!}{k!(m-k)!} \left\| \partial_{x_j}^k u \right\|_{L^2} \left\| \partial_x \partial_{x_j}^{m-k} u \right\|_{L^\infty} \left\| \partial_t \partial_{x_j}^m u \right\|_{L^2}$$
$$=: S_{1,1} + S_{1,2}, \tag{2.26}$$

where $[\frac{m}{2}]$ stands for the largest integer $\leq \frac{m}{2}$. To estimate $S_{1,1}$ and $S_{1,2}$, we will use the following inequalities that follow from straightforward calculation. If $0 \leq k \leq [\frac{m}{2}]$, then

$$\frac{m!}{k!(m-k)!} \frac{L_{\rho,m}}{L_{\rho,k+2} L_{\rho,m-k+1}}$$
$$= \frac{m!}{k!(m-k)!} \frac{[(k+2)!]^2}{\rho^{k+3}(k+3)^7} \frac{[(m-k+1)!]^2}{\rho^{m-k+2}(m-k+2)^7} \frac{\rho^{m+1}(m+1)^7}{(m!)^2}$$
$$\leq C \frac{k!(m-k+1)!(m-k+1)}{m!(k+3)\rho^4} \leq C \frac{(m-k+1)m}{(k+1)\rho^4}$$
$$\leq \frac{C}{\rho^2} \frac{1}{k+1} \frac{(m+1)^{\frac{1}{2}}}{\rho^{\frac{1}{2}}} \frac{(m-k+2)^{\frac{3}{2}}}{\rho^{\frac{3}{2}}}. \tag{2.27}$$

On the other hand, if $[\frac{m}{2}]+1 \leq k \leq m$, then

$$\frac{m!}{k!(m-k)!} \frac{L_{\rho,m}}{L_{\rho,k} L_{\rho,m-k+3}} \leq \frac{C}{\rho^4} \frac{1}{m-k+1}. \tag{2.28}$$

Recalling $S_{1,1}$ given in (2.26), by (2.27) and the definition of $|\cdot|_{Y_\rho}$ in (2.5), we have

$$
\begin{aligned}
S_{1,1} &= \sum_{m=0}^{+\infty} \sum_{k=0}^{[\frac{m}{2}]} \frac{m!}{k!(m-k)!} \frac{L_{\rho,m}}{L_{\rho,k+2} L_{\rho,m-k+1}} \left(L_{\rho,k+2} \left\| \partial_{x_j}^k u \right\|_{L^\infty} \right) \\
&\quad \times \left(L_{\rho,m-k+1} \left\| \partial_x \partial_{x_j}^{m-k} u \right\|_{L^2} \right) L_{\rho,m} \left\| \partial_t \partial_{x_j}^m u \right\|_{L^2} \\
&\le \frac{C}{\rho^2} \sum_{m=0}^{+\infty} \sum_{k=0}^{[\frac{m}{2}]} \frac{L_{\rho,k+2} \left\| \partial_{x_j}^k u \right\|_{L^\infty}}{k+1} \left(\frac{(m-k+2)^{\frac{3}{2}}}{\rho^{\frac{3}{2}}} L_{\rho,m-k+1} \left\| \partial_x \partial_{x_j}^{m-k} u \right\|_{L^2} \right) \\
&\quad \times \left(\frac{(m+1)^{\frac{1}{2}}}{\rho^{\frac{1}{2}}} L_{\rho,m} \left\| \partial_t \partial_{x_j}^m u \right\|_{L^2} \right) \\
&\le \frac{C}{\rho^2} \left[\sum_{m=0}^{+\infty} \left(\sum_{k=0}^{[\frac{m}{2}]} \frac{L_{\rho,k+2} \left\| \partial_{x_j}^k u \right\|_{L^\infty}}{k+1} \times \frac{(m-k+2)^{\frac{3}{2}}}{\rho^{\frac{3}{2}}} L_{\rho,m-k+1} \left\| \partial_x \partial_{x_j}^{m-k} u \right\|_{L^2} \right)^2 \right]^{\frac{1}{2}} |u|_{Y_\rho} \\
&\le \frac{C}{\rho^2} \sum_{k=0}^{\infty} \frac{L_{\rho,k+2} \left\| \partial_{x_j}^k u \right\|_{L^\infty}}{k+1} \left[\sum_{m=0}^{+\infty} \frac{(m+2)^3}{\rho^3} L_{\rho,m+1}^2 \left\| \partial_x \partial_{x_j}^m u \right\|_{L^2}^2 \right]^{\frac{1}{2}} |u|_{Y_\rho}, \qquad (2.29)
\end{aligned}
$$

where we have used Young's inequality for discrete convolution. Moreover, it follows from (1.7) that

$$
\begin{aligned}
& \left[\sum_{m=0}^{+\infty} \frac{(m+2)^3}{\rho^3} L_{\rho,m+1}^2 \left\| \partial_x \partial_{x_j}^m u \right\|_{L^2}^2 \right]^{\frac{1}{2}} \\
& \le \left[\sum_{m=0}^{+\infty} \frac{(m+1)^3}{\rho^3} L_{\rho,m}^2 \left(\left\| \partial_{x_1}^m u \right\|_{L^2}^2 + \left\| \partial_{x_2}^m u \right\|_{L^2}^2 \right) \right]^{\frac{1}{2}} \le |u|_{Y_\rho}.
\end{aligned}
$$

It follows from the Sobolev embedding inequality

$$
\| F \|_{L^\infty} \le C \left(\| F \|_{H_x^2(L_y^2)} + \| \partial_y F \|_{H_x^2(L_y^2)} \right)
$$

and (1.7) that

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \frac{L_{\rho,k+2} \left\| \partial_{x_j}^k u \right\|_{L^\infty}}{k+1} \\
& \le C \sum_{k=0}^{\infty} \frac{1}{k+1} L_{\rho,k+2} \left(\left\| \partial_{x_j}^k u \right\|_{H_x^2(L_y^2)} + \left\| \partial_y \partial_{x_j}^k u \right\|_{H_x^2(L_y^2)} \right)
\end{aligned}
$$

$$\leq C \left[\sum_{k=0}^{\infty} L_{\rho,k+2}^2 \left(\left\| \partial_{x_j}^k u \right\|_{H_x^2(L_y^2)}^2 + \left\| \partial_y \partial_{x_j}^k u \right\|_{H_x^2(L_y^2)}^2 \right) \right]^{\frac{1}{2}} \leq C |u|_{X_\rho}.$$

Hence, combining the above estimates with (2.29) yields

$$S_{1,1} \leq C\rho^{-2} |u|_{X_\rho} |u|_{Y_\rho}^2.$$

Similarly, by using (2.28), we have

$$S_{1,2} \leq C\rho^{-2} |u|_{X_\rho} |u|_{Y_\rho}^2.$$

Substituting the estimates on $S_{1,1}$ and $S_{1,2}$ into (2.26) yields

$$S_1 \leq C\rho^{-2} |u|_{X_\rho} |u|_{Y_\rho}^2.$$

The term S_2 in (2.25) can be estimated similarly and we omit the details for brevity, that is,

$$S_2 \leq C\rho^{-2} |u|_{X_\rho} |u|_{Y_\rho}^2.$$

In summary, in view of (2.24) we have

$$\sum_{m=0}^{+\infty} L_{\rho,m}^2 \left| \left(\sum_{k=0}^m \frac{m!}{k!(m-k)!} \left[(\partial_{x_j}^k u \cdot \partial_x) \partial_{x_j}^{m-k} u + (\partial_{x_j}^k v) \partial_{x_j}^{m-k} \partial_y u \right], \partial_t \partial_{x_j}^m u \right)_{L^2} \right|$$
$$\leq C\rho^{-2} |u|_{X_\rho} |u|_{Y_\rho}^2.$$

Similarly,

$$\sum_{m=0}^{+\infty} L_{\rho,m}^2 \left| \left(\sum_{k=0}^m \frac{m!}{k!(m-k)!} \left[(\partial_{x_j}^k u \cdot \partial_x) \partial_{x_j}^{m-k} u + (\partial_{x_j}^k v) \partial_{x_j}^{m-k} \partial_y u \right], \partial_{x_j}^m u \right)_{L^2} \right|$$
$$\leq C\rho^{-2} |u|_{X_\rho} |u|_{Y_\rho}^2.$$

Therefore, the statement (2.16) holds. $\qquad\square$

2.2   Proof of Theorem 2.1: The second assertion

Since the argument is similar to the one used in the previous section, we now sketch the proof of (2.3) in Theorem 2.1 for brevity. In fact, using the notation

$$\mathcal{U} = (u, \partial_t u), \quad \mathcal{H}_m = (H_m, \partial_t H_m)$$

with H_m defined in (2.10), we have by applying $\partial_t \partial_{x_j}^m$ to (1.5),

$$(\partial_t^2 + \partial_t - \partial_y^2) \partial_{x_j}^m \mathcal{U} = \mathcal{H}_m. \tag{2.30}$$

Similar to (2.16), we conclude

$$\sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho,m}^2 \left| (\mathcal{H}_m, \partial_{x_j}^m \mathcal{U})_{L^2} \right| + 2 \sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho,m}^2 \left| (\mathcal{H}_m, \partial_t \partial_{x_j}^m \mathcal{U})_{L^2} \right|$$
$$\leq C \rho^{-2} |\mathcal{U}|_{X_\rho} |\mathcal{U}|_{Y_\rho}^2. \tag{2.31}$$

In fact, as (2.21), the equations

$$\left(\partial_t \partial_x \partial_{x_j}^m p, \partial_{x_j}^m u \right)_{L^2} = \left(\partial_t \partial_x \partial_{x_j}^m p, \partial_t \partial_{x_j}^m u \right)_{L^2} = \left(\partial_t \partial_x \partial_{x_j}^m p, \partial_t^2 \partial_{x_j}^m u \right)_{L^2} = 0$$

also hold. Hence,

$$\left(\partial_t \partial_x \partial_{x_j}^m p, \partial_{x_j}^m \mathcal{U} + 2 \partial_t \partial_{x_j}^m \mathcal{U} \right)_{L^2} = 0.$$

As a result, (2.31) follows by repeating the argument for obtaining (2.16).

We now take the inner product with $\partial_{x_j}^m \mathcal{U} + 2 \partial_t \partial_{x_j}^m \mathcal{U}$ on both sides of Eq. (2.30), and then repeat the argument for proving (2.2) with $\partial_{x_j}^m u$ replaced by $\partial_{x_j}^m \mathcal{U}$. By (2.31), we have the following estimate that is similar to (2.19):

$$e^{\frac{t}{16}} |\mathcal{U}(t)|_{X_\rho}^2 = e^{\frac{t}{16}} \sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho(t),m}^2 \left(\left\| \partial_{x_j}^m \mathcal{U}(t) \right\|_{L^2}^2 + \left\| \partial_t \partial_{x_j}^m \mathcal{U}(t) \right\|_{L^2}^2 + \left\| \partial_y \partial_{x_j}^m \mathcal{U}(t) \right\|_{L^2}^2 \right)$$

$$\leq 12 \sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho_0,m}^2 \left(\left\| \partial_{x_j}^m \mathcal{U} |_{t=0} \right\|_{L^2}^2 + \left\| \partial_t \partial_{x_j}^m \mathcal{U} |_{t=0} \right\|_{L^2}^2 + \left\| \partial_y \partial_{x_j}^m \mathcal{U} |_{t=0} \right\|_{L^2}^2 \right)$$

$$+ 8 \sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho_0,m}^2 \left\| \partial_{x_j}^m \mathcal{U} |_{t=0} \right\|_{L^2}^2 \left(\rho'(0)^2 \frac{(m+1)^2}{\rho_0^2} - \rho'(0) \frac{m+1}{\rho_0} \right)$$

$$\leq C \left(|u_0|_{X_{2\rho_0}}^2 + |u_1|_{X_{2\rho_0}}^2 \right) + 12 \sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho_0,m}^2 \left\| \partial_t^2 \partial_{x_j}^m u |_{t=0} \right\|_{L^2}^2, \tag{2.32}$$

where we have used the fact in the last inequality

$$\mathcal{U}|_{t=0} = (u_0, u_1), \quad \partial_y \mathcal{U}|_{t=0} = (\partial_y u_0, \partial_y u_1), \quad \partial_t \mathcal{U}|_{t=0} = (u_1, \partial_t^2 u|_{t=0}).$$

It remains to estimate the last term on the right hand side of (2.32). For this, when $t=0$, Eq. (1.5a) gives

$$\partial_t^2 u|_{t=0} = -u_1 - (u_0 \cdot \partial_x) u_0 - v_0 \partial_y u_0 + \partial_y^2 u_0 - \partial_x p|_{t=0}, \quad v_0 = -\int_0^y \partial_x \cdot u_0 d\tilde{y}. \tag{2.33}$$

By applying the divergence operator to Eq. (1.5) and using the compatibility condition (2.20), we have the elliptic equation

$$\Delta_x p = -\partial_x \cdot \int_0^1 \left[(u \cdot \partial_x)u + (\partial_x \cdot u)u \right] dy + \partial_x \cdot \int_0^1 \partial_y^2 u \, dy, \quad x \in \mathbb{R}^2 \tag{2.34}$$

for the pressure function. The standard elliptic theory implies that

$$\sum_{m=0}^{\infty} L_{\rho_0,m}^2 \left\| \partial_{x_j}^m \partial_x p |_{t=0} \right\|_{L_x^2}^2 \leq C \left(|u_0|_{X_{2\rho_0}}^4 + |\partial_y u_0|_{X_{2\rho_0}}^2 \right). \tag{2.35}$$

This together with (2.33) yields

$$\sum_{m=0}^{\infty} L_{\rho_0,m}^2 \left\| \partial_t^2 \partial_{x_j}^m u |_{t=0} \right\|_{L^2}^2 \leq C \left(|u_0|_{X_{2\rho_0}}^4 + |\partial_y u_0|_{X_{2\rho_0}}^2 + |u_1|_{X_{2\rho_0}}^2 \right). \tag{2.36}$$

For completeness, we give the proof of (2.35) and (2.36) in Appendix A. Combining the above estimate with (2.32) gives the second assertion in Theorem 2.1.

2.3   Proof of Theorem 2.1: The third assertion

It remains to prove (2.4). We take the L^2 inner product with $\partial_{x_j}^m \partial_y^2 u$ on both sides of the equation

$$(\partial_t^2 + \partial_t - \partial_y^2)\partial_{x_j}^m u = -\partial_x \partial_{x_j}^m p - \underbrace{\sum_{k=0}^m \binom{m}{k} \left[(\partial_{x_j}^k u \cdot \partial_x)\partial_{x_j}^{m-k} u + (\partial_{x_j}^k v)\partial_{x_j}^{m-k}\partial_y u \right]}_{\stackrel{def}{=} Q_m},$$

and then multiply by $L_{\rho/2,m}^2$ before taking summation for m. This gives

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left\| \partial_y^2 \partial_{x_j}^m u \right\|_{L^2}^2$$

$$\leq \sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left(\left\| \partial_t^2 \partial_{x_j}^m u \right\|_{L^2} + \left\| \partial_t \partial_{x_j}^m u \right\|_{L^2} \right) \left\| \partial_y^2 \partial_{x_j}^m u \right\|_{L^2}$$

$$+ \sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left\| Q_m \right\|_{L^2} \left\| \partial_y^2 \partial_{x_j}^m u \right\|_{L^2} + \sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left\| \partial_{x_j}^m \partial_x p \right\|_{L_x^2} \left\| \partial_y^2 \partial_{x_j}^m u \right\|_{L^2}.$$

Thus

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left\| \partial_y^2 \partial_{x_j}^m u \right\|_{L^2}^2 \leq C \left(|\partial_t u|_{X_{\rho/2}}^2 + |u|_{X_{\rho/2}}^2 \right)$$

$$+4\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|Q_m\big\|_{L^2}\big\|\partial_y^2\partial_{x_j}^m u\big\|_{L^2}+2\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|\partial_{x_j}^m\partial_x p\big\|_{L_x^2}^2. \qquad (2.37)$$

Using a similar argument as in (A.1) of Appendix A gives

$$\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|Q_m\big\|_{L^2}\big\|\partial_y^2\partial_{x_j}^m u\big\|_{L^2}$$

$$\leq C|u|_{X_\rho}^2\left(\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|\partial_y^2\partial_{x_j}^m u\big\|_{L^2}^2\right)^{\frac{1}{2}}\leq\frac{1}{8}\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|\partial_y^2\partial_{x_j}^m u\big\|_{L^2}^2+C|u|_{X_\rho}^4.$$

We use the elliptic theory for (2.34) and a similar computation as in (2.35) to conclude

$$\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|\partial_{x_j}^m\partial_x p\big\|_{L_x^2}^2$$

$$\leq C|u|_{X_\rho}^4+C\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|\partial_y\partial_{x_j}^m u\big\|_{L_x^2 L_y^\infty}^2$$

$$\leq\frac{1}{8}\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|\partial_y^2\partial_{x_j}^m u\big\|_{L^2}^2+C\left(|u|_{X_\rho}^4+|u|_{X_\rho}^2\right).$$

Here, in the last inequality, we have used the fact that

$$\forall r\in[0,1],\quad \big(\partial_y u(r)\big)^2=\int_\xi^r\partial_y\big(\partial_y u(y)\big)^2 dy=2\int_\xi^r\big(\partial_y^2 u(y)\big)\partial_y u(y)dy$$

with $\xi\in[0,1]$ satisfying

$$\partial_y u(\xi)=\int_0^1\partial_y u\,dy=0$$

because of the boundary condition $u|_{y=0,1}=0$. Substituting the above estimates into (2.37) yields

$$\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|\partial_y^2\partial_{x_j}^m u\big\|_{L^2}^2\leq C\left(|u|_{X_\rho}^2+|u|_{X_\rho}^4+|\partial_t u|_{X_\rho}^2\right).$$

Observe

$$|\partial_y u|_{X_{\rho/2}}^2\leq\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\big\|\partial_y^2\partial_{x_j}^m u\big\|_{L^2}^2+|u|_{X_{\rho/2}}^2+|\partial_t u|_{X_{\rho/2}}^2,$$

so that (2.4) follows. The proof of Theorem 2.1 is complete. $\qquad\square$

3   Global well-posedness of original system

We will prove Theorem 1.2 about the global well-posedness of the anisotropic hyperbolic Navier-Stokes system (1.4) in this section. The argument is similar to the one used in the previous section so that we only sketch the proof.

Sketch of the proof of Theorem 1.2. We apply ∂_{x_j} to the evolution equations of u^ε and v^ε in (1.4) to obtain

$$
\left(\partial_t^2 + \partial_t - \varepsilon^2 \Delta_x - \partial_y^2\right)\partial_{x_j}^m u^\varepsilon
$$
$$
= -\partial_x \partial_{x_j}^m p^\varepsilon - \sum_{k=0}^m \binom{m}{k}\left[\left(\partial_{x_j}^k u^\varepsilon \cdot \partial_x\right)\partial_{x_j}^{m-k}u^\varepsilon + \left(\partial_{x_j}^k v^\varepsilon\right)\partial_{x_j}^{m-k}\partial_y u^\varepsilon\right] \overset{def}{=} T_m^\varepsilon, \qquad (3.1)
$$

and

$$
\varepsilon^2\left(\partial_t^2 + \partial_t - \varepsilon^2 \Delta_x - \partial_y^2\right)\partial_{x_j}^m v^\varepsilon
$$
$$
= -\partial_y \partial_{x_j}^m p^\varepsilon - \varepsilon^2 \sum_{k=0}^m \binom{m}{k}\left[\left(\partial_{x_j}^k u^\varepsilon \cdot \partial_x\right)\partial_{x_j}^{m-k}v^\varepsilon + \left(\partial_{x_j}^k v^\varepsilon\right)\partial_{x_j}^{m-k}\partial_y v^\varepsilon\right] \overset{def}{=} N_m^\varepsilon. \qquad (3.2)
$$

It follows from the divergence-free condition that

$$
\left(\partial_x \partial_{x_j}^m p^\varepsilon, \partial_{x_j}^m u^\varepsilon\right)_{L^2} + \left(\partial_y \partial_{x_j}^m p^\varepsilon, \partial_{x_j}^m v^\varepsilon\right)_{L^2}
$$
$$
= \left(\partial_x \partial_{x_j}^m p^\varepsilon, \partial_t \partial_{x_j}^m u^\varepsilon\right)_{L^2} + \left(\partial_y \partial_{x_j}^m p^\varepsilon, \partial_t \partial_{x_j}^m v^\varepsilon\right)_{L^2} = 0.
$$

Let $T_m^\varepsilon, N_m^\varepsilon$ be in (3.1) and (3.2). Then by using the above equality instead of (2.22) and following the argument in the proof of (2.16), we conclude

$$
\sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho,m}^2 \left(T_m^\varepsilon, \partial_{x_j}^m u^\varepsilon + 2\partial_t \partial_{x_j}^m u^\varepsilon\right)_{L^2}
$$
$$
+ \sum_{j=1}^2 \sum_{m=0}^{+\infty} L_{\rho,m}^2 \left(N_m^\varepsilon, \partial_{x_j}^m v^\varepsilon + 2\partial_t \partial_{x_j}^m v^\varepsilon\right)_{L^2}
$$
$$
\leq C\rho^{-2}\left|(u^\varepsilon, \varepsilon v^\varepsilon)\right|_{X_\rho}\left|(u^\varepsilon, \varepsilon v^\varepsilon)\right|_{Y_\rho}^2,
$$

where $\left|(u^\varepsilon, \varepsilon v^\varepsilon)\right|_{X_\rho}$ and $\left|(u^\varepsilon, \varepsilon v^\varepsilon)\right|_{Y_\rho}$ are given by Definition 1.1 and (2.5). Then we take the L^2 inner product with $\partial_{x_j}^m u^\varepsilon + 2\partial_t \partial_{x_j}^m u^\varepsilon$ in (3.1) and with $\partial_{x_j}^m v^\varepsilon + 2\partial_t \partial_{x_j}^m v^\varepsilon$ in (3.2). The following a priori estimate for the system (1.4) can be obtained by using the argument in the previous section:

$$
\forall t \geq 0, \quad \left|(u^\varepsilon(t), \varepsilon v^\varepsilon(t))\right|_{X_{\rho(t)}} \leq 4\delta_0 e^{-\frac{t}{32}},
$$

provided that the initial data satisfy

$$\left|\left(u_0^\varepsilon, \varepsilon v_0^\varepsilon\right)\right|_{X_{2\rho_0}} + \left|\left(u_1^\varepsilon, \varepsilon v_1^\varepsilon\right)\right|_{X_{2\rho_0}} \le \delta_0$$

with δ_0 being sufficiently small. Hence, the proof of Theorem 1.2 is complete. $\square$

4 Hydrostatic limit

In the final section, we will prove Theorem 1.3 about the hydrostatic limit from (1.4) to (1.5) as $\varepsilon \to 0$.

Suppose the assumptions in Theorem 1.3 hold. Let $(u^\varepsilon, v^\varepsilon, p^\varepsilon)$ and (u, p) solve the anisotropic hyperbolic Navier-Stokes system (1.4) and the hyperbolic hydrostatic Navier-Stokes system (1.5), respectively. In the following discussion, we use the notation

$$U^\varepsilon = u^\varepsilon - u, \quad V^\varepsilon = v^\varepsilon - v, \quad P^\varepsilon = p^\varepsilon - p,$$

where

$$v(t, x, y) = -\int_0^y \partial_x \cdot u(t, x, \tilde{y}) d\tilde{y}.$$

Then it follows from (1.4) and (1.5) that

$$\begin{cases} \left(\partial_t^2 + \partial_t - \varepsilon^2 \Delta_x - \partial_y^2\right) U^\varepsilon + \partial_x P^\varepsilon = \varepsilon^2 \Delta_x u + R_\varepsilon, \\ \varepsilon^2 \left(\partial_t^2 + \partial_t - \varepsilon^2 \Delta_x - \partial_y^2\right) V^\varepsilon + \partial_y P^\varepsilon = -\varepsilon^2 \left(\partial_t^2 + \partial_t + u \cdot \partial_x + v \partial_y - \varepsilon^2 \Delta_x - \partial_y^2\right) v + S_\varepsilon, \\ \partial_x \cdot U^\varepsilon + \partial_y V^\varepsilon = 0, \\ U^\varepsilon|_{y=0,1} = 0, \ V^\varepsilon|_{y=0,1} = 0, \\ (U^\varepsilon, V^\varepsilon)|_{t=0} = \left(u_0^\varepsilon - u_0, v_0^\varepsilon - v_0\right), \ (\partial_t U^\varepsilon, \partial_t V^\varepsilon)|_{t=0} = \left(u_1^\varepsilon - u_1, v_1^\varepsilon - v_1\right), \end{cases}$$

where

$$v_j = -\int_0^y \partial_x \cdot u_j(t, x, \tilde{y}) d\tilde{y}, \quad j = 0, 1,$$

and

$$R_\varepsilon = -(U^\varepsilon \cdot \partial_x) u^\varepsilon - (u \cdot \partial_x) U^\varepsilon - V^\varepsilon \partial_y u^\varepsilon - v \partial_y U^\varepsilon,$$
$$S_\varepsilon = -\varepsilon^2 (U^\varepsilon \cdot \partial_x) v^\varepsilon - \varepsilon^2 (u \cdot \partial_x) V^\varepsilon - \varepsilon^2 V^\varepsilon \partial_y v^\varepsilon - \varepsilon^2 v \partial_y V^\varepsilon.$$

In the following, we will use C to denote a generic constant depending only on ρ_0 and the Sobolev embedding constant but independent of ε. Similar to the argument used in Section 2, we have

$$\sum_{m=0}^{+\infty} L_{\rho/2, m}^2 \left(\partial_{x_j}^m R_\varepsilon, \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon\right)_{L^2} \le C\left(|u^\varepsilon|_{X_\rho} + |u|_{X_\rho}\right) |U^\varepsilon|_{Y_{\rho/2}}^2, \qquad (4.1)$$

and

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left(\partial_{x_j}^m S_\varepsilon, \partial_{x_j}^m V^\varepsilon + 2\partial_t \partial_{x_j}^m V^\varepsilon\right)_{L^2}$$
$$\leq C|\varepsilon v^\varepsilon|_{X_\rho} |U^\varepsilon|_{Y_{\rho/2}}^2 + C\left(|\varepsilon v^\varepsilon|_{X_\rho} + |u^\varepsilon|_{X_\rho} + |u|_{X_\rho}\right) |\varepsilon V^\varepsilon|_{Y_{\rho/2}}^2, \tag{4.2}$$

where $|\cdot|_{Y_{\rho/2}}$ is defined in (2.5). Please refer to Appendix A for the detailed proof. Moreover, direct calculation gives

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left(\varepsilon^2 \partial_{x_j}^m \Delta_x u, \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon\right)_{L^2} \leq C\varepsilon^2 |u|_{X_\rho} |U^\varepsilon|_{X_{\rho/2}},$$

and

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left(-\varepsilon^2 \partial_{x_j}^m \left(\partial_t^2 + \partial_t + u\cdot\partial_x + v\partial_y - \varepsilon^2 \Delta_x - \partial_y^2\right) v, \partial_{x_j}^m V^\varepsilon + 2\partial_t \partial_{x_j}^m V^\varepsilon\right)_{L^2}$$
$$\leq C\varepsilon \left(|u|_{X_\rho} + |u|_{X_\rho}^2 + |\partial_t u|_{X_\rho}\right) |\varepsilon V^\varepsilon|_{X_{\rho/2}}, \tag{4.3}$$

where we used the relation

$$\partial_t^2 v = -\int_0^y \partial_t^2 \partial_x \cdot u(t,x,\tilde{y}) d\tilde{y}$$

(see Appendix A for details). By using the argument in Section 2, we can derive estimates on U^ε and $\varepsilon V^\varepsilon$ as those given in (2.15). Hence, by using the estimates in Theorems 1.1 and 1.2, we conclude that

$$\sup_{t\geq 0} \left(|U^\varepsilon(t)|_{X_{\rho(t)/2}}^2 + |\varepsilon V^\varepsilon(t)|_{X_{\rho(t)/2}}^2\right) + \int_0^{+\infty} \left(|U^\varepsilon(s)|_{X_{\rho(s)/2}}^2 + |\varepsilon V^\varepsilon(s)|_{X_{\rho(s)/2}}^2\right) ds$$
$$\leq C \left(|u_0^\varepsilon - u_0|_{X_{\rho_0}}^2 + |u_1^\varepsilon - u_1|_{X_{\rho_0}}^2 + \varepsilon|v_0^\varepsilon - v_0|_{X_{\rho_0}}^2 + \varepsilon|v_1^\varepsilon - v_1|_{X_{\rho_0}}^2\right)$$
$$\quad + \varepsilon^2 C \int_0^{+\infty} \left(|u|_{X_\rho} + |u|_{X_\rho}^2 + |\partial_t u|_{X_\rho}\right)^2 ds$$
$$\leq C \left(|u_0^\varepsilon - u_0|_{X_{2\rho_0}}^2 + |u_1^\varepsilon - u_1|_{X_{2\rho_0}}^2\right) + \varepsilon^2 C,$$

where we have used Theorem 1.1 in the last inequality. Then this completes the proof of Theorem 1.3. $\square$

Appendix A. Some computation

We now present the proof of the estimates (4.1)-(4.3) and (2.35)-(2.36).

Proof of (4.1) *and* (4.2). In the proof C denotes a generic constant depending only on ρ_0 and the Sobolev embedding constant but independent of ε. To estimate the first term on the right of

$$R_\varepsilon = -(U^\varepsilon \cdot \partial_x)u^\varepsilon - (u \cdot \partial_x)U^\varepsilon - V^\varepsilon \partial_y u^\varepsilon - v\partial_y U^\varepsilon,$$

we use the estimate

$$\forall k \geq 0, \quad \frac{m!}{k!(m-k)!} \frac{L_{\rho/2,m}}{L_{\rho,k+3}L_{\rho/2,m-k}} \leq 2^{7-k}(k+3)^6 \rho^{-4}$$

to compute

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left| \left(\partial_{x_j}^m((U^\varepsilon \cdot \partial_x)u^\varepsilon), \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon \right)_{L^2} \right|$$

$$\leq \sum_{m=0}^{+\infty} \sum_{k=0}^m \frac{m!}{k!(m-k)!} \frac{L_{\rho/2,m}}{L_{\rho,k+3}L_{\rho/2,m-k}} L_{\rho,k+3} \left\| \partial_x \partial_{x_j}^k u^\varepsilon \right\|_{L^\infty}$$

$$\times \left(L_{\rho/2,m-k} \left\| \partial_{x_j}^{m-k} U^\varepsilon \right\|_{L^2} \right) \times \left(L_{\rho/2,m} \left\| \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon \right\|_{L^2} \right)$$

$$\leq C\rho^{-4} \left[\sum_{m=0}^{+\infty} \left(\sum_{k=0}^m 2^{-k}k^6 L_{\rho,k+2} \left\| \partial_x \partial_{x_j}^k u^\varepsilon \right\|_{L^\infty} L_{\rho/2,m-k} \left\| \partial_{x_j}^{m-k} U^\varepsilon \right\|_{L^2} \right)^2 \right]^{\frac{1}{2}} |U^\varepsilon|_{X_{\rho/2}}$$

$$\leq \frac{C}{\rho^4} \sum_{k=0}^\infty 2^{-k}k^6 L_{\rho,k+3} \left\| \partial_x \partial_{x_j}^k u^\varepsilon \right\|_{L^\infty} \left[\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left\| \partial_{x_j}^m U^\varepsilon \right\|_{L^2}^2 \right]^{\frac{1}{2}} |U^\varepsilon|_{X_{\rho/2}}$$

$$\leq \frac{C}{\rho^4} |u^\varepsilon|_{X_\rho} |U^\varepsilon|_{X_{\rho/2}}^2 \leq C|u^\varepsilon|_{X_\rho} |U^\varepsilon|_{Y_{\rho/2}}^2, \tag{A.1}$$

where we have used Young's inequality for discrete convolution in the third inequality and the fact that $\frac{\rho_0}{2} \leq \rho \leq \rho_0$ in the last line. Similarly,

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left| \left(\partial_{x_j}^m(v\partial_y U^\varepsilon), \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon \right)_{L^2} \right| \leq C|u|_{X_\rho} |U^\varepsilon|_{Y_{\rho/2}}^2.$$

As for the third term in R_ε, we use the fact

$$V^\varepsilon = -\int_0^y \partial_x \cdot U^\varepsilon(t,x,\tilde{y})d\tilde{y}$$

to write

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left| \left(\partial_{x_j}^m (V^\varepsilon \partial_y u^\varepsilon), \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon \right)_{L^2} \right| \leq J_1 + J_2$$

with

$$J_1 = \sum_{m=0}^{+\infty} \sum_{k=0}^{\left[\frac{m}{2}\right]} \frac{m!}{k!(m-k)!} \frac{L_{\rho/2,m}}{L_{\rho/2,k+2} L_{\rho/2,m-k+1}} \left(L_{\rho/2,k+2} \left\| \partial_{x_j}^k \partial_y u^\varepsilon \right\|_{L_x^\infty L_y^2} \right)$$
$$\times \left(L_{\rho/2,m-k+1} \left\| \partial_x \partial_{x_j}^{m-k} U^\varepsilon \right\|_{L^2} \right) \times \left(L_{\rho/2,m} \left\| \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon \right\|_{L^2} \right),$$

$$J_2 = \sum_{m=0}^{+\infty} \sum_{k=\left[\frac{m}{2}\right]+1}^{m} \frac{m!}{k!(m-k)!} \frac{L_{\rho/2,m}}{L_{\rho,k+2} L_{\rho/2,m-k+1}} \left(L_{\rho,k+2} \left\| \partial_{x_j}^k \partial_y u^\varepsilon \right\|_{L_x^\infty L_y^2} \right)$$
$$\times \left(L_{\rho/2,m-k+1} \left\| \partial_x \partial_{x_j}^{m-k} U^\varepsilon \right\|_{L^2} \right) \times \left(L_{\rho/2,m} \left\| \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon \right\|_{L^2} \right).$$

Repeating the computation in (2.29) yields

$$J_1 \leq C |u^\varepsilon|_{X_{\rho/2}} |U^\varepsilon|_{Y_{\rho/2}}^2.$$

Moreover, by observing

$$\forall k \geq \left[\frac{m}{2}\right]+1, \quad \frac{m!}{k!(m-k)!} \frac{L_{\rho/2,m}}{L_{\rho,k+2} L_{\rho/2,m-k+1}} \leq 2^{7-k}(k+2)^6 \rho^{-4},$$

and by using a similar argument as in (A.1), we conclude

$$J_2 \leq C |u^\varepsilon|_{X_\rho} |U^\varepsilon|_{Y_{\rho/2}}^2.$$

Thus,

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left| \left(\partial_{x_j}^m (V^\varepsilon \partial_y u^\varepsilon), \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon \right)_{L^2} \right| \leq C |u^\varepsilon|_{X_\rho} |U^\varepsilon|_{Y_{\rho/2}}^2.$$

Similarly, we have

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left| \left(\partial_{x_j}^m ((u \cdot \partial_x) U^\varepsilon), \partial_{x_j}^m U^\varepsilon + 2\partial_t \partial_{x_j}^m U^\varepsilon \right)_{L^2} \right| \leq C |u|_{X_\rho} |U^\varepsilon|_{Y_{\rho/2}}^2.$$

In summary, we have (4.1).

  Next we prove (4.2). Recall

$$S_\varepsilon = -\varepsilon^2 (U^\varepsilon \cdot \partial_x) v^\varepsilon - \varepsilon^2 (u \cdot \partial_x) V^\varepsilon - \varepsilon^2 V^\varepsilon \partial_y v^\varepsilon - \varepsilon^2 v \partial_y V^\varepsilon.$$

Following the argument used in (A.1) with slight modification, we can show that

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left| \left(\partial_{x_j}^m (\varepsilon^2 (U^\varepsilon \cdot \partial_x) v^\varepsilon), \partial_{x_j}^m V^\varepsilon + 2\partial_t \partial_{x_j}^m V^\varepsilon \right)_{L^2} \right|$$
$$\leq C |\varepsilon v^\varepsilon|_{X_\rho} |U^\varepsilon|_{X_{\rho/2}} |\varepsilon V^\varepsilon|_{X_{\rho/2}} \leq C |\varepsilon v^\varepsilon|_{X_\rho} |U^\varepsilon|_{Y_{\rho/2}}^2 + C |\varepsilon v^\varepsilon|_{X_\rho} |\varepsilon V^\varepsilon|_{Y_{\rho/2}}^2.$$

Similarly, by observing $\partial_y v^\varepsilon = -\partial_x \cdot u^\varepsilon$, we have

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left| \left(\partial_{x_j}^m (\varepsilon^2 V^\varepsilon \partial_y v^\varepsilon), \partial_{x_j}^m V^\varepsilon + 2\partial_t \partial_{x_j}^m V^\varepsilon \right)_{L^2} \right|$$
$$\leq C |u^\varepsilon|_{X_\rho} |\varepsilon V^\varepsilon|_{X_{\rho/2}}^2 \leq C |u^\varepsilon|_{X_\rho} |\varepsilon V^\varepsilon|_{Y_{\rho/2}}^2.$$

Using a similar computation as in (2.29) and (A.1) gives

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left| \left(\partial_{x_j}^m (\varepsilon^2 (u \cdot \partial_x) V^\varepsilon), \partial_{x_j}^m V^\varepsilon + 2\partial_t \partial_{x_j}^m V^\varepsilon \right)_{L^2} \right| \leq C |u|_{X_\rho} |\varepsilon V^\varepsilon|_{Y_{\rho/2}}^2.$$

Finally,

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left| \left(\partial_{x_j}^m (\varepsilon^2 v \partial_y V^\varepsilon), \partial_{x_j}^m V^\varepsilon + 2\partial_t \partial_{x_j}^m V^\varepsilon \right)_{L^2} \right| \leq C |u|_{X_\rho} |\varepsilon V^\varepsilon|_{Y_{\rho/2}}^2.$$

Combining the above estimates yields (4.2). $\qquad\square$

Proof of (4.3). Observe that

$$\partial_t^2 v = - \int_0^y \partial_t^2 \partial_x \cdot u \, d\tilde{y}.$$

Then it follows from Definition 1.1 that

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left\| \partial_{x_j}^m \partial_t^2 v \right\|_{L^2}^2 \leq \sum_{m=0}^{+\infty} \frac{L_{\rho/2,m}^2}{L_{\rho,m+1}^2} L_{\rho,m+1}^2 \left\| \partial_{x_j}^m \partial_t^2 v \right\|_{L^2}^2$$
$$\leq C \sum_{m=0}^{+\infty} \frac{m+1}{2^{m+1} \rho} L_{\rho,m+1}^2 \left\| \partial_x \partial_{x_j}^m \partial_t^2 u \right\|_{L^2}^2 \leq C |\partial_t u|_{X_\rho}^2.$$

Similarly, using again the relation $\partial_y v = -\partial_x \cdot u$, we have

$$\sum_{m=0}^{+\infty} L_{\rho/2,m}^2 \left(\left\| \partial_{x_j}^m \partial_t v \right\|_{L^2}^2 + \left\| \partial_{x_j}^m \Delta_x v \right\|_{L^2}^2 + \left\| \partial_{x_j}^m \partial_y^2 v \right\|_{L^2}^2 \right)$$

$$\leq C\sum_{m=0}^{+\infty}L_{\rho,m+1}^2\left(\left\|\partial_x\partial_{x_j}^m\partial_t u\right\|_{L^2}^2+\left\|\partial_x\partial_{x_j}^m\partial_y u\right\|_{L^2}^2\right)$$
$$+C\sum_{m=0}^{+\infty}L_{\rho,m+3}^2\left\|\partial_x\partial_{x_j}^m\Delta_x u\right\|_{L^2}^2\leq C|u|_{X_\rho}^2.$$

Combining the above estimates gives

$$\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\left(-\varepsilon^2\partial_{x_j}^m(\partial_t^2+\partial_t-\varepsilon^2\Delta_x-\partial_y^2)v,\partial_{x_j}^m V^\varepsilon+2\partial_t\partial_{x_j}^m V^\varepsilon\right)_{L^2}$$
$$\leq C\varepsilon\left(|u|_{X_\rho}+|\partial_t u|_{X_\rho}\right)|\varepsilon V^\varepsilon|_{X_{\rho/2}}.$$

Moreover, using a similar computation as in (A.1) yields

$$\sum_{m=0}^{+\infty}L_{\rho/2,m}^2\left(-\varepsilon^2\partial_{x_j}^m(u\cdot\partial_x+v\partial_y)v,\partial_{x_j}^m V^\varepsilon+2\partial_t\partial_{x_j}^m V^\varepsilon\right)_{L^2}\leq C\varepsilon|u|_{X_\rho}^2|\varepsilon V^\varepsilon|_{X_{\rho/2}}.$$

Thus the estimate (4.3) follows. □

Proof of (2.35) *and* (2.36). Letting $t=0$ in (2.34) and then applying $\partial_{x_j}^m$ to the both sides of the equation, we obtain by standard elliptic theory that

$$\left\|\partial_{x_j}^m\partial_x p|_{t=0}\right\|_{L_x^2}^2\leq\left(\left\|\partial_{x_j}^m\left[(u_0\cdot\partial_x)u_0+(\partial_x\cdot u_0)u_0\right]\right\|_{L^2}+\left\|\partial_{x_j}^m\partial_y^2 u_0\right\|_{L^2}\right)$$
$$\times\left\|\partial_{x_j}^m\partial_x p|_{t=0}\right\|_{L_x^2}.$$

Then using a similar computation as in (A.1) gives

$$\sum_{m=0}^{\infty}L_{\rho_0,m}^2\left\|\partial_{x_j}^m\partial_x p|_{t=0}\right\|_{L_x^2}^2$$
$$\leq C\left(|u_0|_{X_{2\rho_0}}^2+|\partial_y u_0|_{X_{\rho_0}}\right)\left(\sum_{m=0}^{\infty}L_{\rho_0,m}^2\left\|\partial_{x_j}^m\partial_x p|_{t=0}\right\|_{L_x^2}^2\right)^{\frac{1}{2}}.$$

Thus (2.35) follows. Similar argument holds for (2.36). □

Acknowledgements

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