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Intrinsic Formulation of the Kirchhoff-Love Theory of Nonlinearly Elastic Shallow Shells

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Abstract. The classical formulation of the Kirchhoff-Love theory of nonlinearly elastic shallow shells consists of a system of nonlinear partial differential equations and boundary conditions whose unknowns are the Cartesian components of the displacement field of the middle surface of the shell subjected to applied forces. We show that this system is equivalent to a system whose sole unknowns are the bending moments and stress resultants inside the middle surface of the shell. This system thus provides a direct method for computing the stresses appearing in such a shell, without any recourse to the displacement field. To this end, we first establish specific compatibility conditions of Saint-Venant type for the bending moments and stress resultants; we then identify the boundary conditions that these fields must satisfy.

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1 Introduction

A shallow shell is a thin shell whose middle surface is "almost planar", in the sense that the principal curvatures of the middle surface of the shell are of the order of its thickness (the precise definition is given in Section 2).

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If a shallow shell is made of an elastic material and is subjected to external forces, the shell will undergo a deformation to reach an equilibrium state, where the corresponding internal stresses are given in terms of the displacement field and its partial derivatives by means of the constitutive equation of the elastic material from which the shell is made. The displacement field then satisfies a specific boundary value problem formed by a system of partial differential equations and boundary conditions defined over a three dimensional domain representing the reference configuration of the shell.

The classical Kirchhoff-Love theory for nonlinearly elastic shallow shells provides a way to compute the internal stresses and the displacement field inside a shallow shell by solving a boundary value problem defined over a two-dimensional domain, whose unknown is the vector field formed by the Cartesian components of the displacement field of the middle surface of the reference configuration of the shell.

More specifically, the classical formulation of the Kirchhoff-Love theory computes the internal stresses in the deformed shallow shell in two stages: First, the displacement field of the middle surface of the shallow shell is computed by solving a specific boundary value problem (Section 2); Second, the internal stresses are computed in terms of this displacement field by using the constitutive equation of the elastic material constituting the shell.

The objective of this paper is to provide a simpler way to compute the internal stresses in the deformed shallow shell by means of an intrinsic formulation, the main feature of which is to entirely eliminate the need of computing the displacement field. This is done by introducing a new boundary value problem whose sole unknowns are the two-dimensional stresses, or equivalently the strains, of the middle surface of the shell and by proving that the two-dimensional stresses found in this way coincide with those found by solving the classical boundary value problem of Kirchhoff-Love (Theorems 4.1 and 4.2).

More specifically, we show that the bending moments and stress resultants of the middle surface of the deformed shell are the symmetric tensor fields (all the notation used in this introduction is defined in Section 2)

$$(M_{\alpha\beta}):\overline{\omega}\to\mathbb{S}^2,\quad (N_{\alpha\beta}):\overline{\omega}\to\mathbb{S}^2$$

that satisfy the following boundary value problem (see Theorem 4.1):

$$\begin{aligned}
&-\partial_\beta N_{\alpha\beta}=p_\alpha && \text{in } \omega,\\
&-\partial_{\alpha\beta}M_{\alpha\beta}-\partial_\alpha\big(N_{\alpha\beta}[\partial_\beta\zeta_3+\varepsilon\partial_\beta h]\big)=p_3+\partial_\alpha q_\alpha && \text{in } \omega,\\
&\quad N_{\alpha\beta}\nu_\beta=M_{\alpha\beta}\nu_\alpha\nu_\beta=0 && \text{on } \gamma_1,\\
&\quad N_{\alpha\beta}\nu_\alpha(\partial_\beta\zeta_3+\varepsilon\partial_\beta h)+(\partial_\alpha M_{\alpha\beta})\nu_\beta+\partial_\tau(M_{\alpha\beta}\nu_\alpha\tau_\beta)=-q_\alpha\nu_\alpha && \text{on } \gamma_1,
\end{aligned}$$

$$
\begin{aligned}
&\partial_\alpha F_{\beta\sigma} - \partial_\beta F_{\alpha\sigma} = 0 && \text{in } \omega, \\
&\partial_{\alpha\beta} E_{\sigma\tau} + \partial_{\sigma\tau} E_{\alpha\beta} - \partial_{\alpha\sigma} E_{\beta\tau} - \partial_{\beta\tau} E_{\alpha\sigma} \\
= &F_{\alpha\sigma} F_{\beta\tau} - F_{\alpha\beta} F_{\sigma\tau} + \varepsilon\big(\partial_{\alpha\sigma} h F_{\beta\tau} + \partial_{\beta\tau} h F_{\alpha\sigma} - \partial_{\alpha\beta} h F_{\sigma\tau} - \partial_{\sigma\tau} h F_{\alpha\beta}\big) && \text{in } \omega, \\
&E_{\alpha\beta}\tau_\alpha\tau_\beta = \partial_\sigma E_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma - 2\tau_\sigma\nu_\beta) - \kappa E_{\alpha\beta}\nu_\alpha\nu_\beta = F_{\alpha\beta}\tau_\beta = 0 && \text{on } \gamma_0,
\end{aligned}
$$

where

$$b_{\alpha\beta\sigma\tau} := \frac{1}{8\mu}(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma}) - \frac{\lambda}{4\mu(3\lambda+2\mu)}\delta_{\sigma\tau}\delta_{\alpha\beta}$$

denote the components of the inverse of the two-dimensional elasticity tensor of the shell, and

$$E_{\alpha\beta} := \frac{1}{\varepsilon} b_{\alpha\beta\sigma\tau} N_{\sigma\tau}, \quad F_{\alpha\beta} := -\frac{3}{\varepsilon^3} b_{\alpha\beta\sigma\tau} M_{\sigma\tau}$$

denote the components of the strain tensors inside the middle surface of the shell.

The above system constitute our intrinsic formulation of the Kirchhoff-Love theory of a nonlinearly elastic shallow shell.

The components $\zeta_i : \overline{\omega} \to \mathbb{R}$ of the displacement field of the middle surface of the shell found in the classical formulation of the same theory can then be recovered a posteriori from the solution of our intrinsic formulation above by solving the system

$$
\begin{aligned}
&\frac{1}{2}\big(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha + \varepsilon(\partial_\alpha h \partial_\beta\zeta_3 + \partial_\beta h \partial_\alpha\zeta_3) + \partial_\alpha\zeta_3\partial_\beta\zeta_3\big) = E_{\alpha\beta} && \text{in } \omega, \\
&\partial_{\alpha\beta}\zeta_3 = F_{\alpha\beta} && \text{in } \omega, \\
&\zeta_i = \partial_\nu\zeta_3 = 0 && \text{on } \gamma_0.
\end{aligned}
$$

The main objective of this paper thus consists in showing that the intrinsic theory of elasticity, whose origin goes back to the founding papers of Chien [3,4] and Antman [2], can also be applied to nonlinearly elastic shallow shells, thus complementing earlier works where it has been shown to apply to three-dimensional linearly elastic bodies [7], three-dimensional nonlinearly elastic bodies [9], linearly elastic plates [10], nonlinearly elastic plates [8, 12], and linearly elastic shells [11].

2 The classical formulation of the Kirchhoff-Love theory of a nonlinearly elastic shallow shell

We briefly describe here the classical Kirchhoff-Love theory for nonlinearly elastic shallow shells. For more details, see, e.g., Ciarlet [5].

In all that follows, Greek indices vary in the set $\{1,2\}$, save in the notations ∂_τ and ∂_ν which respectively designate the tangential and normal derivative operators along the boundary of a domain $\omega \subset \mathbb{R}^2$, Latin indices vary in the set $\{1,2,3\}$, and the summation convention for repeated Greek or Latin indices is used. Vectors and vector-valued functions are denoted by boldface letters.

The notations $\mathbb{E}^3$, $\cdot$, $\wedge$, and $|\cdot|$, respectively designate the three-dimensional Euclidean space, the inner product in $\mathbb{E}^3$, the vector product in $\mathbb{E}^3$, and the Euclidean norm in $\mathbb{E}^3$. The notation $\boldsymbol{e}_i$ designate the i-th vector of a Cartesian basis in $\mathbb{E}^3$. A generic point in the "horizontal" plane spanned by the vectors $\boldsymbol{e}_\alpha$ is denoted $y = (y_\alpha)$ and the variable along the "vertical" vector $\boldsymbol{e}_3$ is denoted x_3. Partial derivative operators are denoted $\partial_\alpha := \partial/\partial y_\alpha$, $\partial_3 := \partial/\partial x_3$, and $\partial_{\alpha\beta} := \partial^2/\partial y_\alpha \partial y_\beta$. The space of all real 2×2 symmetric matrices is denoted $\mathbb{S}^2$.

We adopt the definition of a shallow shell from Ciarlet and Miara [13] expressed in Cartesian coordinates (other definitions are possible, for instance using curvilinear coordinates as in, e.g., Ciarlet [6]). A shallow shell is a three-dimensional body whose reference configuration is the closure of a set of the form

$$\hat{\Omega} := \big\{ \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y);\ y \in \omega,\ x_3 \in]-\varepsilon,\varepsilon[\big\},$$

where ω is a bounded and connected open subset of $\mathbb{R}^2$, $\varepsilon > 0$, $h \in \mathcal{C}^2(\overline{\omega})$,

$$\begin{aligned} \boldsymbol{\theta}(y) &:= \big(y_1, y_2, \varepsilon h(y)\big) && \text{for all} \quad y = (y_1,y_2) \in \omega, \\ \boldsymbol{a}_3(y) &:= \frac{\partial_1 \boldsymbol{\theta}(y) \wedge \partial_2 \boldsymbol{\theta}(y)}{|\partial_1 \boldsymbol{\theta}(y) \wedge \partial_2 \boldsymbol{\theta}(y)|} && \text{for all} \quad y \in \omega. \end{aligned}$$

The set $\boldsymbol{\theta}(\overline{\omega})$ is called the middle surface of the shell and $2\varepsilon > 0$ is the thickness; note that, at each $y \in \overline{\omega}$, $\boldsymbol{a}_3(y)$ is a unit vector, normal to the middle surface at the point $\boldsymbol{\theta}(y)$.

We consider a shallow shell made of a homogeneous and isotropic nonlinearly elastic material whose reference configuration is a natural state (i.e., stress-free). Thus the two-dimensional constitutive equation relating the strains and stresses inside the middle surface of the deformed shell is governed by two Lamé constants $\lambda \geq 0$ and $\mu > 0$ by means of the two-dimensional elasticity tensor whose components are defined by

$$a_{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda + 2\mu} \delta_{\sigma\tau}\delta_{\alpha\beta} + 2\mu(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma}),$$

where the notation $\delta_{\alpha\beta}$ designates the Kronecker symbol.

We assume that the shallow shell under consideration is kept fixed on the subset

$$\hat{\Gamma}_0 := \big\{ \boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y);\ y \in \gamma_0,\ x_3 \in]-\varepsilon,\varepsilon[\big\}$$

of its lateral face, where γ_0 is a relatively open subset of the boundary γ of ω, and that the shell is subjected to applied body and surface forces whose densities per unit volume and per unit area are respectively denoted by $\hat{f}_i \boldsymbol{e}_i \in L^2(\hat{\Omega};\mathbb{E}^3)$ and $\hat{g}_i \boldsymbol{e}_i \in L^2(\hat{\Gamma}_- \cup \hat{\Gamma}_+;\mathbb{E}^3)$, where

$$\hat{\Gamma}_- := \big\{\boldsymbol{\theta}(y) - \varepsilon \boldsymbol{a}_3(y);\ y \in \omega\big\},$$
$$\hat{\Gamma}_+ := \big\{\boldsymbol{\theta}(y) + \varepsilon \boldsymbol{a}_3(y);\ y \in \omega\big\},$$

respectively denote the lower and upper faces of the shallow shell. Then the resulting two-dimensional forces and momentums along the middle surface of the shallow shell are defined by their densities $p_i \boldsymbol{e}_i \in L^2(\omega;\mathbb{E}^3)$ and $q_\alpha \boldsymbol{e}_\alpha \in L^2(\omega;\mathbb{E}^3)$ per unit area along ω, where

$$p_i(y) := \int_{-\varepsilon}^{\varepsilon} f_i(y,x_3)\,\mathrm{d}x_3 + g_i(y,\varepsilon) + g_i(y,-\varepsilon),$$
$$q_\alpha(y) := \int_{-\varepsilon}^{\varepsilon} x_3 f_\alpha(y,x_3)\,\mathrm{d}x_3 + \varepsilon\big(g_\alpha(y,\varepsilon) - g_\alpha(y,-\varepsilon)\big)$$

for all $y \in \omega$, with

$$f_i(y,x_3) := a(y,x_3)\hat{f}_i\big(\boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y)\big),$$
$$g_i(y,x_3) := a(y,x_3)\hat{g}_i\big(\boldsymbol{\theta}(y) + x_3 \boldsymbol{a}_3(y)\big),$$
$$a(y,x_3) := \big|\big(\partial_1\boldsymbol{\theta}(y) + x_3 \partial_1 \boldsymbol{a}_3(y)\big) \wedge \big(\partial_2\boldsymbol{\theta}(y) + x_3 \partial_2 \boldsymbol{a}_3(y)\big)\big|$$

for all $y \in \omega$ and $x_3 \in]-\varepsilon,\varepsilon[$.

Under the above assumptions, the classical Kirchhoff-Love theory (so named after Kirchhoff [16] and Love [17]) asserts that the Cartesian components $\boldsymbol{\zeta} = (\zeta_i)$ of the unknown displacement field $\zeta_i \boldsymbol{e}_i : \overline{\omega} \to \mathbb{E}^3$ of the middle surface of the shallow shell is a solution to the following minimization problem:

$$\boldsymbol{\zeta} \in \boldsymbol{V}(\omega) \quad \text{and} \quad J(\boldsymbol{\zeta}) = \inf\big\{J(\boldsymbol{\eta});\ \boldsymbol{\eta} \in \boldsymbol{V}(\omega)\big\},$$

where

$$\boldsymbol{V}(\omega) := \big\{\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0\big\},$$
$$J(\boldsymbol{\eta}) := \int_\omega \left(\frac{\varepsilon}{2} a_{\alpha\beta\sigma\tau} \hat{E}_{\sigma\tau}(\boldsymbol{\eta}) \hat{E}_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{6} a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau}\eta_3 \partial_{\alpha\beta}\eta_3 - p_i \eta_i + q_\alpha \partial_\alpha \eta_3\right) dy,$$

and

$$\hat{E}_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{2}\big(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \varepsilon(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) + \partial_\alpha \eta_3 \partial_\beta \eta_3\big).$$

The space $\boldsymbol{V}(\omega)$ is thus spanned by the Cartesian components η_i of the admissible displacement fields $\eta_i\boldsymbol{e}_i$ of the middle surface of the shell, the functional J represents the total energy of the shallow shell, and the functions $\hat{E}_{\alpha\beta}(\boldsymbol{\eta})$ and $\partial_{\alpha\beta}\eta_3$ denote the Cartesian components of the strain tensor fields associated with the displacement field $\eta_i\boldsymbol{e}_i$ of the middle surface of the shallow shell, where $\boldsymbol{\eta}=(\eta_i)\in\boldsymbol{V}(\omega)$.

The equations of the Kirchhoff-Love theory of a nonlinearly elastic shallow shell are the Euler-Lagrange equations (e.g., [14, 15]) associated with the above minimisation problem. These Euler-Lagrange equations can be recast as a boundary value problem of partial differential equations and boundary conditions if the boundary of the set ω and the minimiser $\boldsymbol{\zeta}$ are sufficiently smooth.

More specifically, assume that ω is locally on only one side of its boundary $\gamma:=\partial\omega$ and that γ is a Lipschitz-continuous boundary in sense of Adams [1], and that $\boldsymbol{\zeta}=(\zeta_i)\in\mathcal{C}^2(\overline{\omega})\times\mathcal{C}^2(\overline{\omega})\times\mathcal{C}^4(\overline{\omega})$. Since for all $\boldsymbol{\eta}=(\eta_i)\in\boldsymbol{V}(\omega)$,

$$\begin{aligned}\hat{E}_{\alpha\beta}(\boldsymbol{\zeta}+\boldsymbol{\eta})=\hat{E}_{\alpha\beta}(\boldsymbol{\zeta})+\frac{1}{2}\big(&\partial_\alpha\eta_\beta+\partial_\beta\eta_\alpha+\varepsilon(\partial_\alpha h\partial_\beta\eta_3+\partial_\beta h\partial_\alpha\eta_3)\\&+\partial_\alpha\zeta_3\partial_\beta\eta_3+\partial_\beta\zeta_3\partial_\alpha\eta_3\big)+\frac{1}{2}\partial_\alpha\eta_3\partial_\beta\eta_3,\end{aligned}$$

one deduces that the Gâteaux derivative of J at $\boldsymbol{\zeta}$ (it is easily seen that $J:\boldsymbol{V}(\omega)\to\mathbb{R}$ is Fréchet differentiable) in the direction $\boldsymbol{\eta}$ is given by

$$\begin{aligned}J'(\boldsymbol{\zeta})(\boldsymbol{\eta})=&\int_\omega\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})\partial_\beta\eta_\alpha dy+\int_\omega\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})(\partial_\beta\zeta_3+\varepsilon\partial_\beta h)\partial_\alpha\eta_3 dy\\&-\int_\omega\hat{M}_{\alpha\beta}(\boldsymbol{\zeta})\partial_{\alpha\beta}\eta_3 dy-\int_\omega p_i\eta_i dy+\int_\omega q_\alpha\partial_\alpha\eta_3 dy,\end{aligned}$$

where

$$\hat{N}_{\alpha\beta}(\boldsymbol{\zeta}):=\varepsilon a_{\alpha\beta\sigma\tau}\hat{E}_{\sigma\tau}(\boldsymbol{\zeta}),\quad \hat{M}_{\alpha\beta}(\boldsymbol{\zeta}):=-\frac{\varepsilon^3}{3}a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3.$$

Note that the functions $\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})$ and $\hat{M}_{\alpha\beta}(\boldsymbol{\zeta})$, which respectively represent the stress resultants and the bending moments of the middle surface of the deformed shallow shell associated with the displacement field $\zeta_i\boldsymbol{e}_i$, satisfy the symmetry conditions

$$\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})=\hat{N}_{\beta\alpha}(\boldsymbol{\zeta}),\quad \hat{M}_{\alpha\beta}(\boldsymbol{\zeta})=\hat{M}_{\beta\alpha}(\boldsymbol{\zeta}).$$

Consequently, if $\boldsymbol{\zeta}\in\boldsymbol{V}(\omega)$ is a minimiser of J over $\boldsymbol{V}(\omega)$, then it must satisfies the Euler-Lagrange equations

$$J'(\boldsymbol{\zeta})(\boldsymbol{\eta})=0\quad\text{for all}\quad\boldsymbol{\eta}\in\boldsymbol{V}(\omega).$$

Let (ν_α) denote the Cartesian coordinates of the unit inner normal vector field along γ, let (τ_α) denote the Cartesian coordinates of the unit tangent vector field along γ defined by

$$\tau_1 := \nu_2 \quad \text{and} \quad \tau_2 := -\nu_1 \quad \text{on } \gamma,$$

and let $\partial_\nu := \nu_\alpha \partial_\alpha$ and $\partial_\tau := \tau_\alpha \partial_\alpha$ respectively denote the normal derivative operator and the tangent derivative operator along γ. Then the above Euler-Lagrange equations are equivalent to the boundary value problem

$$\begin{aligned}
&-\partial_\beta \hat{N}_{\alpha\beta}(\boldsymbol{\zeta}) = p_\alpha && \text{in } \omega,\\
&-\partial_{\alpha\beta} \hat{M}_{\alpha\beta}(\boldsymbol{\zeta}) - \partial_\alpha\big(\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})[\partial_\beta \zeta_3 + \varepsilon \partial_\beta h]\big) = p_3 + \partial_\alpha q_\alpha && \text{in } \omega,\\
&\quad \zeta_i = \partial_\nu \zeta_3 = 0 && \text{on } \gamma_0,\\
&\quad \hat{N}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\beta = 0 && \text{on } \gamma_1,\\
&\quad \hat{M}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\alpha \nu_\beta = 0 && \text{on } \gamma_1,\\
&\quad \hat{N}_{\alpha\beta}(\boldsymbol{\zeta})[\partial_\beta \zeta_3 + \varepsilon \partial_\beta h]\nu_\alpha + \big(\partial_\alpha \hat{M}_{\alpha\beta}(\boldsymbol{\zeta})\big)\nu_\beta + \partial_\tau\big(\hat{M}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\alpha \tau_\beta\big) = -q_\alpha \nu_\alpha && \text{on } \gamma_1.
\end{aligned}$$

This equivalence is a straightforward consequence of the integration by parts formula and of the fundamental theorem in the calculus of variations.

3 Nonlinear Saint-Venant equations and boundary conditions for the stress resultants and bending moments

Given any smooth enough vector field $\boldsymbol{\eta} = (\eta_i) : \omega \to \mathbb{R}^3$, the corresponding stress resultants and bending moments are defined by the functions (Section 2)

$$\hat{N}_{\alpha\beta}(\boldsymbol{\eta}) := \varepsilon a_{\alpha\beta\sigma\tau} \hat{E}_{\sigma\tau}(\boldsymbol{\eta}), \quad \hat{M}_{\alpha\beta}(\boldsymbol{\eta}) := -\frac{\varepsilon^3}{3} a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \eta_3,$$

where

$$\begin{aligned}
a_{\alpha\beta\sigma\tau} &:= \frac{4\lambda\mu}{\lambda + 2\mu} \delta_{\sigma\tau}\delta_{\alpha\beta} + 2\mu(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma}),\\
\hat{E}_{\alpha\beta}(\boldsymbol{\eta}) &:= \frac{1}{2}\big(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \varepsilon(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) + \partial_\alpha \eta_3 \partial_\beta \eta_3\big).
\end{aligned}$$

The next theorem shows that the functions $\hat{N}_{\alpha\beta}(\boldsymbol{\eta})$ and $\hat{M}_{\alpha\beta}(\boldsymbol{\eta})$ characterise a vector field $\boldsymbol{\eta}$ belonging to the space $\boldsymbol{V}(\omega)$ appearing in the definition of the classical Kirchhoff-Love theory for nonlinearly elastic shallow shells (see Section 2).

Theorem 3.1. *Let $\omega \subset \mathbb{R}^2$ be a connected open set with a Lipschitz-continuous boundary γ, let γ_0 be a non-empty relatively open subset of γ, and let $h \in \mathcal{C}^2(\overline{\omega})$. Define the space*

$$\boldsymbol{V}(\omega) := \left\{ \boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega);\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma_0 \right\}.$$

Then the following assertions hold:

(i) If $\boldsymbol{\eta}, \boldsymbol{\zeta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ satisfy

$$\hat{M}_{\alpha\beta}(\boldsymbol{\eta}) = \hat{M}_{\alpha\beta}(\boldsymbol{\zeta}) \quad \text{and} \quad \hat{N}_{\alpha\beta}(\boldsymbol{\eta}) = \hat{N}_{\alpha\beta}(\boldsymbol{\zeta}) \quad \text{in } L^2(\omega),$$

then

$$\hat{E}_{\alpha\beta}(\boldsymbol{\eta}) = \hat{E}_{\alpha\beta}(\boldsymbol{\zeta}) \quad \text{and} \quad \partial_{\alpha\beta}\eta_3 = \partial_{\alpha\beta}\zeta_3 \quad \text{in } L^2(\omega).$$

(ii) If $\boldsymbol{\eta}, \boldsymbol{\zeta} \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ satisfy

$$\hat{E}_{\alpha\beta}(\boldsymbol{\eta}) = \hat{E}_{\alpha\beta}(\boldsymbol{\zeta}) \quad \text{and} \quad \partial_{\alpha\beta}\eta_3 = \partial_{\alpha\beta}\zeta_3 \quad \text{in } L^2(\omega),$$

then there exist six constants $a_1, a_2, a_3, b, d_1, d_2 \in \mathbb{R}$ such that

$$\begin{aligned}
\zeta_1(y) &= \eta_1(y) - d_1\eta_3(y) + a_1 - by_2 - \frac{d_1}{2}(d_1y_1 + d_2y_2) - \varepsilon d_1 h(y), \\
\zeta_2(y) &= \eta_2(y) - d_2\eta_3(y) + a_2 + by_1 - \frac{d_2}{2}(d_1y_1 + d_2y_2) - \varepsilon d_2 h(y), \\
\zeta_3(y) &= \eta_3(y) + a_3 + (d_1y_1 + d_2y_2)
\end{aligned}$$

for almost all $y = (y_1, y_2) \in \omega$.

(iii) If $\boldsymbol{\eta}, \boldsymbol{\zeta} \in \boldsymbol{V}(\omega)$ satisfy

$$\hat{E}_{\alpha\beta}(\boldsymbol{\eta}) = \hat{E}_{\alpha\beta}(\boldsymbol{\zeta}) \quad \text{and} \quad \partial_{\alpha\beta}\eta_3 = \partial_{\alpha\beta}\zeta_3 \quad \text{in } \omega,$$

then $\boldsymbol{\eta} = \boldsymbol{\zeta}$.

Proof. If two symmetric tensor fields $(S_{\alpha\beta})$ and $(T_{\alpha\beta})$ satisfy

$$T_{\alpha\beta} = a_{\alpha\beta\sigma\tau} S_{\alpha\beta},$$

then the definition of the functions $a_{\alpha\beta\sigma\tau}$ implies that

$$T_{\alpha\beta} = \frac{4\lambda\mu}{\lambda + 2\mu} S_{\sigma\sigma}\delta_{\alpha\beta} + 4\mu S_{\alpha\beta}.$$

In particular then,

$$T_{\sigma\sigma}=\left(\frac{8\lambda\mu}{\lambda+2\mu}+4\mu\right)S_{\sigma\sigma}.$$

Therefore, replacing $S_{\sigma\sigma}$ by this expression in the previous relation yields

$$T_{\alpha\beta}=\frac{\lambda}{3\lambda+2\mu}T_{\sigma\sigma}\delta_{\alpha\beta}+4\mu S_{\alpha\beta},$$

or equivalently,

$$S_{\alpha\beta}=b_{\alpha\beta\sigma\tau}T_{\sigma\tau},$$

where

$$b_{\alpha\beta\sigma\tau}:=\frac{1}{8\mu}(\delta_{\alpha\sigma}\delta_{\beta\tau}+\delta_{\alpha\tau}\delta_{\beta\sigma})-\frac{\lambda}{4\mu(3\lambda+2\mu)}\delta_{\sigma\tau}\delta_{\alpha\beta}.$$

Assertion (i) is proved by replacing the pair $(S_{\alpha\beta},T_{\alpha\beta})$ in the above relations successively by the pairs $(\hat{E}_{\alpha\beta}(\boldsymbol{\eta}),\hat{N}_{\alpha\beta}(\boldsymbol{\eta}))$, $(\hat{E}_{\alpha\beta}(\boldsymbol{\zeta}),\hat{N}_{\alpha\beta}(\boldsymbol{\zeta}))$, $(\partial_{\alpha\beta}\eta_3,\hat{M}_{\alpha\beta}(\boldsymbol{\eta}))$, and $(\partial_{\alpha\beta}\zeta_3,\hat{M}_{\alpha\beta}(\boldsymbol{\zeta}))$.

To prove assertion (ii) let $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ be two vector fields in $H^1(\omega)\times H^1(\omega)\times H^2(\omega)$ satisfying

$$\hat{E}_{\alpha\beta}(\boldsymbol{\eta})=\hat{E}_{\alpha\beta}(\boldsymbol{\zeta}) \quad \text{and} \quad \partial_{\alpha\beta}\eta_3=\partial_{\alpha\beta}\zeta_3 \quad \text{in } \omega.$$

Since ω is connected, the last relations imply that there exist constants $a_3,d_1,d_2 \in \mathbb{R}$ such that

$$\zeta_3(y)=\eta_3(y)+a_3+(d_1y_1+d_2y_2) \quad \text{for all} \quad y=(y_\alpha)\in\omega.$$

Define the vector field $\boldsymbol{\xi}:=(\xi_i):\omega\to\mathbb{R}^3$ by

$$\xi_\alpha(y):=\zeta_\alpha(y)+\varepsilon d_\alpha h(y), \quad \xi_3(y):=\zeta_3(y), \quad y\in\omega.$$

Then the relations $\hat{E}_{\alpha\beta}(\boldsymbol{\eta})=\hat{E}_{\alpha\beta}(\boldsymbol{\zeta})$ imply that

$$\frac{1}{2}(\partial_\alpha\eta_\beta+\partial_\beta\eta_\alpha+\partial_\alpha\eta_3\partial_\beta\eta_3)=\frac{1}{2}(\partial_\alpha\xi_\beta+\partial_\beta\xi_\alpha+\partial_\alpha\xi_3\partial_\beta\xi_3) \quad \text{in } \omega.$$

By a theorem due to Ciarlet and Mardare (see [8, Theorem 4.2]; the theorem is stated in *ibid.* for simply-connected domains, but the proof mentions that the one conclusion used here holds in fact under the weaker assumption that ω is only connected, like here), there then exist constants $a_1,a_2,b\in\mathbb{R}$ such that

$$\xi_1(y)=\eta_1(y)-d_1\eta_3(y)+a_1-by_2-\frac{d_1}{2}(d_1y_1+d_2y_2),$$
$$\xi_2(y)=\eta_2(y)-d_2\eta_3(y)+a_2+by_1-\frac{d_2}{2}(d_1y_1+d_2y_2)$$

for almost all $y=(y_1,y_2)\in\omega$. Consequently,

$$\begin{aligned}
&\zeta_1(y)=\eta_1(y)-d_1\eta_3(y)+a_1-by_2-\frac{d_1}{2}(d_1y_1+d_2y_2)-\varepsilon d_1h(y),\\
&\zeta_2(y)=\eta_2(y)-d_2\eta_3(y)+a_2+by_1-\frac{d_2}{2}(d_1y_1+d_2y_2)-\varepsilon d_2h(y),\\
&\zeta_3(y)=\eta_3(y)+a_3+(d_1y_1+d_2y_2)
\end{aligned}$$

for almost all $y=(y_1,y_2)\in\omega$.

Assertion (iii) is a consequence of assertion (ii) together with the boundary conditions appearing in the definition of the space $\boldsymbol{V}(\omega)$. To see this, let $\boldsymbol{\eta},\boldsymbol{\zeta}\in\boldsymbol{V}(\omega)$ be such that

$$\hat{E}_{\alpha\beta}(\boldsymbol{\eta})=\hat{E}_{\alpha\beta}(\boldsymbol{\zeta}) \quad \text{and} \quad \partial_{\alpha\beta}\eta_3=\partial_{\alpha\beta}\zeta_3 \quad \text{in } \omega.$$

Then assertion (ii) shows that there exist constants $a_1,a_2,a_3,b,d_1,d_2\in\mathbb{R}$ such that the above relations hold.

Furthermore, the assumption that the vector fields $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ belong to the space $\boldsymbol{V}(\omega)$ implies in particular that their components satisfy boundary conditions

$$\partial_\alpha\zeta_3=\partial_\alpha\eta_3=0 \quad \text{and} \quad \zeta_3=\eta_3=0 \quad \text{on } \gamma_0.$$

Since γ_0 is non-empty by assumption, this implies $d_\alpha=0$ and $a_3=0$.

Besides, the assumption that the vector fields $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$ belong to the space $\boldsymbol{V}(\omega)$ implies that their components satisfy boundary conditions

$$\zeta_\alpha=\eta_\alpha=0 \quad \text{on } \gamma_0.$$

Therefore $a_1=a_2=b=0$. The proof is complete. $\square$

The next theorem shows that the stress resultants and bending moments

$$\hat{N}_{\alpha\beta}(\boldsymbol{\eta}):=\varepsilon a_{\alpha\beta\sigma\tau}\hat{E}_{\sigma\tau}(\boldsymbol{\eta}), \quad \hat{M}_{\alpha\beta}(\boldsymbol{\eta}):=-\frac{\varepsilon^3}{3}a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\eta_3,$$

or equivalently the components

$$\hat{E}_{\alpha\beta}(\boldsymbol{\eta}):=\frac{1}{\varepsilon}b_{\alpha\beta\sigma\tau}\hat{N}_{\sigma\tau}(\boldsymbol{\eta}), \quad \partial_{\alpha\beta}\eta_3:=-\frac{3}{\varepsilon^3}b_{\alpha\beta\sigma\tau}\hat{M}_{\sigma\tau}(\boldsymbol{\eta})$$

of the strain tensor fields necessarily satisfy specific compatibility conditions of Saint-Venant type, and that these conditions become sufficient if the set ω is simply-connected.

Theorem 3.2. *(i) Let $\omega \subset \mathbb{R}^2$ be a connected open set, let $\varepsilon > 0$, let $h \in \mathcal{C}^2(\overline{\omega})$, and let $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$.*

Then the functions

$$
\begin{aligned}
&E_{\alpha\beta} := \frac{1}{2}\big(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \varepsilon(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) + \partial_\alpha \eta_3 \partial_\beta \eta_3\big) \in L^2(\omega),\\
&F_{\alpha\beta} := \partial_{\alpha\beta}\eta_3 \in L^2(\omega)
\end{aligned}
$$

satisfy the relations

$$
\begin{aligned}
&E_{\alpha\beta} = E_{\beta\alpha} && \text{in } L^2(\omega),\\
&F_{\alpha\beta} = F_{\beta\alpha} && \text{in } L^2(\omega),\\
&\partial_\alpha F_{\beta\sigma} = \partial_\beta F_{\alpha\sigma} && \text{in } H^{-1}(\omega),\\
&\partial_{\alpha\beta} E_{\sigma\tau} + \partial_{\sigma\tau} E_{\alpha\beta} - \partial_{\alpha\sigma} E_{\beta\tau} - \partial_{\beta\tau} E_{\alpha\sigma}\\
&= F_{\alpha\sigma} F_{\beta\tau} - F_{\alpha\beta} F_{\sigma\tau} + \varepsilon\big(\partial_{\alpha\sigma} h F_{\beta\tau} + \partial_{\beta\tau} h F_{\alpha\sigma} - \partial_{\alpha\beta} h F_{\sigma\tau} - \partial_{\sigma\tau} h F_{\alpha\beta}\big) && \text{in } H^{-2}(\omega).
\end{aligned}
$$

(ii) Let $\omega \subset \mathbb{R}^2$ be a simply-connected open set, let $\varepsilon > 0$, let $h \in \mathcal{C}^2(\overline{\omega})$, and let $E_{\alpha\beta} \in L^2(\omega)$ and $F_{\alpha\beta} \in L^2(\omega)$ be such that

$$
\begin{aligned}
&E_{\alpha\beta} = E_{\beta\alpha} && \text{in } L^2(\omega),\\
&F_{\alpha\beta} = F_{\beta\alpha} && \text{in } L^2(\omega),\\
&\partial_\alpha F_{\beta\sigma} = \partial_\beta F_{\alpha\sigma} && \text{in } H^{-1}(\omega),\\
&\partial_{\alpha\beta} E_{\sigma\tau} + \partial_{\sigma\tau} E_{\alpha\beta} - \partial_{\alpha\sigma} E_{\beta\tau} - \partial_{\beta\tau} E_{\alpha\sigma}\\
&= F_{\alpha\sigma} F_{\beta\tau} - F_{\alpha\beta} F_{\sigma\tau} + \varepsilon\big(\partial_{\alpha\sigma} h F_{\beta\tau} + \partial_{\beta\tau} h F_{\alpha\sigma} - \partial_{\alpha\beta} h F_{\sigma\tau} - \partial_{\sigma\tau} h F_{\alpha\beta}\big) && \text{in } H^{-2}(\omega).
\end{aligned}
$$

Then there exists a vector field $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ such that

$$
\begin{aligned}
&E_{\alpha\beta} = \frac{1}{2}\big(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \varepsilon(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) + \partial_\alpha \eta_3 \partial_\beta \eta_3\big) && \text{in } L^2(\omega),\\
&F_{\alpha\beta} = \partial_{\alpha\beta}\eta_3 && \text{in } L^2(\omega).
\end{aligned}
$$

Proof. (i) Given any vector field $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$, let

$$
\begin{aligned}
&A_{\alpha\beta} := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) && \text{in } L^2(\omega),\\
&B_{\alpha\beta} := \partial_{\alpha\beta}\eta_3 && \text{in } L^2(\omega).
\end{aligned}
$$

Then, by [8, Theorem 4.1], the functions $A_{\alpha\beta} \in L^2(\omega)$ and $B_{\alpha\beta} \in L^2(\omega)$ satisfy the compatibility conditions

$$\begin{aligned}
&A_{\alpha\beta} = A_{\beta\alpha} && \text{in } L^2(\omega),\\
&B_{\alpha\beta} = B_{\beta\alpha} && \text{in } L^2(\omega),\\
&\partial_\alpha B_{\beta\sigma} = \partial_\beta B_{\alpha\sigma} && \text{in } H^{-1}(\omega),\\
&\partial_{\alpha\beta} A_{\sigma\tau} + \partial_{\sigma\tau} A_{\alpha\beta} - \partial_{\alpha\sigma} A_{\beta\tau} - \partial_{\beta\tau} A_{\alpha\sigma}\\
&= B_{\alpha\sigma} B_{\beta\tau} - B_{\alpha\beta} B_{\sigma\tau} && \text{in } H^{-2}(\omega).
\end{aligned}$$

Next, the definition of the functions $E_{\alpha\beta}$ and $F_{\alpha\beta}$ in assertion (i) implies that

$$\begin{aligned}
&A_{\alpha\beta} = E_{\alpha\beta} - \frac{\varepsilon}{2}(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) && \text{in } L^2(\omega),\\
&B_{\alpha\beta} = F_{\alpha\beta} && \text{in } L^2(\omega).
\end{aligned}$$

Replacing the functions $A_{\alpha\beta}$ and $B_{\alpha\beta}$ by these expressions in the above compatibility conditions yields the compatibility conditions for $E_{\alpha\beta}$ and $F_{\alpha\beta}$ stated in assertion (i).

(ii) Let $E_{\alpha\beta} \in L^2(\omega)$ and $F_{\alpha\beta} \in L^2(\omega)$ be functions that satisfy the compatibility conditions stated in assertion (ii). In particular then

$$\begin{aligned}
&F_{\alpha\beta} = F_{\beta\alpha} && \text{in } L^2(\omega),\\
&\partial_\alpha F_{\beta\sigma} = \partial_\beta F_{\alpha\sigma} && \text{in } H^{-1}(\omega).
\end{aligned}$$

Since ω is simply-connected, [8, Theorem 4.2] shows that there exists a function $\eta_3 \in H^2(\omega)$ such that

$$F_{\alpha\beta} = \partial_{\alpha\beta}\eta_3 \quad \text{in } L^2(\omega).$$

Define the functions

$$A_{\alpha\beta} := E_{\alpha\beta} - \frac{\varepsilon}{2}(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) \quad \text{in } L^2(\omega).$$

Then replacing the functions $E_{\alpha\beta}$ by $(A_{\alpha\beta} + \frac{\varepsilon}{2}(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3))$ in the compatibility conditions stated in assertion (ii) shows that the functions $A_{\alpha\beta}$ and $F_{\alpha\beta}$ satisfy the relations

$$\begin{aligned}
&A_{\alpha\beta} = A_{\beta\alpha} && \text{in } L^2(\omega),\\
&F_{\alpha\beta} = F_{\beta\alpha} && \text{in } L^2(\omega),
\end{aligned}$$

$$\begin{aligned}
&\partial_\alpha F_{\beta\sigma} = \partial_\beta F_{\alpha\sigma} && \text{in } H^{-1}(\omega),\\
&\partial_{\alpha\beta} A_{\sigma\tau} + \partial_{\sigma\tau} A_{\alpha\beta} - \partial_{\alpha\sigma} A_{\beta\tau} - \partial_{\beta\tau} A_{\alpha\sigma}\\
&= F_{\alpha\sigma} F_{\beta\tau} - F_{\alpha\beta} F_{\sigma\tau} && \text{in } H^{-2}(\omega).
\end{aligned}$$

Consequently, since ω is simply-connected, [8, Theorem 4.2] can again be applied to prove the existence of functions $\eta_\alpha \in H^1(\omega)$ such that

$$A_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \partial_\alpha \eta_3 \partial_\beta \eta_3) \quad \text{in } L^2(\omega).$$

Thus the vector field $\boldsymbol{\eta} = (\eta_i) \in H^1(\omega) \times H^1(\omega) \times H^2(\omega)$ found in this fashion satisfies

$$\begin{aligned}
&E_{\alpha\beta} = \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \varepsilon(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) + \partial_\alpha \eta_3 \partial_\beta \eta_3) && \text{in } L^2(\omega),\\
&F_{\alpha\beta} = \partial_{\alpha\beta} \eta_3 && \text{in } L^2(\omega).
\end{aligned}$$

The proof is complete. $\square$

The next theorem shows that if a smooth enough vector field $\boldsymbol{\eta} = (\eta_i) : \overline{\omega} \to \mathbb{R}^3$ satisfies the boundary conditions

$$\eta_i = \partial_\nu \eta_3 = 0 \quad \text{on } \gamma_0,$$

then the stress resultants and bending moments

$$\hat{N}_{\alpha\beta}(\boldsymbol{\eta}) := \varepsilon a_{\alpha\beta\sigma\tau} \hat{E}_{\sigma\tau}(\boldsymbol{\eta}), \quad \hat{M}_{\alpha\beta}(\boldsymbol{\eta}) := -\frac{\varepsilon^3}{3} a_{\alpha\beta\sigma\tau} \partial_{\sigma\tau} \eta_3,$$

or equivalently the strain tensor fields

$$\hat{E}_{\alpha\beta}(\boldsymbol{\eta}) := \frac{1}{\varepsilon} b_{\alpha\beta\sigma\tau} \hat{N}_{\sigma\tau}(\boldsymbol{\eta}), \quad \partial_{\alpha\beta} \eta_3 := -\frac{3}{\varepsilon^3} b_{\alpha\beta\sigma\tau} \hat{M}_{\sigma\tau}(\boldsymbol{\eta})$$

necessarily satisfy specific boundary conditions.

Theorem 3.3. *Let $\omega \subset \mathbb{R}^2$ be a connected open set with a boundary γ of class $\mathcal{C}^2$, let γ_0 be a non-empty relatively open subset of γ, and let $h \in \mathcal{C}^2(\overline{\omega})$. Let $(\eta_i) \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ be a vector field that satisfies the boundary conditions*

$$\eta_i = \partial_\nu \eta_3 = 0 \quad \text{on } \gamma_0.$$

Then the functions

$$E_{\alpha\beta} := \frac{1}{2}\big(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha + \varepsilon(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) + \partial_\alpha \eta_3 \partial_\beta \eta_3\big) = E_{\beta\alpha} \in \mathcal{C}^1(\overline{\omega}),$$
$$F_{\alpha\beta} := \partial_{\alpha\beta}\eta_3 = F_{\beta\alpha} \in \mathcal{C}^0(\overline{\omega})$$

satisfy the boundary conditions

$$\begin{aligned} &E_{\alpha\beta}\tau_\alpha\tau_\beta = \partial_\sigma E_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma - 2\tau_\sigma\nu_\beta) - \kappa E_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \quad && \textit{on } \gamma_0, \\ &F_{\alpha\beta}\tau_\beta = 0 && \textit{on } \gamma_0, \end{aligned}$$

where (ν_α) *denotes the inner normal unit vector field along* γ, $\tau_1 := \nu_2$, $\tau_2 := -\nu_1$, *and*

$$\kappa := \nu_\alpha \cdot \partial_\tau \tau_\alpha$$

denotes the signed curvature of the planar curve γ.

Proof. Let $\boldsymbol{\eta} = (\eta_i) \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ be such that

$$\eta_i = \partial_\nu \eta_3 = 0 \quad \text{on } \gamma_0.$$

Then [10, Theorem 4.1] implies that the functions

$$c_{\alpha\beta} := \frac{1}{2}(\partial_\alpha \eta_\beta + \partial_\beta \eta_\alpha) \in \mathcal{C}^1(\overline{\omega}), \quad F_{\alpha\beta} := \partial_{\alpha\beta}\eta_3 \in \mathcal{C}^0(\overline{\omega})$$

satisfy the boundary conditions

$$c_{\alpha\beta}\tau_\alpha\tau_\beta = \partial_\sigma c_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma - 2\tau_\sigma\nu_\beta) - \kappa c_{\alpha\beta}\nu_\alpha\nu_\beta = 0 \quad \text{on } \gamma_0, \tag{3.1a}$$
$$F_{\alpha\beta}\tau_\alpha\tau_\beta = F_{\alpha\beta}\tau_\alpha\nu_\beta = 0 \quad \text{on } \gamma_0. \tag{3.1b}$$

The relation (3.1b), the definition of the vector fields (ν_α) and (τ_α), and the symmetries $F_{\alpha\beta} = F_{\beta\alpha}$ in $\overline{\omega}$, together imply that

$$F_{\alpha\beta}\tau_\beta = 0 \quad \text{on } \gamma_0$$

on the one hand.

The boundary conditions $\eta_3 = \partial_\nu \eta_3 = 0$ on γ_0 imply that

$$\begin{aligned} &\partial_\alpha \eta_3 = 0 && \text{on } \gamma_0, \\ &\partial_\sigma(\partial_\alpha \eta_3 \partial_\beta \eta_3) = \partial_{\sigma\alpha}\eta_3 \partial_\beta \eta_3 + \partial_\alpha \eta_3 \partial_{\sigma\beta}\eta_3 = 0 \quad && \text{on } \gamma_0. \end{aligned}$$

Then the definition of the functions

$$E_{\alpha\beta} := c_{\alpha\beta} + \frac{\varepsilon}{2}(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) + \frac{1}{2}(\partial_\alpha \eta_3 \partial_\beta \eta_3) \quad \text{in } \overline{\omega}$$

implies that

$$E_{\alpha\beta} = c_{\alpha\beta} \quad \text{and} \quad \partial_\sigma E_{\alpha\beta} = \partial_\sigma c_{\alpha\beta} \quad \text{on } \gamma_0.$$

Therefore,

$$\begin{aligned} &E_{\alpha\beta}\tau_\alpha\tau_\beta = c_{\alpha\beta}\tau_\alpha\tau_\beta = 0 && \text{on } \gamma_0, \\ &\partial_\sigma E_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma - 2\tau_\sigma\nu_\beta) - \kappa E_{\alpha\beta}\nu_\alpha\nu_\beta \\ &= \partial_\sigma c_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma - 2\tau_\sigma\nu_\beta) - \kappa c_{\alpha\beta}\nu_\alpha\nu_\beta = 0 && \text{on } \gamma_0 \end{aligned}$$

on the other hand. The proof is complete. □

The next theorem shows that the converse of Theorem 3.3 holds under the additional assumption that γ_0 is connected.

Theorem 3.4. *Let $\omega \subset \mathbb{R}^2$ be a connected open set with a boundary γ of class $\mathcal{C}^2$, let γ_0 be a non-empty, connected and relatively open subset of γ, and let $h \in \mathcal{C}^2(\overline{\omega})$.*

Let functions $E_{\alpha\beta} \in \mathcal{C}^1(\overline{\omega})$ and $F_{\alpha\beta} \in \mathcal{C}^0(\overline{\omega})$ satisfy the following properties: There exists a vector field $(\eta_i) \in \mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ such that

$$\begin{aligned} &E_{\alpha\beta} = \frac{1}{2}\big(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha + \varepsilon(\partial_\alpha h \partial_\beta \eta_3 + \partial_\beta h \partial_\alpha \eta_3) + \partial_\alpha\eta_3\partial_\beta\eta_3\big) && \textit{in } \overline{\omega}, \\ &F_{\alpha\beta} = \partial_{\alpha\beta}\eta_3 && \textit{in } \overline{\omega}, \end{aligned}$$

and the following boundary conditions are satisfied:

$$\begin{aligned} &E_{\alpha\beta}\tau_\alpha\tau_\beta = \partial_\sigma E_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma - 2\tau_\sigma\nu_\beta) - \kappa E_{\alpha\beta}\nu_\alpha\nu_\beta = 0 && \textit{on } \gamma_0, \\ &F_{\alpha\beta}\tau_\beta = 0 && \textit{on } \gamma_0, \end{aligned}$$

where (ν_α) denote the inner normal unit vector field along the planar curve γ, $\tau_1 := \nu_2$, $\tau_2 := -\nu_1$ and $\kappa := \nu_\alpha \cdot \partial_\tau \tau_\alpha$.

Then there exist constants $a_1, a_2, a_3, b, d_1, d_2 \in \mathbb{R}$ such that the vector field $\boldsymbol{\zeta} = (\zeta_i) : \overline{\omega} \to \mathbb{R}^3$, defined by

$$\begin{aligned} &\zeta_1(y) := \eta_1(y) - d_1\eta_3(y) + a_1 - b y_2 - \frac{d_1}{2}(d_1 y_1 + d_2 y_2) - \varepsilon d_1 h(y), \\ &\zeta_2(y) := \eta_2(y) - d_2\eta_3(y) + a_2 + b y_1 - \frac{d_2}{2}(d_1 y_1 + d_2 y_2) - \varepsilon d_2 h(y), \\ &\zeta_3(y) := \eta_3(y) + a_3 + (d_1 y_1 + d_2 y_2), \quad y = (y_1, y_2) \in \overline{\omega} \end{aligned}$$

satisfies

$$\begin{aligned}
&\frac{1}{2}\big(\partial_\alpha\zeta_\beta+\partial_\beta\zeta_\alpha+\varepsilon(\partial_\alpha h\partial_\beta\zeta_3+\partial_\beta h\partial_\alpha\zeta_3)+\partial_\alpha\zeta_3\partial_\beta\zeta_3\big)=E_{\alpha\beta} && \textit{in } \overline{\omega},\\
&\partial_{\alpha\beta}\zeta_3=F_{\alpha\beta} && \textit{in } \overline{\omega},\\
&\zeta_i=\partial_\nu\zeta_3=0 && \textit{on } \gamma_0.
\end{aligned}$$

Proof. Let functions $E_{\alpha\beta}$ and $F_{\alpha\beta}$ satisfy the assumptions of the theorem. In particular then

$$F_{\alpha\beta}\tau_\beta=0 \quad \text{on } \gamma_0.$$

Consequently,

$$\partial_\tau(\partial_\alpha\eta_3)=F_{\alpha\beta}\tau_\beta=0 \quad \text{on } \gamma_0,$$

which means that the tangential derivative of each function $(\partial_\alpha\eta_3)$ vanishes along the connected curve γ_0; hence there exist two constants $d_1,d_2\in\mathbb{R}$ such that

$$\partial_\alpha\eta_3(y)+d_\alpha=0 \quad \text{for all} \quad y\in\gamma_0.$$

Define the vector field $(\xi_i)\in\mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ by

$$\begin{aligned}
&\xi_1(y):=\eta_1(y)-d_1\eta_3(y)-\frac{d_1}{2}(d_1y_1+d_2y_2)-\varepsilon d_1h(y),\\
&\xi_2(y):=\eta_2(y)-d_2\eta_3(y)-\frac{d_2}{2}(d_1y_1+d_2y_2)-\varepsilon d_2h(y),\\
&\xi_3(y):=\eta_3(y)+(d_1y_1+d_2y_2), \quad y=(y_1,y_2)\in\overline{\omega}.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{2}\big(\partial_\alpha\xi_\beta+\partial_\beta\xi_\alpha+\varepsilon(\partial_\alpha h\partial_\beta\xi_3+\partial_\beta h\partial_\alpha\xi_3)+\partial_\alpha\xi_3\partial_\beta\xi_3\big)=E_{\alpha\beta} && \text{in } \overline{\omega},\\
&\partial_{\alpha\beta}\xi_3=F_{\alpha\beta} && \text{in } \overline{\omega},\\
&\partial_\alpha\xi_3=\partial_\alpha\eta_3+d_\alpha=0 && \text{on } \gamma_0,\\
&\partial_\sigma(\partial_\alpha\xi_3\partial_\beta\xi_3)=\partial_{\sigma\alpha}\xi_3\partial_\beta\xi_3+\partial_\alpha\xi_3\partial_{\sigma\beta}\xi_3=0 && \text{on } \gamma_0.
\end{aligned}$$

Define the functions $c_{\alpha\beta}\in\mathcal{C}^1(\overline{\omega})$ by

$$c_{\alpha\beta}:=\frac{1}{2}(\partial_\alpha\xi_\beta+\partial_\beta\xi_\alpha).$$

Then, combining the relations

$$c_{\alpha\beta}=E_{\alpha\beta}-\frac{\varepsilon}{2}(\partial_\alpha h\partial_\beta\xi_3+\partial_\beta h\partial_\alpha\xi_3)-\frac{1}{2}\partial_\alpha\xi_3\partial_\beta\xi_3$$

with the above boundary conditions satisfied by ξ_3, we infer that

$$c_{\alpha\beta}=E_{\alpha\beta} \quad \text{and} \quad \partial_\sigma c_{\alpha\beta}=\partial_\sigma E_{\alpha\beta} \quad \text{on } \gamma_0.$$

Consequently, the boundary conditions for the functions $E_{\alpha\beta}$ and $F_{\alpha\beta}$ appearing in the statement of the theorem imply that the functions $c_{\alpha\beta}:=\frac{1}{2}(\partial_\alpha\xi_\beta+\partial_\beta\xi_\alpha)$ and $F_{\alpha\beta}:=\partial_{\alpha\beta}\xi_3$ associated with the vector field (ξ_i) satisfy the boundary conditions

$$\begin{aligned} &c_{\alpha\beta}\tau_\alpha\tau_\beta=\partial_\sigma c_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma-2\tau_\sigma\nu_\beta)-\kappa c_{\alpha\beta}\nu_\alpha\nu_\beta=0 && \text{on } \gamma_0,\\ &F_{\alpha\beta}\tau_\alpha\tau_\beta=F_{\alpha\beta}\tau_\alpha\nu_\beta=0 && \text{on } \gamma_0. \end{aligned}$$

Then a theorem due to Ciarlet & Mardare (see [10, Theorem 4.1]) implies that there exists constants $a_1,a_2,a_3,b\in\mathbb{R}$ such that

$$\zeta_i=\partial_\nu\zeta_3=0 \quad \text{on } \gamma_0,$$

where

$$\begin{aligned} &\zeta_1(y):=\xi_1(y)+a_1-by_2, && y=(y_\alpha)\in\overline{\omega},\\ &\zeta_2(y):=\xi_2(y)+a_2+by_1, && y=(y_\alpha)\in\overline{\omega},\\ &\zeta_3(y):=\xi_3(y)+a_3, && y=(y_\alpha)\in\overline{\omega}. \end{aligned}$$

This proves that the vector field $\boldsymbol{\zeta}=(\zeta_i)\in\mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ satisfies all the announced properties. The proof is complete. $\square$

4 New intrinsic formulation of the Kirchhoff-Love theory of a nonlinearly elastic shallow shell

We are now in a position to introduce our new intrinsic formulation of the Kirchhoff-Love theory of a nonlinearly elastic shallow shell and to justify it by proving its equivalence to the classical formulation of the same equations, cf. Theorems 4.1 and 4.2 below.

An intrinsic formulation of the Kirchhoff-Love theory for a nonlinearly elastic shallow shell consists in replacing the unknown displacement field appearing in the classical approach by new unknowns, which are in effect either "measures of strain" or "measures of stress" inside the deformed shell, with the property that the displacement field can be recovered a posteriori from them, once the problem is solved in its intrinsic formulation. As we already proved in Section 3 that the displacement field can be recovered uniquely from the stress resultants and bending moments of the middle surface of the deformed shell, it remains to find the equations satisfied by these new unknowns. This is the object of the next theorem.

Recall that $\lambda \geq 0$ and $\mu > 0$ denote the Lamé constants of the elastic material constituting the shell,

$$a_{\alpha\beta\sigma\tau} := \frac{4\lambda\mu}{\lambda+2\mu}\delta_{\sigma\tau}\delta_{\alpha\beta} + 2\mu(\delta_{\alpha\sigma}\delta_{\beta\tau} + \delta_{\alpha\tau}\delta_{\beta\sigma})$$

denote the components of the corresponding two-dimensional elasticity tensor, $2\varepsilon > 0$ denotes the thickness of the shell, and $h \in \mathcal{C}^2(\overline{\omega})$ denotes the function defining the middle surface of the shell as the graph of the function

$$x_3 = \varepsilon h(y), \quad y \in \overline{\omega}, \quad \omega \subset \mathbb{R}^2.$$

The stress resultants and bending moments associated with a displacement field $\boldsymbol{\zeta} = (\zeta_i) \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ of the middle surface of the shell are respectively denoted and defined by

$$\hat{N}_{\alpha\beta}(\boldsymbol{\zeta}) := \varepsilon a_{\alpha\beta\sigma\tau}\hat{E}_{\sigma\tau}(\boldsymbol{\zeta}), \quad \hat{M}_{\alpha\beta}(\boldsymbol{\zeta}) := -\frac{\varepsilon^3}{3}a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3,$$

where

$$\hat{E}_{\alpha\beta}(\boldsymbol{\zeta}) := \frac{1}{2}\big(\partial_\alpha\zeta_\beta + \partial_\beta\zeta_\alpha + \varepsilon(\partial_\alpha h\partial_\beta\zeta_3 + \partial_\beta h\partial_\alpha\zeta_3) + \partial_\alpha\zeta_3\partial_\beta\zeta_3\big).$$

Theorem 4.1. *Let $\omega \subset \mathbb{R}^2$ be an open and connected set with a boundary γ of class $\mathcal{C}^2$, let γ_0 be a nonempty relatively open subset of γ, let $\gamma_1 := \gamma \setminus \gamma_0$, and let $h \in \mathcal{C}^2(\overline{\omega})$.*

Assume that a vector field $\boldsymbol{\zeta} = (\zeta_i) \in \mathcal{C}^2(\overline{\omega}; \mathbb{R}^3)$ satisfies the boundary value problem

$$\begin{aligned}
&-\partial_\beta\hat{N}_{\alpha\beta}(\boldsymbol{\zeta}) = p_\alpha && \textit{in } \omega, && (4.1a)\\
&-\partial_{\alpha\beta}\hat{M}_{\alpha\beta}(\boldsymbol{\zeta}) - \partial_\alpha\big(\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})[\partial_\beta\zeta_3 + \varepsilon\partial_\beta h]\big) = p_3 + \partial_\alpha q_\alpha && \textit{in } \omega, && (4.1b)\\
&\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\beta = \hat{M}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\alpha\nu_\beta = 0 && \textit{on } \gamma_1, && (4.1c)\\
&\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\alpha(\partial_\beta\zeta_3 + \varepsilon\partial_\beta h) + (\partial_\alpha\hat{M}_{\alpha\beta}(\boldsymbol{\zeta}))\nu_\beta \\
&\quad + \partial_\tau(\hat{M}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\alpha\tau_\beta) = -q_\alpha\nu_\alpha && \textit{on } \gamma_1, && (4.1d)\\
&\zeta_i = \partial_\nu\zeta_3 = 0 && \textit{on } \gamma_0. && (4.1e)
\end{aligned}$$

Then the functions

$$E_{\alpha\beta} := \hat{E}_{\alpha\beta}(\boldsymbol{\zeta}), \quad F_{\alpha\beta} := \partial_{\alpha\beta}\zeta_3, \quad M_{\alpha\beta} := \hat{M}_{\alpha\beta}(\boldsymbol{\zeta}), \quad N_{\alpha\beta} := \hat{N}_{\alpha\beta}(\boldsymbol{\zeta})$$

satisfy the boundary value problem

$$
\begin{aligned}
&-\partial_\beta N_{\alpha\beta}=p_\alpha && \textit{in } \omega, && (4.2a)\\
&-\partial_{\alpha\beta}M_{\alpha\beta}-\partial_\alpha(N_{\alpha\beta}[\partial_\beta\zeta_3+\varepsilon\partial_\beta h])=p_3+\partial_\alpha q_\alpha && \textit{in } \omega, && (4.2b)\\
&\quad N_{\alpha\beta}\nu_\beta=M_{\alpha\beta}\nu_\alpha\nu_\beta=0 && \textit{on } \gamma_1, && (4.2c)\\
&\quad N_{\alpha\beta}\nu_\alpha(\partial_\beta\zeta_3+\varepsilon\partial_\beta h)+(\partial_\alpha M_{\alpha\beta})\nu_\beta+\partial_\tau(M_{\alpha\beta}\nu_\alpha\tau_\beta)=-q_\alpha\nu_\alpha && \textit{on } \gamma_1, && (4.2d)\\
&\quad \partial_\alpha F_{\beta\sigma}-\partial_\beta F_{\alpha\sigma}=0 && \textit{in } \omega, && (4.2e)\\
&\quad \partial_{\alpha\beta}E_{\sigma\tau}+\partial_{\sigma\tau}E_{\alpha\beta}-\partial_{\alpha\sigma}E_{\beta\tau}-\partial_{\beta\tau}E_{\alpha\sigma}\\
&=F_{\alpha\sigma}F_{\beta\tau}-F_{\alpha\beta}F_{\sigma\tau}+\varepsilon(\partial_{\alpha\sigma}hF_{\beta\tau}+\partial_{\beta\tau}hF_{\alpha\sigma}-\partial_{\alpha\beta}hF_{\sigma\tau}-\partial_{\sigma\tau}hF_{\alpha\beta}) && \textit{in } \omega, && (4.2f)\\
&\quad E_{\alpha\beta}\tau_\alpha\tau_\beta=\partial_\sigma E_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma-2\tau_\sigma\nu_\beta)-\kappa E_{\alpha\beta}\nu_\alpha\nu_\beta=F_{\alpha\beta}\tau_\beta=0 && \textit{on } \gamma_0. && (4.2g)
\end{aligned}
$$

Proof. Let $\boldsymbol{\zeta}=(\zeta_i)\in\mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ be a vector field that satisfies the boundary value problem (4.1). Then

i) Eqs. (4.2a)-(4.2d) are deduced from Eqs. (4.1a)-(4.1d) by using the definition of the functions $M_{\alpha\beta}:=\hat{M}_{\alpha\beta}(\boldsymbol{\zeta})$ and $N_{\alpha\beta}:=\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})$.

ii) Eqs. (4.2e) and (4.2f) are satisfied thanks to the definition of the functions $E_{\alpha\beta}:=\hat{E}_{\alpha\beta}(\boldsymbol{\zeta})$ and $F_{\alpha\beta}:=\partial_{\alpha\beta}\zeta_3$ combined with Theorem 3.2 (i).

iii) Eq. (4.2g) was shown to hold in Theorem 3.3.

The proof is complete. □

The next theorem shows that the converse of Theorem 4.1 holds under the additional assumptions that ω is simply-connected and that γ_0 is connected. Recall that

$$b_{\alpha\beta\sigma\tau}:=\frac{1}{8\mu}(\delta_{\alpha\sigma}\delta_{\beta\tau}+\delta_{\alpha\tau}\delta_{\beta\sigma})-\frac{\lambda}{4\mu(3\lambda+2\mu)}\delta_{\sigma\tau}\delta_{\alpha\beta}$$

are the components of the inverse of the two-dimensional elasticity tensor with components $a_{\alpha\beta\sigma\tau}$.

Theorem 4.2. *Let $\omega\subset\mathbb{R}^2$ be an open and simply-connected set with a boundary γ of class $\mathcal{C}^2$, let γ_0 be a nonempty connected and relatively open subset of γ, and let $h\in\mathcal{C}^2(\overline{\omega})$. Let $\gamma_1:=\gamma\setminus\gamma_0$.*

Assume that symmetric tensor fields $(M_{\alpha\beta})\in\mathcal{C}^0(\overline{\omega};\mathbb{S}^2)$ and $(N_{\alpha\beta})\in\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$ satisfy the boundary value problem

$$
\begin{aligned}
&-\partial_\beta N_{\alpha\beta}=p_\alpha && \textit{in } \omega, && \text{(4.3a)}\\
&-\partial_{\alpha\beta}M_{\alpha\beta}-\partial_\alpha\big(N_{\alpha\beta}[\partial_\beta\zeta_3+\varepsilon\partial_\beta h]\big)=p_3+\partial_\alpha q_\alpha && \textit{in } \omega, && \text{(4.3b)}\\
&\quad N_{\alpha\beta}\nu_\beta=M_{\alpha\beta}\nu_\alpha\nu_\beta=0 && \textit{on } \gamma_1, && \text{(4.3c)}\\
&\quad N_{\alpha\beta}\nu_\alpha(\partial_\beta\zeta_3+\varepsilon\partial_\beta h)+(\partial_\alpha M_{\alpha\beta})\nu_\beta+\partial_\tau(M_{\alpha\beta}\nu_\alpha\tau_\beta)=-q_\alpha\nu_\alpha && \textit{on } \gamma_1, && \text{(4.3d)}\\
&\quad \partial_\alpha F_{\beta\sigma}-\partial_\beta F_{\alpha\sigma}=0 && \textit{in } \omega, && \text{(4.3e)}\\
&\quad \partial_{\alpha\beta}E_{\sigma\tau}+\partial_{\sigma\tau}E_{\alpha\beta}-\partial_{\alpha\sigma}E_{\beta\tau}-\partial_{\beta\tau}E_{\alpha\sigma}\\
&=F_{\alpha\sigma}F_{\beta\tau}-F_{\alpha\beta}F_{\sigma\tau}+\varepsilon\big(\partial_{\alpha\sigma}hF_{\beta\tau}+\partial_{\beta\tau}hF_{\alpha\sigma}-\partial_{\alpha\beta}hF_{\sigma\tau}-\partial_{\sigma\tau}hF_{\alpha\beta}\big) && \textit{in } \omega, && \text{(4.3f)}\\
&\quad E_{\alpha\beta}\tau_\alpha\tau_\beta=\partial_\sigma E_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma-2\tau_\sigma\nu_\beta)-\kappa E_{\alpha\beta}\nu_\alpha\nu_\beta=F_{\alpha\beta}\tau_\beta=0 && \textit{on } \gamma_0, && \text{(4.3g)}
\end{aligned}
$$

where

$$E_{\alpha\beta}:=\frac{1}{\varepsilon}b_{\alpha\beta\sigma\tau}N_{\sigma\tau},\quad F_{\alpha\beta}:=-\frac{3}{\varepsilon^3}b_{\alpha\beta\sigma\tau}M_{\sigma\tau}.$$

Then there exists a unique vector field $\boldsymbol{\zeta}=(\zeta_i)\in\mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$ *such that*

$$
\begin{aligned}
&M_{\alpha\beta}=\hat{M}_{\alpha\beta}(\boldsymbol{\zeta}) && \textit{in } \omega,\\
&N_{\alpha\beta}=\hat{N}_{\alpha\beta}(\boldsymbol{\zeta}) && \textit{in } \omega,\\
&\zeta_i=\partial_\nu\zeta_3=0 && \textit{on } \gamma_0.
\end{aligned}
$$

Consequently, the vector field $\boldsymbol{\zeta}$ *satisfies the boundary value problem*

$$
\begin{aligned}
&-\partial_\beta\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})=p_\alpha && \textit{in } \omega, && \text{(4.4a)}\\
&-\partial_{\alpha\beta}\hat{M}_{\alpha\beta}(\boldsymbol{\zeta})-\partial_\alpha\big(\hat{N}_{\alpha\beta}(\boldsymbol{\zeta})[\partial_\beta\zeta_3+\varepsilon\partial_\beta h]\big)=p_3+\partial_\alpha q_\alpha && \textit{in } \omega, && \text{(4.4b)}\\
&\quad \hat{N}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\beta=\hat{M}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\alpha\nu_\beta=0 && \textit{on } \gamma_1, && \text{(4.4c)}\\
&\quad \hat{N}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\alpha(\partial_\beta\zeta_3+\varepsilon\partial_\beta h)+\big(\partial_\alpha\hat{M}_{\alpha\beta}(\boldsymbol{\zeta})\big)\nu_\beta\\
&\qquad +\partial_\tau\big(\hat{M}_{\alpha\beta}(\boldsymbol{\zeta})\nu_\alpha\tau_\beta\big)=-q_\alpha\nu_\alpha && \textit{on } \gamma_1, && \text{(4.4d)}\\
&\quad \zeta_i=\partial_\nu\zeta_3=0 && \textit{on } \gamma_0. && \text{(4.4e)}
\end{aligned}
$$

Proof. Let $(N_{\alpha\beta})\in\mathcal{C}^1(\overline{\omega};\mathbb{S}^2)$ and $(M_{\alpha\beta})\in\mathcal{C}^0(\overline{\omega};\mathbb{S}^2)$ be tensor fields satisfying Eqs. (4.3). In particular then, the functions

$$E_{\alpha\beta}:=\frac{1}{\varepsilon}b_{\alpha\beta\sigma\tau}N_{\sigma\tau},\quad F_{\alpha\beta}:=-\frac{3}{\varepsilon^3}b_{\alpha\beta\sigma\tau}M_{\sigma\tau}$$

satisfy the compatibility conditions

$$
\begin{aligned}
&\quad \partial_\alpha F_{\beta\sigma}-\partial_\beta F_{\alpha\sigma}=0 && \text{in } \omega,\\
&\quad \partial_{\alpha\beta}E_{\sigma\tau}+\partial_{\sigma\tau}E_{\alpha\beta}-\partial_{\alpha\sigma}E_{\beta\tau}-\partial_{\beta\tau}E_{\alpha\sigma}\\
&=F_{\alpha\sigma}F_{\beta\tau}-F_{\alpha\beta}F_{\sigma\tau}+\varepsilon(\partial_{\alpha\sigma}hF_{\beta\tau}+\partial_{\beta\tau}hF_{\alpha\sigma}-\partial_{\alpha\beta}hF_{\sigma\tau}-\partial_{\sigma\tau}hF_{\alpha\beta}) && \text{in } \omega,
\end{aligned}
$$

which are precisely the nonlinear Saint-Venant compatibility conditions appearing in Theorem 3.2 (ii) established in Section 3. Hence, there exists a vector field $\boldsymbol{\eta}=(\eta_i)\in H^1(\omega)\times H^1(\omega)\times H^2(\omega)$ such that

$$\hat{E}_{\alpha\beta}(\boldsymbol{\eta})=E_{\alpha\beta} \quad \text{and} \quad \partial_{\alpha\beta}\eta_3=F_{\alpha\beta} \quad \text{in } \omega.$$

Furthermore, $\eta_3\in\mathcal{C}^2(\overline{\omega})$ since $M_{\alpha\beta}\in\mathcal{C}^0(\overline{\omega})$ by assumption. Then the definition of the functions $\hat{E}_{\alpha\beta}(\boldsymbol{\eta})$ given at the beginning of this section and the assumption that $N_{\alpha\beta}\in\mathcal{C}^1(\overline{\omega})$ together imply that the functions

$$c_{\alpha\beta}:=\frac{1}{2}(\partial_\alpha\eta_\beta+\partial_\beta\eta_\alpha)$$

belong to the space $\mathcal{C}^1(\overline{\omega})$. Consequently, $\eta_\alpha\in\mathcal{C}^2(\overline{\omega})$ since

$$\partial_{\alpha\beta}\eta_\sigma=\partial_\alpha c_{\alpha\sigma}+\partial_\beta c_{\beta\sigma}-\partial_\sigma c_{\alpha\beta}.$$

The functions

$$E_{\alpha\beta}=\hat{E}_{\alpha\beta}(\boldsymbol{\eta}), \quad F_{\alpha\beta}=\partial_{\alpha\beta}\eta_3$$

satisfy in particular the boundary conditions (see Eq. (4.3g))

$$E_{\alpha\beta}\tau_\alpha\tau_\beta=\partial_\sigma E_{\alpha\beta}\tau_\alpha(\tau_\beta\nu_\sigma-2\tau_\sigma\nu_\beta)-\kappa E_{\alpha\beta}\nu_\alpha\nu_\beta=F_{\alpha\beta}\tau_\beta=0 \quad \text{on } \gamma_0.$$

Then Theorem 3.4 implies that there exist constants $a_1,a_2,a_3,b,d_1,d_2\in\mathbb{R}$ such that the vector field $\boldsymbol{\zeta}=(\zeta_i)\in\mathcal{C}^2(\overline{\omega};\mathbb{R}^3)$, defined for each $y=(y_1,y_2)\in\overline{\omega}$ by

$$\begin{aligned}
&\zeta_1(y):=\eta_1(y)-d_1\eta_3(y)+a_1-by_2-\frac{d_1}{2}(d_1y_1+d_2y_2)-\varepsilon d_1h(y),\\
&\zeta_2(y):=\eta_2(y)-d_2\eta_3(y)+a_2+by_1-\frac{d_2}{2}(d_1y_1+d_2y_2)-\varepsilon d_2h(y),\\
&\zeta_3(y):=\eta_3(y)+a_3+(d_1y_1+d_2y_2)
\end{aligned}$$

satisfies the boundary conditions

$$\zeta_i=\partial_\nu\zeta_3=0 \quad \text{on } \gamma_0.$$

Besides, Theorem 3.4 shows that

$$\begin{aligned}
&E_{\alpha\beta}=\frac{1}{2}\big(\partial_\alpha\zeta_\beta+\partial_\beta\zeta_\alpha+\varepsilon(\partial_\alpha h\partial_\beta\zeta_3+\partial_\beta h\partial_\alpha\zeta_3)+\partial_\alpha\zeta_3\partial_\beta\zeta_3\big) && \text{in } \overline{\omega},\\
&F_{\alpha\beta}=\partial_{\alpha\beta}\zeta_3 && \text{in } \overline{\omega},
\end{aligned}$$

which in turn implies that

$$\begin{aligned} \hat{M}_{\alpha\beta}(\boldsymbol{\zeta}) &:= -\frac{1}{3}\varepsilon^3 a_{\alpha\beta\sigma\tau}\partial_{\sigma\tau}\zeta_3 = -\frac{1}{3}\varepsilon^3 a_{\alpha\beta\sigma\tau}F_{\sigma\tau} = M_{\alpha\beta} \quad \text{in } \overline{\omega}, \\ \hat{N}_{\alpha\beta}(\boldsymbol{\zeta}) &:= \varepsilon a_{\alpha\beta\sigma\tau}\hat{E}_{\sigma\tau}(\boldsymbol{\zeta}) = \varepsilon a_{\alpha\beta\sigma\tau}E_{\sigma\tau} = N_{\alpha\beta} \quad \text{in } \overline{\omega}. \end{aligned}$$

Then Eqs. (4.4a)-(4.4d) follow from Eqs. (4.3a)-(4.3d). The proof is complete. □

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