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# Boundedness and Asymptotic Stability in a Chemotaxis Model with Signal-Dependent Motility and Nonlinear Signal Secretion

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**Abstract.** In the present study, we consider the following parabolic-elliptic chemotaxis system:

$$\begin{cases} u_t = \nabla\cdot\big(\gamma(v)\nabla u - u\chi(v)\nabla v\big) + \lambda u - \mu u^{\sigma}, & x\in\Omega, \quad t>0,\\ 0=\Delta v - v + u^{\kappa}, & x\in\Omega, \quad t>0, \end{cases}$$

where $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ is a smooth and bounded domain, $\lambda>0,\mu>0,\sigma>1,\kappa>0$. Under appropriate assumptions on $\gamma(v)$ and $\chi(v)$, we obtain the global boundedness of solutions when $\kappa n<2$ or $\kappa n\geq 2, \sigma\geq\kappa+1$, which generalize the previous result to the case with nonlinear signal secretion and superlinear logistic term when $n\geq 2$. Moreover, if adding additional conditions $\sigma\geq 2\kappa$ and $\mu$ is sufficiently large, it is shown that the global solution $(u,v)$ converges to

$$\left(\left(\frac{\lambda}{\mu}\right)^{\frac{1}{\sigma-1}}, \left(\frac{\lambda}{\mu}\right)^{\frac{\kappa}{\sigma-1}}\right)$$

exponentially as $t\to\infty$.

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# 1 Introduction

Chemotaxis, a kind of taxis, which refers to the phenomenon that cells, bacteria or multicellular organisms direct their movements according to certain chemicals. To describe the aggregation phase of amoeba cells in response to the chemical signal emitted by cells, Keller and Segel [20] introduced the following system:

$$\begin{cases} u_t = \nabla\cdot\big(\gamma(v)\nabla u - u\chi(v)\nabla v\big), & x\in\Omega, \quad t>0, & (1.1a)\\ \tau v_t = \Delta v - v + u, & x\in\Omega, \quad t>0, & (1.1b)\end{cases}$$

where $\tau\in\{0,1\}$, $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ is a bounded domain, $u(x,t),v(x,t)$ denote the cell density and the concentration of chemical signal, $\gamma(v)$, $\chi(v)$ are the cell diffusion function and chemo-sensitivity function respectively, which have the following relation:

$$\chi(v) = (\alpha-1)\gamma'(v), \tag{1.2}$$

where $\alpha$ is the ratio of effective body length to step size.

For the case that $\gamma(v)=d$, $\chi(v)=\chi$, where $d$ and $\chi$ are positive constants, (1.1) can be reduced to the minimal Keller-Segel model, whether model (1.1) or its related variants, there has been a large number of research with regard to the existence, boundedness, finite-time blow-up, asymptotic behavior et al. (see the review literatures [2,3,10,12] and the references therein). In particular, considering system (1.1) with nonlinear signal production and general growth source, that is, adding the logistic term $f(u)=\lambda u-\mu u^\sigma$ $(\lambda\in\mathbb{R},\mu>0,\sigma>1)$ on the right hand side of Eq. (1.1a), and replacing the linear term $u$ in Eq. (1.1b) by $u^k$ $(k>0)$. When $k\geq 1$, Galakhov *et al.* [9] considered the global dynamics of solutions, thereinto, by assuming that $\sigma>k+1$ or $\sigma=k+1,\mu>((nk-2)/nk)\chi$, they obtained the global boundedness result; and this boundedness result was extended to the borderline case $\sigma=k+1$, $\mu=((nk-2)/nk)\chi$, $n\geq 3$ by Hu and Tao [13]; then Xiang [33] removed the restrictions $k\geq 1$ and $n\geq 3$, under the condition $k+1<\max\{\sigma,1+2/n\}$ or $k+1=\sigma$, $\mu\geq((nk-2)/nk)\chi$, he proved that the solution is globally bounded; afterwards, Xiang *et al.* [30] further extended the result in [33] to the case with nonlinear diffusion function $D(u)$ and nonlinear sensitivity function $S(u)$; lately, considering the chemo-repulsion case, Hu *et al.* [15] established the global boundedness of solutions for the quasilinear case without any conditions on parameters. As for the parabolic-parabolic case, one can refer to the literatures [14,24,26] and the reference thereinto.

When the motility function $\gamma(v)$ and $\chi(v)$ satisfy (1.2) with $\alpha=0$, system (1.1) can be written as

$$\begin{cases} u_t=\Delta(\gamma(v)u), \quad x\in\Omega, \quad t>0, & (1.3a)\\ \tau v_t=\Delta v-v+u, \ x\in\Omega, \quad t>0, & (1.3b)\end{cases}$$

which can also be used to describe the stripe pattern formation of bacterial movements in the experiment when adding the logistic term $f(u)=\mu u(1-u)$ [5]. In such case, considering the cell growth or not, the global dynamics of solutions were detected when $\gamma(v)$ satisfies certain conditions, or for special $\gamma(v)$, such as $\gamma(v)=c_0/v^k, e^{-\chi v}, 1/(c+v^k)$ et al.. On the one hand, considering $\mu=0$ (i.e. no cell growth), when $\tau=1$, if $\gamma(v)\in C^3([0,\infty))$, $\gamma_1\le\gamma(s)\le\gamma_2$, $|\gamma'(s)|\le\gamma'$ $(\gamma_1,\gamma_2,\gamma'>0)$ for all $s\ge 0$, Tao and Winkler [27] demonstrated that system (1.3) admits a globally bounded classical solution for all suitably regular initial data in two-dimensional setting, and for $n\ge 3$, they also obtained the existence of global weak solutions; while for $\gamma(v)$ without lower or upper bound, such as, $\gamma(v)=c_0/v^k$ $(c_0>0,k>0)$, with a smallness assumption on $c_0$, Yoon and Kim [34] proved that system (1.3) possesses global classical solutions in all dimensions; choosing $\gamma(v)=e^{-\chi v}$, the critical mass phenomenon was detected for $n=2$ [8, 18], especially in [8], the infinite-time blow-up was identified; lately, the existence and uniqueness of global weak solutions when $n\ge 1$, and the regularity as well as blow-up of solution when $n\le 2$ were discussed in [4]. When $\tau=0$, if $\gamma(v)=1/v^k$, the global boundedness of classical solutions was established for $0<k<2/(n-2)_+$ [1]; considering general motility function $\gamma(v)$, if $\gamma(v)\in C^3([0,\infty))$, $\gamma(v)>0$, $\gamma'(v)\le 0$ for all $v>0$, and there exists $l>0$ such that $\lim_{s\to\infty}s^l\gamma(s)=+\infty$, then the uniform boundedness of classical solution was derived when $n=2$ [6], in addition, for the specific case $\gamma(v)=e^{-v}$, the critical mass phenomenon was also observed for $n=2$, and it was shown that the global solution becomes unbounded as $t\to\infty$; moreover, for $\gamma(v)\in C^3([0,\infty))$, $\gamma(v)>0$, $\gamma'(v)\le 0$ on $(0,\infty)$, if $\lim_{s\to\infty}e^{\alpha s}\gamma(s)=+\infty,\forall\alpha>0$ when $n=2$ or $l_1|\gamma'(s)|^2\le\gamma(s)\gamma''(s),\forall s>0$ with some $l_1>n/2$ when $n\ge 3$, it was proved in [7] that the classical solution is globally bounded, typically, for $\gamma(v)=v^{-k}$, $n\ge 3$, the uniform boundedness of solutions was also established under the condition $k<2/(n-2)$; and very recently, assuming that $\gamma(v)\in C^3([0,\infty))$, $\gamma(v)>0$, $\gamma'(v)\le 0$ on $(0,\infty)$, $\lim_{s\to\infty}\gamma(s)=0$, and $\gamma(s)+s\gamma'(s)\ge 0,\forall s>0$, if there exists $l_2>(n+2)/4$ such that $l_2|\gamma'(s)|^2\le\gamma(s)\gamma''(s)$ for all $s>0$, Jiang [16] proved that the solution is globally bounded for $n\ge 4$, and the solution converges to the average of the initial data $u_0$ exponentially, specifically, when $\gamma(v)=v^{-k}$, the same result holds provided that $k\in(0,1)$ for $n=4,5$ or $k\in(0,4/(n-2))$ for $n\ge 6$.

On the other hand, adding logistic source $f(u)=\mu u(1-u)$ $(\mu>0)$ in (1.3), for the parabolic-parabolic case, by imposing the hypothesis $\gamma(v)\in C^3([0,\infty))$, $\gamma(v)>0$, $\gamma'(v)<0$ on $[0,\infty)$, $\lim_{v\to\infty}\gamma(v)=0$, and $\lim_{v\to\infty}(\gamma'(v)/\gamma(v))$ exists, Jin *et al.* [17] established the global boundedness of solutions for (1.3) when $n=2$, and further assuming that $\mu>L_0/16$ with $L_0=\max_{0\le v\le\infty}(|\gamma'(v)|^2/\gamma(v))$, the asymptotic behavior of solution was detected; as for the case $n\ge 3$, when $\Omega$ is convex and $\mu$ is large, it was shown that the solution is globally bounded by assuming that $|\gamma'(v)|\le L$ in $[0,\infty)$ with $L>0$ [29], if $n=3,\gamma_1\le\gamma(v)\le\gamma_2,|\gamma'(v)|\le L$, the global boundedness result was derived for sufficiently large $\mu$, moreover, the large time behavior of solution was proved under the same constraints on $\mu$ as in [17]. As for the parabolic-elliptic case, when $n=2$, Fujie and Jiang [6] established the global boundedness result, where the existence of $\lim_{v\to\infty}(\gamma'(v)/\gamma(v))$ required in [17] was removed, and for the asymptotic behavior of solutions, the same result as the fully parabolic case was obtained; in three or lower dimensions, the existence or nonexistence of nonconstant steady states of (1.3) was derived for small $\mu$ [23]; recently, if $\gamma(v)\in C^3([0,\infty))$, $\gamma(v)>0$, $\gamma'(v)\le 0$, $\gamma''(v)\ge 0$, $\gamma'''(v)\le 0$, $-2\gamma'(v)+\gamma''(v)v\le\mu_0<\mu$, and $|\gamma'(v)|^2/\gamma(v)\le c$ for all $v>0$, Tello [28] established the global boundedness and asymptotic behavior of weak solution when $n\ge 1$.

Lately, adding the logistic term $f(u)=au-bu^\sigma$ $(a>0,b>0,\sigma>1)$ on the right hand side of Eq. (1.3a), Lyu and Wang [22] explored the global boundedness and large time behavior of solutions, it was shown that if $n\le 2,\sigma>1$, or $n\ge 3,\sigma>2$, or $n\ge 3,\sigma=2$ and $b$ is greater than some constant, then the corresponding solution is globally bounded, also, the large time behavior of solutions was detected. Furthermore, replacing Eq. (1.3b) by $v_t=\Delta v-v+u^\kappa$, Tao and Fang [25] established the boundedness result when $\kappa<(2/(n+2))\sigma$, and under the additional assumption that $\mu$ is sufficiently large, it was shown that the solution converges to the constant steady state exponentially.

So far, most of the research results related to (1.1) are focused on the case $\chi(v)=-\gamma'(v)$, there are few results on general $\gamma(v)$, $\chi(v)$. Irrespective of the logistic source, for the case $\tau=0$, if $(\gamma(v),\chi(v))\in[C^2([0,\infty))]^2$, $\gamma(v)>0$, $\chi(v)\ge 0$, $\chi'(v)<0$ for all $v\ge 0$, it was shown in [31] that the solution of (1.1) is globally bounded when $n=1$, moreover, the global boundedness result was extended to $n\ge 2$ by attaching the additional assumptions that

$$\inf_{v\ge 0}\frac{\gamma(v)|\chi'(v)|}{|\chi(v)|^2}>\frac{n}{2},\quad \lim_{v\to\infty}v\chi(v)<\infty\quad\text{for}\quad n>3,$$

and $\int_\Omega\chi(v)^{-p}dx<\infty$ for some $p>n/2$; for $\tau=1$, under the hypothesis that $(\gamma(v),\chi(v))\in[C^2([0,\infty))]^2$, $\gamma(v)>0$, $\chi(v)\ge 0$, $\chi'(v)<0$ for all $v>0$, and

$$\inf_{v\geq 0}\frac{\gamma(v)}{v\chi(v)(v\chi(v)+1-\gamma(v))_+}>\frac{n}{2}$$

the uniform boundedness was investigated for $2\leq n\leq 4$ [32]. Taking into account the growth term, namely, adding $\mu u(1-u)(\mu>0)$ on the right side of Eq. (1.1a), the existence of globally bounded solutions and asymptotic behavior were established by Jin and Wang [19], where the boundedness result was detected by assuming that $(\gamma(v),\chi(v))\in[C^2([0,\infty))]^2$, $\gamma(v)>0$, and $|\chi(v)|^2/\gamma(v)$ is bounded for all $v\geq 0$.

Yet there are no results for systems with both nonlinear signal production and general logistic source when $\tau=0$, as compared with the work in [22], the presence of the nonlinear signal production makes it hard to obtain the boundedness of $v$ at the first time, so the method in [22] fails. Luckily, the method in [33] seems to work, but the interaction between the motility function and the nonlinear signal production brings new difficulty, which needs finer analysis and estimates.

Motivated by the works in [19, 30, 31, 33], we consider the following chemotaxis system with nonlinear signal production and general logistic source:

$$\begin{cases} u_t=\nabla\cdot\big(\gamma(v)\nabla u-u\chi(v)\nabla v\big)+\lambda u-\mu u^{\sigma}, & x\in\Omega, \quad t>0, & (1.4a)\\ 0=\Delta v-v+u^{\kappa}, & x\in\Omega, \quad t>0, & (1.4b)\\ \dfrac{\partial u}{\partial \nu}=\dfrac{\partial v}{\partial \nu}=0, & x\in\partial\Omega, \ t>0, & (1.4c)\\ u(x,0)=u_0(x), & x\in\Omega, & (1.4d)\end{cases}$$

where $\lambda>0,\mu>0,\sigma>1,\kappa>0$. Let the initial data satisfy

$$u_0\in C^0(\overline{\Omega}),\quad u_0>0,\quad u_0\not\equiv 0. \tag{1.5}$$

And the motility functions $\gamma(v)$, $\chi(v)$ satisfy the following assumptions:

(H1) $\big(\gamma(v),\chi(v)\big)\in\big[C^2([0,\infty))\big]^2$, $\gamma(v)>0$, $\chi(v)>0$, $\chi'(v)<0$ for all $v\geq 0$.

(H2) $\inf\limits_{v\geq 0}\dfrac{\gamma(v)|\chi'(v)|}{|\chi(v)|^2}>\dfrac{nk}{2}$, if $nk\geq 2$.

The main result may now be enunciated.

**Theorem 1.1.** *Let $\Omega\subset\mathbb{R}^n(n\geq 2)$ be a bounded domain with smooth boundary, $\lambda>0,\mu>0,\sigma>1$, and the initial data satisfy* (1.5)*. Assume that* (H1) *holds, if $\kappa n<2$, or $\kappa n\geq 2,\sigma\geq\kappa+1,\gamma(v),\chi(v)$ satisfy* (H2)*, then system* (1.4) *possesses a globally bounded classical solution $(u,v)$ satisfying*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\leq C\quad \textit{for all}\quad t>0, \tag{1.6}$$

*where $C>0$ is independent of $t$.*

**Remark 1.1.** It is obvious that the set of functions satisfying (H1)-(H2) is not empty, such as $\gamma(v)=(v+1)^{-m}, \chi(v)=-\gamma'(v)$ with $m>0, 1+1/m>n\kappa/2$.

**Remark 1.2.** If the solution $(u,v)$ of system (1.4) blows up at some time $\hat{T}>0$, from Lemma 3.2, it is obvious that $u$ and $v$ blow up at the same time, in the sense that

$$\begin{aligned}&\limsup_{t\to\hat{T}^-}\|u(\cdot,t)\|_{L^p(\Omega)}=\infty \quad \text{for all} \quad p>\frac{\kappa n}{2},\\&\limsup_{t\to\hat{T}^-}\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}=\infty.\end{aligned}$$

**Theorem 1.2.** *Let $\Omega\subset\mathbb{R}^n\,(n\geq 2)$ be a bounded domain with smooth boundary, $\lambda>0, \mu>0, \sigma>1$, and the initial data satisfy* (1.5)*. Suppose that the assumptions in Theorem* 1.1 *hold, $(u,v)$ is the globally bounded solution of* (1.4) *obtained in Theorem* 1.1*. If $\sigma\geq 2\kappa$, then one can find $\mu_0>0, \tilde{m}>0, \mathcal{C}>0$, whenever $\mu>\mu_0$, for all $t>0$ the following holds:*

$$\begin{aligned}&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}\leq \mathcal{C}e^{-\tilde{m}t},\\&\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}\leq \mathcal{C}e^{-\tilde{m}t},\end{aligned} \tag{1.7}$$

*where*

$$u_*=\left(\frac{\lambda}{\mu}\right)^{\frac{1}{\sigma-1}}, \quad v_*=\left(\frac{\lambda}{\mu}\right)^{\frac{\kappa}{\sigma-1}}.$$

**Remark 1.3.** Actually, if the motility functions satisfy $\chi^2(v)/\gamma(v)\leq K_1$ for some $K_1>0$, then the lower bound of $\mu$ and the convergence rates in Theorem 1.2 can be given precisely, that is,

$$\begin{aligned}&\mu_0=\frac{u_*^{2\kappa+1-\sigma}K_1}{16}\sup_{s\in(0,1)\cup(1,\infty)}\frac{(s^\kappa-1)^2}{(s-1)(s^{\sigma-1}-1)},\\&\tilde{m}=\frac{(\sigma-1)u_*^{\sigma-1}(\mu-\mu_0)}{n+2}.\end{aligned}$$

This paper is organized as follows: in Section 2, we present the local well-posedness result and some significant estimates for future use. In Section 3, we establish the global boundedness of solutions for system (1.4). In Section 4, we detect the large time behavior for system (1.4).

# 2 Preliminaries

In this section, we first state the local well-posedness result, which can be attained by a similar discussion as in [1, 17], also the Gagliardo-Nirenberg interpolation inequality [33] is given in preparation for the later proof.

**Lemma 2.1.** *Let $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ be a bounded domain with smooth boundary, $\lambda>0$, $\mu>0$, $\sigma>1$, $\kappa>0$, and the initial data satisfy* (1.5). *Suppose that the assumptions* (H1) *and* (H2) *hold. Then there exists $T_{\max}\in(0,\infty]$ such that system* (1.4) *possesses a unique non-negative classical solution $(u,v)\in C^0(\overline{\Omega}\times[0,T_{\max}))\cap C^{2,1}(\overline{\Omega}\times(0,T_{\max}))\times C^{2,1}(\overline{\Omega}\times(0,T_{\max}))$, and if $T_{\max}<\infty$, then*

$$\limsup_{t\nearrow T_{\max}}\|u(\cdot,t)\|_{L^\infty(\Omega)}=\infty.$$

*In addition, there exists $C_1>0$ such that $\int_\Omega udx\leq C_1$ for all $t\in(0,T_{\max})$.*

**Lemma 2.2.** *Let $\Omega\subset\mathbb{R}^n$ $(n\geq 1)$ be a bounded domain with smooth boundary, $1\leq r\leq\infty$, $0<q<\infty$. Then for any $\alpha\in(0,1)$, fixing*

$$\frac{1}{p}=\alpha\left(\frac{1}{r}-\frac{1}{n}\right)+(1-\alpha)\frac{1}{q},$$

*for any $\varphi\in W^{1,r}(\Omega)\cap L^q(\Omega)$ one has*

$$\|\varphi\|_{L^p(\Omega)}\leq C_{GN}\|\varphi\|^{\alpha}_{W^{1,r}(\Omega)}\|\varphi\|^{1-\alpha}_{L^q(\Omega)}.$$

The following lemma is an extension of [11, Lemma 2.2], which is essential for the boundedness result.

**Lemma 2.3.** *Suppose that the assumptions in Lemma* 2.1 *hold. Then for all $t\in(0,T_{\max})$, the solution $(u,v)$ of* (1.4) *satisfies*

$$\int_\Omega v^\zeta dx\leq\eta\int_\Omega u^{\kappa\zeta}dx+C_2 \tag{2.1}$$

*for any $\eta>0$ and $\zeta>1$, where $C_2>0$.*

*Proof.* Testing Eq. (1.4b) by $v^{\zeta-1}$, $\zeta>1$, one has

$$\begin{aligned}&\frac{4(\zeta-1)}{\zeta^2}\int_\Omega|\nabla v^{\frac{\zeta}{2}}|^2dx+\int_\Omega v^\zeta dx\\=&\int_\Omega u^\kappa v^{\zeta-1}dx\leq\frac{1}{\zeta}\int_\Omega u^{\kappa\zeta}dx+\frac{\zeta-1}{\zeta}\int_\Omega v^\zeta dx,\end{aligned} \tag{2.2}$$

which indicates that

$$\frac{4(\zeta-1)}{\zeta}\int_\Omega|\nabla v^{\frac{\zeta}{2}}|^2dx\leq\int_\Omega u^{\kappa\zeta}dx. \tag{2.3}$$

It follows from Ehrling's lemma that

$$\begin{aligned}\int_\Omega v^\zeta dx &\le \eta_1 \frac{4(\zeta-1)}{\zeta}\int_\Omega |\nabla v^{\frac{\zeta}{2}}|^2 dx + C(\eta_1,\zeta)\left(\int_\Omega v dx\right)^\zeta \\ &\le \eta_1 \int_\Omega u^{\kappa\zeta} dx + C(\eta_1,\zeta)\left(\int_\Omega u^\kappa dx\right)^\zeta \end{aligned} \tag{2.4}$$

for any $\eta_1 > 0$ and some $C(\eta,\zeta) > 0$, where we have used (2.3) and the relation $\int_\Omega v dx = \int_\Omega u^\kappa dx$. If $\kappa \le 1$, applying the $L^1$ bound for $u$ in Lemma 2.1, it is obvious that (2.1) holds. If $\kappa > 1$, for $\zeta > 1$, choosing positive constants $a, b, \kappa_1, \kappa_2$,

$$\frac{1}{a} = \frac{\kappa-1}{\kappa\zeta-1}, \quad \frac{1}{b} = 1-\frac{1}{a} = \frac{\kappa\zeta-\kappa}{\kappa\zeta-1}, \quad \kappa_1 = \kappa - \frac{1}{b}, \quad \kappa_2 = \frac{1}{b},$$

then combing the Hölder inequality and Young inequality, and utilizing the fact that $\kappa_2 b = 1, \kappa_1 a = \zeta\kappa, \zeta/a < 1$ and $\int_\Omega u dx \le C_1$, we find the following relation:

$$\begin{aligned}\left(\int_\Omega u^\kappa dx\right)^\zeta &= \left(\int_\Omega u^{\kappa_1+\kappa_2} dx\right)^\zeta \le \left(\int_\Omega u^{\kappa_1 a} dx\right)^{\frac{\zeta}{a}} \left(\int_\Omega u^{\kappa_2 b} dx\right)^{\frac{\zeta}{b}} \\ &\le C_1^{\frac{\zeta}{b}} \left(\int_\Omega u^{\kappa_1 a} dx\right)^{\frac{\zeta}{a}} \le \eta_2 \int_\Omega u^{\kappa\zeta} dx + C(a,b,\zeta,\eta_2,C_1)\end{aligned}$$

holds for any $\eta_2 > 0$ and some $C(a,b,\zeta,\eta_2,C_1) > 0$. This completes the proof of the lemma. □

# 3 Global boundedness of solutions

To obtain the global boundedness result, we first establish the following inequality.

**Lemma 3.1.** *Let $\Omega \subset \mathbb{R}^n (n \ge 1)$ be a bounded domain with smooth boundary, $\lambda > 0, \mu > 0, \sigma > 1, \kappa > 0$, the initial data satisfy* (1.5). *Assume that $(u,v)$ is the classical solution of* (1.4), *and* (H1) *holds. Then for $z = u^{p/2}$, $p > 1$, one has*

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega z^2 dx + \frac{p-1}{p^2}\int_\Omega \left|\nabla\left(\gamma^{\frac{1}{2}}(v) z\right)\right|^2 dx + \mu \int_\Omega z^{\frac{2(p+\sigma-1)}{p}} dx \\ \le& \frac{p-1}{2}\int_\Omega \frac{|\chi(v)|^2}{\gamma(v)} z^2 |\nabla v|^2 dx + \frac{p-1}{2p^2}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)} z^2 |\nabla v|^2 \\ &+ \lambda \int_\Omega z^2 dx \quad \textit{for all} \quad t \in (0, T_{\max}).\end{aligned} \tag{3.1}$$

*Proof.* Testing Eq. (1.4a) by $u^{p-1}$, and use the Young inequality, one obtains

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx =& -(p-1)\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 dx + (p-1)\int_\Omega \chi(v)u^{p-1}\nabla u\cdot\nabla v dx\\ &+\int_\Omega \left(\lambda u^p - \mu u^{p+\sigma-1}\right)dx\\ \le& -\frac{p-1}{2}\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2 dx + \frac{p-1}{2}\int_\Omega \frac{|\chi(v)|^2}{\gamma(v)}u^p|\nabla v|^2 dx\\ &+\int_\Omega \left(\lambda u^p - \mu u^{p+\sigma-1}\right)dx\\ =& -\frac{2(p-1)}{p^2}\int_\Omega \gamma(v)|\nabla u^{\frac{p}{2}}|^2 dx + \frac{p-1}{2}\int_\Omega \frac{|\chi(v)|^2}{\gamma(v)}u^p|\nabla v|^2 dx\\ &+\int_\Omega \left(\lambda u^p - \mu u^{p+\sigma-1}\right)dx. \end{aligned} \tag{3.2}$$

Denoting $z = u^{p/2}$, it follows from (3.2) that

$$\begin{aligned}\frac{1}{p}\frac{d}{dt}\int_\Omega z^2 dx \le& -\frac{2(p-1)}{p^2}\int_\Omega \gamma(v)|\nabla z|^2 dx + \frac{p-1}{2}\int_\Omega \frac{|\chi(v)|^2}{\gamma(v)}z^2|\nabla v|^2 dx\\ &+\lambda\int_\Omega z^2 dx - \mu\int_\Omega z^{\frac{2(p+\sigma-1)}{p}}dx. \end{aligned} \tag{3.3}$$

Here, we notice that

$$\begin{aligned}\gamma(v)|\nabla z|^2 =& \left|\gamma^{\frac{1}{2}}(v)\nabla z\right|^2 = \left(\nabla\left(\gamma^{\frac{1}{2}}(v)z\right) - \frac{1}{2}\frac{\gamma'(v)}{\gamma^{\frac{1}{2}}(v)}z\nabla v\right)^2\\ \ge& \frac{1}{2}\left|\nabla\left(\gamma^{\frac{1}{2}}(v)z\right)\right|^2 - \frac{1}{4}\frac{|\gamma'(v)|^2}{\gamma(v)}z^2|\nabla v|^2. \end{aligned} \tag{3.4}$$

Consequently, utilizing the assumption (H2), it is thereby inferred that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega z^2 dx + \frac{p-1}{p^2}\int_\Omega \left|\nabla\left(\gamma^{\frac{1}{2}}(v)z\right)\right|^2 dx + \mu\int_\Omega z^{\frac{2(p+\sigma-1)}{p}}dx\\ \le& \frac{p-1}{2}\int_\Omega \frac{|\chi(v)|^2}{\gamma(v)}z^2|\nabla v|^2 dx + \frac{p-1}{2p^2}\int_\Omega \frac{|\gamma'(v)|^2}{\gamma(v)}z^2|\nabla v|^2 + \lambda\int_\Omega z^2 dx. \end{aligned}$$

The proof is complete. □

Next, based on Lemma 3.1, we derive a boundedness criterion for system (1.4) as follows.

**Lemma 3.2.** *Let $\Omega\subset\mathbb{R}^n$ $(n\geq 2)$ be a bounded domain with smooth boundary, and the assumptions in Lemma* 3.1 *hold. If there exist $\epsilon\in(0,\kappa n/2)$ and $L>0$ such that*

$$\|u(\cdot,t)\|_{L^{\frac{\kappa n}{2}+\epsilon}}\leq L \tag{3.5}$$

*for all $t\in(0,T_{\max})$, then $(u,v)$ is a global classical solution of problem* (1.4)*, which is uniformly bounded in the sense that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}+\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\leq C_3 \quad \textit{for all} \quad t\in(0,T_{\max}),$$

*where $C_3>0$ is independent of $t$.*

*Proof.* For any $\epsilon\in(0,\kappa n/2)$, let $l:=\kappa n/2+\epsilon$, due to the fact that $n\geq 2$, one finds

$$\frac{2kn}{n+2}<l<kn. \tag{3.6}$$

Under the hypotheses that $\|u(\cdot,t)\|_{L^l(\Omega)}\leq L$, we have $\|u^\kappa(\cdot,t)\|_{L^{\frac{l}{\kappa}}(\Omega)}\leq L^\kappa$, then from the elliptic regularity estimate for the second equation, we deduce that

$$\|v(\cdot,t)\|_{W^{2,\frac{l}{\kappa}}(\Omega)}\leq c_0 \tag{3.7}$$

with $c_0>0$. Therefore, in view of the Sobolev embedding theorem

$$W^{2,\frac{l}{\kappa}}(\Omega)\hookrightarrow W^{1,\hat{q}}(\Omega) \quad \text{with} \quad \hat{q}=\frac{ln}{n\kappa-l},$$

we obtain that $\|v(\cdot,t)\|_{W^{1,\hat{q}}(\Omega)}$ is bounded, and a direct calculation yields that $\hat{q}>n\geq 2$. Employing the Sobolev embedding theorem again, one finds

$$\|v(\cdot,t)\|_{L^\infty(\Omega)}\leq c_1$$

with $c_1>0$, therefore, one can find positive constants $\gamma_1,\gamma_2,K$ such that

$$\gamma_1\leq\gamma(v)\leq\gamma_2, \quad \frac{\chi(v)+|\gamma'(v)|}{\gamma(v)}\leq K. \tag{3.8}$$

Then from (3.1) and (3.8), we arrive at

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega z^2dx+\frac{p-1}{p^2}\int_\Omega\left|\nabla(\gamma^{\frac{1}{2}}(v)z)\right|^2dx+\mu\int_\Omega z^{\frac{2(p+\sigma-1)}{p}}dx\\ \leq&\left(\frac{p-1}{2}+\frac{p-1}{2p^2}\right)K^2\int_\Omega\gamma(v)z^2|\nabla v|^2dx+\lambda\int_\Omega z^2dx\end{aligned} \tag{3.9}$$

with $z=u^{p/2}$. To handle the first term in the right-hand side of (3.9), we set

$$q=\frac{\hat{q}/2}{\hat{q}/2-1}=\frac{ln}{(n+2)l-2\kappa n}.$$

It is clear that $q>1$. By the Hölder inequality and the boundedness of $\|v(\cdot,t)\|_{W^{1,\hat{q}}}$, we can find $c_2>0$ such that

$$\begin{aligned}\int_\Omega \gamma(v)z^2|\nabla v|^2dx&=\big\|\big(\gamma^{\frac{1}{2}}(v)z\big)^2|\nabla v|^2\big\|_{L^1(\Omega)}\\&\le\big\|\big(\gamma^{\frac{1}{2}}(v)z\big)^2\big\|_{L^q(\Omega)}\big\||\nabla v|^2\big\|_{L^{\frac{\hat{q}}{2}}(\Omega)}\\&\le c_2\big\|\gamma^{\frac{1}{2}}(v)z\big\|^2_{L^{2q}(\Omega)}.\end{aligned}\tag{3.10}$$

Choose

$$\xi=\frac{2(\kappa n-l)p-\big((n+2)l-2\kappa n\big)(\sigma-1)}{\big(2p-(n-2)(\sigma-1)\big)l},$$

Since $l>\kappa n/2$, for

$$p>\max\left\{1,\frac{(n-2)(\sigma-1)}{2},\frac{\big((n+2)l-2\kappa n\big)(\sigma-1)}{2(\kappa n-l)}\right\}$$

it is clear that $\xi\in(0,1)$, then we can apply the Gagliardo-Nirenberg interpolation inequality in Lemma 2.2 and the Young inequality to get

$$\begin{aligned}&\int_\Omega \gamma(v)z^2|\nabla v|^2dx\le c_2\big\|\gamma^{\frac{1}{2}}(v)z\big\|^2_{L^{2q}(\Omega)}\\&\le c_3\big\|\gamma^{\frac{1}{2}}(v)z\big\|^{2\xi}_{W^{1,2}(\Omega)}\big\|\gamma^{\frac{1}{2}}(v)z\big\|^{2(1-\xi)}_{L^{\frac{2(p+\sigma-1)}{p}}(\Omega)}\\&\le\epsilon_1\big\|\gamma^{\frac{1}{2}}(v)z\big\|^{\frac{2(p+\sigma-1)}{p}}_{L^{\frac{2(p+\sigma-1)}{p}}(\Omega)}+c_4\big\|\gamma^{\frac{1}{2}}(v)z\big\|^{\frac{2\xi(p+\sigma-1)}{\xi p+\sigma-1}}_{W^{1,2}(\Omega)}\\&\le\epsilon_1\big\|\gamma^{\frac{1}{2}}(v)z\big\|^{\frac{2(p+\sigma-1)}{p}}_{L^{\frac{2(p+\sigma-1)}{p}}(\Omega)}+\epsilon_2\big\|\gamma^{\frac{1}{2}}(v)z\big\|^2_{W^{1,2}(\Omega)}+c_5\\&\le(\epsilon_1+\epsilon_2)\big\|\gamma^{\frac{1}{2}}(v)z\big\|^{\frac{2(p+\sigma-1)}{p}}_{L^{\frac{2(p+\sigma-1)}{p}}(\Omega)}+\epsilon_2\big\|\nabla\big(\gamma^{\frac{1}{2}}(v)z\big)\big\|^2_{L^2(\Omega)}+c_6,\end{aligned}\tag{3.11}$$

where $c_3,c_4,c_5,c_6>0$, $\epsilon_1,\epsilon_2$ are arbitrary positive constants. Note that we have used the relations

$$\begin{aligned}\|\gamma^{\frac{1}{2}}(v)z\|^2_{W^{1,2}(\Omega)}&=\|\gamma^{\frac{1}{2}}(v)z\|^2_{L^2(\Omega)}+\|\nabla(\gamma^{\frac{1}{2}}(v)z)\|^2_{L^2(\Omega)}\\&\le\|\gamma^{\frac{1}{2}}(v)z\|^{\frac{2(p+\sigma-1)}{p}}_{\frac{2(p+\sigma-1)}{p}}+\|\nabla(\gamma^{\frac{1}{2}}(v)z)\|^2_{L^2(\Omega)}+c(|\Omega|).\end{aligned}$$

Utilizing the upper bound of $\gamma(v)$ in (3.8), one can select sufficiently small $\epsilon_1,\epsilon_2>0$ such that

$$\begin{aligned}&\left(\frac{p-1}{2}+\frac{p-1}{2p^2}\right)K^2\epsilon_2\le\frac{p-1}{p^2},\\&\left(\frac{p-1}{2}+\frac{p-1}{2p^2}\right)K^2\gamma_2^{\frac{p+\sigma-1}{p}}(\epsilon_1+\epsilon_2)\le\frac{\mu}{2},\end{aligned}$$

now combining (3.9)-(3.11), one can apply the young inequality to obtain that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega z^2dx+\int_\Omega z^2dx\le c_7 \tag{3.12}$$

with $c_7>0$, which implies that

$$\begin{aligned}&\|u(\cdot,t)\|_{L^p(\Omega)}\le c_8\quad\text{for}\\&p>\max\left\{1,\frac{(n-2)(\sigma-1)}{2},\frac{\big((n+2)l-2\kappa n\big)(\sigma-1)}{2(\kappa n-l)}\right\}\end{aligned}\tag{3.13}$$

with $c_8>0$. Thanks to (3.13), the elliptic regularity estimate for the second equation directly yields

$$\|v(\cdot,t)\|_{W^{2,\frac{p}{\kappa}}(\Omega)}\le c_9\quad\text{for}\quad p>\max\left\{1,\frac{(n-2)(\sigma-1)}{2},\frac{\big((n+2)l-2\kappa n\big)(\sigma-1)}{2(\kappa n-l)}\right\}$$

with $c_9>0$. In consequence, for

$$p>\max\left\{n\kappa,1,\frac{(n-2)(\sigma-1)}{2},\frac{\big((n+2)l-2\kappa n\big)(\sigma-1)}{2(\kappa n-l)}\right\}$$

it is found from the Sobolev embedding theorem $W^{2,\frac{p}{\kappa}}\hookrightarrow C^{1,1-n\kappa/p}$ that

$$\|v(\cdot,t)\|_{W^{1,\infty}(\Omega)}\le c_{10}$$

for all $t\in(0,T_{\max})$ with $c_{10}>0$. Then utilizing the boundedness of $\|v(\cdot,t)\|_{L^\infty(\Omega)}$, one can find $\gamma_0>0$ such that $\gamma_0\le\gamma(v)$, thus, by virtue of the fact $\chi'(v)<0$ and $\|\nabla v(\cdot,t)\|_{L^\infty(\Omega)}\le c_{10}$, it follows from (3.2) that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega u^p dx+(p-1)\gamma_0\int_\Omega u^{p-2}|\nabla u|^2dx\\ \le&(p-1)c_{10}\chi(0)\int_\Omega u^{p-1}|\nabla u|dx+\lambda\int_\Omega u^p dx-\mu\int_\Omega u^{p+\sigma-1}dx\\ \le&\frac{p-1}{2}\gamma_0\int_\Omega u^{p-2}|\nabla u|^2dx+\left(\frac{c_{10}^2\chi^2(0)}{2\gamma_0}(p-1)+\lambda\right)\int_\Omega u^p dx.\end{aligned}\tag{3.14}$$

Therefore, applying the similar method in [19, Lemma 3.6], we can see that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)}\le c_{11}$$

for all $t\in(0,T_{\max})$ with $c_{11}>0$. This completes the proof of the lemma. □

From Lemma 3.2 and the $L^1$ bound for $u$ in Lemma 2.1, it is obvious that the boundedness result holds for $kn<2$, so we just need to consider the case $kn\ge2$ in the rest of the paper.

**Lemma 3.3.** *Let $\Omega\subset\mathbb{R}^n$ $(n\ge2)$ be a bounded domain with smooth boundary, $\lambda>0,\mu>0,\sigma\ge\kappa+1,kn\ge2$, the initial data satisfy* (1.5). *Assume that $(u,v)$ is the classical solution of* (1.4), *and* (H1)-(H2) *hold. Then there exist $\epsilon\in(0,\kappa n/2)$ and $C_4>0$ such that*

$$\|u(\cdot,t)\|_{L^{\frac{nk}{2}+\epsilon}(\Omega)}\le C_4.$$

*Proof.* Testing Eq. (1.4a) by $pu^{p-1}$, one obtains

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p dx+p(p-1)\int_\Omega\gamma(v)u^{p-2}|\nabla u|^2dx\\ =&p(p-1)\int_\Omega\chi(v)u^{p-1}\nabla u\cdot\nabla vdx+\lambda p\int_\Omega u^p dx-\mu p\int_\Omega u^{p+\sigma-1}dx.\end{aligned}\tag{3.15}$$

Multiplying Eq. (1.4b) by $\chi(v)u^p$, it holds that

$$\begin{aligned}&-p\int_\Omega\chi(v)u^{p-1}\nabla u\cdot\nabla vdx-\int_\Omega u^p\chi'(v)|\nabla v|^2dx\\ &-\int_\Omega u^p v\chi(v)+\int_\Omega u^{p+\kappa}\chi(v)dx=0,\end{aligned}\tag{3.16}$$

which along with (3.15), gives

$$\begin{aligned}&\frac{d}{dt}\int_\Omega u^p dx+(p-1)\left(p\int_\Omega \gamma(v)u^{p-2}|\nabla u|^2dx-2p\int_\Omega \chi(v)u^{p-1}\nabla u\cdot\nabla v dx\right.\\&\qquad\qquad\qquad\qquad\left.-\int_\Omega u^p\chi'(v)|\nabla v|^2dx\right)\\&=(p-1)\int_\Omega \chi(v)vu^pdx-(p-1)\int_\Omega u^{p+\kappa}\chi(v)dx\\&\quad+\lambda p\int_\Omega u^pdx-\mu p\int_\Omega u^{p+\sigma-1}dx. \qquad (3.17)\end{aligned}$$

Let $\vec{\omega}_1=u^{p/2-1}\nabla u$, $\vec{\omega}_2=u^{p/2}\nabla v$, we find

$$\begin{aligned}&p\gamma(v)u^{p-2}|\nabla u|^2-2p\chi(v)u^{p-1}\nabla u\cdot\nabla v-u^p\chi'(v)|\nabla v|^2\\&=p\gamma(v)|\vec{\omega}_1|^2-2p\chi(v)\vec{\omega}_1\cdot\vec{\omega}_2-\chi'(v)|\vec{\omega}_2|^2\geq 0 \qquad (3.18)\end{aligned}$$

provided that $p|\chi(v)|^2\leq-\gamma(v)\chi'(v)$. From (H2), we know that

$$\inf_{v\geq 0}\frac{\gamma(v)|\chi'(v)|}{|\chi(v)|^2}>\frac{nk}{2}.$$

Therefore, there exists $\epsilon\in(0,\kappa n/2)$ such that

$$1\leq\frac{nk}{2}<\frac{nk}{2}+\epsilon<\inf_{v\geq 0}\frac{\gamma(v)|\chi'(v)|}{|\chi(v)|^2}\leq-\frac{\gamma(v)\chi'(v)}{|\chi(v)|^2}.$$

Choosing $p=nk/2+\epsilon$, we conclude that

$$\frac{d}{dt}\int_\Omega u^pdx\leq(p-1)\int_\Omega\chi(v)vu^pdx+\lambda p\int_\Omega u^pdx-\mu p\int_\Omega u^{p+\sigma-1}dx. \qquad (3.19)$$

Then utilizing the Young inequality and Lemma 2.3, we find

$$\begin{aligned}\int_\Omega\chi(v)vu^pdx&\leq\chi(0)\hat{\epsilon}\int_\Omega u^{p+\sigma-1}dx+\chi(0)c(\hat{\epsilon})\int_\Omega v^{\frac{p+\sigma-1}{\sigma-1}}dx\\&\leq\chi(0)\hat{\epsilon}\int_\Omega u^{p+\sigma-1}dx+\chi(0)c(\hat{\epsilon})\eta\int_\Omega u^{\frac{p+\sigma-1}{\sigma-1}\kappa}dx+\hat{c}_1 \qquad (3.20)\end{aligned}$$

holds for any $\hat{\epsilon}>0$ and $\eta>0$, where $c_1>0$ and $c(\hat{\epsilon})>0$. Due to the fact that $\sigma\geq\kappa+1$, choosing appropriate parameters $\hat{\epsilon}$ and $\eta$, it follows from the Young inequality that

$$\chi(0)\hat{\epsilon}\int_\Omega u^{p+\sigma-1}dx+\chi(0)c(\hat{\epsilon})\eta\int_\Omega u^{\frac{p+\sigma-1}{\sigma-1}\kappa}dx\leq\frac{\mu}{2}\int_\Omega u^{p+\sigma-1}dx+\hat{c}_2$$

with $c_2>0$, which gives rise to

$$\frac{d}{dt}\int_\Omega u^p dx \leq \lambda p\int_\Omega u^p dx - \frac{\mu p}{2}\int_\Omega u^{p+\sigma-1}dx + \hat{c}_2(p-1). \tag{3.21}$$

Since $\sigma>1$, using the Young inequality and the Gronwall inequality, we can derive that $\|u(\cdot,t)\|_{L^p(\Omega)} \leq \hat{c}_2$ with $\hat{c}_2>0$. This finishes the proof. □

*Proof of Theorem* 1.1. Theorem 1.1 is a direct result of Lemmas 3.2 and 3.3. □

# 4 Asymptotic behavior of solutions

In this section, motivated by the method in [30], we are devoted to establishing the long time dynamics of solutions. We aim to show that the solution of (1.4) converges to $(u_*,v_*)$ exponentially, where

$$u_* = \left(\frac{\lambda}{\mu}\right)^{\frac{1}{\sigma-1}}, \quad v_* = \left(\frac{\lambda}{\mu}\right)^{\frac{\kappa}{\sigma-1}}. \tag{4.1}$$

To this end, we introduce the following Lyapunov functional:

$$F(t) = \int_\Omega \left(u - u_* - u_* \ln\frac{u}{u_*}\right) dx. \tag{4.2}$$

**Lemma 4.1.** *Let $(u,v)$ be the globally bounded solution of* (1.4) *obtained in Theorem* 1.1*, $(u_*,v_*)$ and $F(t)$ be given by* (4.1) *and* (4.2) *respectively. If $\sigma \geq 2\kappa$, then there exists $\mu_0>0$, whenever $\mu>\mu_0$, one has*

$$\|u(\cdot,t)-u_*\|_{L^2(\Omega)} \to 0, \quad \|v(\cdot,t)-v_*\|_{L^2(\Omega)} \to 0 \quad as \quad t\to\infty. \tag{4.3}$$

*Proof.* A simple computation directly yields that $F(t)\geq 0$ for all $t>0$. It follows from Eq. (1.4a) that

$$\begin{aligned}
&\frac{d}{dt}F(t) = \int_\Omega \frac{u-u_*}{u} u_t dx \\
&= -u_*\int_\Omega \gamma(v)\frac{|\nabla u|^2}{u^2}dx + u_*\int_\Omega \chi(v)\frac{\nabla u\cdot\nabla v}{u}dx + \int_\Omega (u-u_*)(\lambda-\mu u^{\sigma-1})dx \\
&= -u_*\int_\Omega \left(\gamma^{\frac{1}{2}}(v)\frac{\nabla u}{u} - \frac{\chi(v)}{2\gamma^{\frac{1}{2}}(v)}\nabla v\right)^2 dx + \frac{u_*}{4}\int_\Omega \frac{\chi^2(v)}{\gamma(v)}|\nabla v|^2 dx \\
&\quad + \int_\Omega (u-u_*)(\lambda-\mu u^{\sigma-1})dx \\
&\leq \frac{u_*}{4}\int_\Omega \frac{\chi^2(v)}{\gamma(v)}|\nabla v|^2 dx + \int_\Omega (u-u_*)(\lambda-\mu u^{\sigma-1})dx.
\end{aligned} \tag{4.4}$$

Testing Eq. (1.4b) by $v-v_*$, and using the relation $v_*=u_*^\kappa$, we find

$$\begin{aligned}\int_\Omega |\nabla v|^2dx=&-\int_\Omega (v-u^\kappa)(v-v_*)dx\\ =&-\int_\Omega (v-v_*)^2dx+\int_\Omega (u^\kappa-u_*^\kappa)(v-v_*)dx,\end{aligned} \tag{4.5}$$

moreover, in view of the fact that $v$ is bounded, it follows form the assumption (H1) that

$$\frac{\chi^2(v)}{\gamma(v)}\le K_1 \tag{4.6}$$

with $K_1>0$. Therefore, combining (4.4)-(4.6), we can achieve that

$$\begin{aligned}\frac{d}{dt}F(t)\le&-\frac{u_*K_1}{4}\int_\Omega (v-v_*)^2dx+\frac{u_*K_1}{4}\int_\Omega (u^\kappa-u_*^\kappa)(v-v_*)dx\\ &+\int_\Omega (u-u_*)(\lambda-\mu u^{\sigma-1})dx\\ \le&\frac{u_*K_1}{16}\int_\Omega (u^\kappa-u_*^\kappa)^2dx-\mu\int_\Omega (u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})dx.\end{aligned} \tag{4.7}$$

We notice that for $u\neq u_*$, the following relation:

$$\begin{aligned}&\frac{u_*K_1}{16}\frac{(u^\kappa-u_*^\kappa)^2}{(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})}\\ =&\frac{u_*^{2\kappa+1-\sigma}K_1}{16}\frac{\big((u/u_*)^\kappa-1\big)^2}{(u/u_*-1)\big((u/u_*)^{\sigma-1}-1\big)}\\ \le&\frac{u_*^{2\kappa+1-\sigma}K_1}{16}\sup_{s\in(0,1)\cup(1,\infty)}\frac{(s^\kappa-1)^2}{(s-1)(s^{\sigma-1}-1)}\end{aligned} \tag{4.8}$$

holds, and by a direct calculation, we have

$$\sup_{s\in(0,1)\cup(1,\infty)}\frac{(s^\kappa-1)^2}{(s-1)(s^{\sigma-1}-1)}<\infty$$

provided that $\sigma\ge 2\kappa$. Choosing

$$\mu>\frac{u_*^{2\kappa+1-\sigma}K_1}{16}\sup_{s\in(0,1)\cup(1,\infty)}\frac{(s^\kappa-1)^2}{(s-1)(s^{\sigma-1}-1)}, \tag{4.9}$$

and denoting

$$\varsigma=\mu-\frac{u_*^{2\kappa+1-\sigma}K_1}{16}\sup_{s\in(0,1)\cup(1,\infty)}\frac{(s^\kappa-1)^2}{(s-1)(s^{\sigma-1}-1)},\tag{4.10}$$

it is obvious that

$$(u^\kappa-u_*^\kappa)^2<\frac{16\mu}{u_*K_1}(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})\tag{4.11}$$

and

$$\frac{d}{dt}F(t)\le-\varsigma\int_\Omega(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})dx.\tag{4.12}$$

Integrating (4.12) over $(t_0,\infty)$ and utilizing the nonnegativity of $F(t)$, we have

$$\varsigma\int_{t_0}^\infty\int_\Omega(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})dxdt\le F(t_0)<\infty,$$

which along with (4.11) yields that

$$\int_\Omega(u^\kappa-u_*^\kappa)^2<\frac{16\mu}{u_*K_1}\int_\Omega(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})dx\to0\quad\text{as}\quad t\to\infty.\tag{4.13}$$

From the second equation of (1.4) and the Young inequality, we have

$$\begin{aligned}\int_\Omega|\nabla v|^2dx&=\int_\Omega(u^\kappa-u_*^\kappa)(v-v_*)dx-\int_\Omega(v-v_*)^2dx\\&\le\frac{1}{2}\int_\Omega(u^\kappa-u_*^\kappa)^2dx-\frac{1}{2}\int_\Omega(v-v_*)^2dx.\end{aligned}$$

This indicates that

$$\int_\Omega(v-v_*)^2dx\le\int_\Omega(u^\kappa-u_*^\kappa)^2dx\to0\quad\text{as}\quad t\to\infty.\tag{4.14}$$

Let $R>\max\{u,u_*\}$, for the case $0<\kappa\le1$, let $f(y)=y^{1/\kappa}$, in light of the mean value theorem that

$$u-u_*=f(u^\kappa)-f(u_*^\kappa)=\frac{1}{\kappa}\zeta^{\frac{1}{\kappa}-1}(u^\kappa-u_*^\kappa)$$

for some $\zeta$ between $u$ and $u_*$, consequently,

$$(u-u_*)^2\le\frac{1}{\kappa^2}R^{\frac{2}{\kappa}-2}(u^\kappa-u_*^\kappa)^2.\tag{4.15}$$

As for the case $\kappa>1$, let

$$M=\sup_{s\in(0,R)}\frac{(s-u_*)^2}{(s^\kappa-u_*^\kappa)^2}$$

it is clear that $M<\infty$, therefore

$$(u-u_*)^2\le M(u^\kappa-u_*^\kappa)^2. \tag{4.16}$$

It then follows from (4.13), (4.15) and (4.16) that

$$\int_\Omega (u-u_*)^2dx\to 0 \quad \text{as} \quad t\to\infty. \tag{4.17}$$

Thus, for $\mu$ satisfying (4.9), we obtain that (4.3) holds. ☐

Now, we give the explicit convergence rates.

**Lemma 4.2.** *Let* $(u,v)$ *be the globally bounded solution of* (1.4) *obtained in Theorem* 1.1, $(u_*,v_*)$ *and* $F(t)$ *be given by* (4.1) *and* (4.2) *respectively. If* $\sigma\ge 2\kappa$, *then one can find* $\mu_0>0, m_1>0, \mathcal{C}>0$, *whenever* $\mu>\mu_0$, *for all* $t>0$ *the following holds:*

$$\begin{aligned}&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}\le \mathcal{C}e^{-m_1t},\\ &\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}\le \mathcal{C}e^{-m_1t}.\end{aligned} \tag{4.18}$$

*Proof.* By the known regularity argument in [21], there exist $\delta\in(0,1)$ and $\tilde{C}>0$ such that

$$\|u(\cdot,t)\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])}+\|v(\cdot,t)\|_{C^{2+\sigma,1+\frac{\sigma}{2}}(\bar\Omega\times[t,t+1])}\le\tilde{C},\quad \forall t\ge 1,$$

thereupon, in light of the Gagliardo-Nirenberg interpolation inequality in Lemma 2.2, the above inequality and (4.13)-(4.16), we have

$$\begin{aligned}\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}&\le C_{GN}\|u(\cdot,t)-u_*\|^{\frac{n}{n+2}}_{W^{1,\infty}(\Omega)}\|u(\cdot,t)-u_*\|^{\frac{2}{n+2}}_{L^2(\Omega)}\\ &\le \mathcal{C}_1\big\|u^\kappa(\cdot,t)-u_*^\kappa\big\|^{\frac{2}{n+2}}_{L^2(\Omega)},\end{aligned} \tag{4.19}$$

$$\begin{aligned}\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}&\le C_{GN}\|v(\cdot,t)-v_*\|^{\frac{n}{n+2}}_{W^{1,\infty}(\Omega)}\|v(\cdot,t)-v_*\|^{\frac{2}{n+2}}_{L^2(\Omega)}\\ &\le \mathcal{C}_2\big\|u^\kappa(\cdot,t)-u_*^\kappa\big\|^{\frac{2}{n+2}}_{L^2(\Omega)}\end{aligned} \tag{4.20}$$

with $\mathcal{C}_1,\mathcal{C}_2>0$. By virtue of L'Hôpital's rule, we find that

$$\lim_{u\to u_*}\frac{u-u_*-u_*\ln(u/u_*)}{(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})}=\frac{1}{2(\sigma-1)u_*^{\sigma-1}},$$

which implies that there exists $t_1>0$ such that

$$\begin{aligned}&\frac{1}{4(\sigma-1)u_*^{\sigma-1}}(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})\\ \le& u-u_*-u_*\ln\frac{u}{u_*}\le\frac{1}{(\sigma-1)u_*^{\sigma-1}}(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})\end{aligned}$$

for $t\ge t_1$, it then follows from (4.12) that

$$\begin{aligned}\frac{d}{dt}F(t)&\le-\varsigma\int_\Omega(u-u_*)(u^{\sigma-1}-u_*^{\sigma-1})dx\\ &\le-(\sigma-1)u_*^{\sigma-1}\varsigma F(t)\quad\text{for all}\quad t\ge t_1,\end{aligned}\tag{4.21}$$

where $\varsigma$ is given by (4.10). Therefore, we have

$$F(t)\le F(t_1)e^{-(\sigma-1)u_*^{\sigma-1}\varsigma(t-t_1)},\quad t\ge t_1.\tag{4.22}$$

Accordingly, applying (4.13), (4.19) and (4.20), we arrive at

$$\begin{aligned}&\|u(\cdot,t)-u_*\|_{L^\infty(\Omega)}\le\mathcal{C}_1\|u^\kappa(\cdot,t)-u_*^\kappa\|_{L^2(\Omega)}^{\frac{2}{n+2}}\\ \le&\mathcal{C}_1\left(\frac{64\mu(\sigma-1)u_*^{\sigma-2}}{K_1}F(t_1)\right)^{\frac{1}{n+2}}e^{-\frac{(\sigma-1)u_*^{\sigma-1}\varsigma}{n+2}(t-t_1)}\quad\text{for all}\quad t\ge t_1,\end{aligned}\tag{4.23}$$

$$\begin{aligned}&\|v(\cdot,t)-v_*\|_{L^\infty(\Omega)}\\ \le&\mathcal{C}_2\left(\frac{64\mu(\sigma-1)u_*^{\sigma-2}}{K_1}F(t_1)\right)^{\frac{1}{n+2}}e^{-\frac{(\sigma-1)u_*^{\sigma-1}\varsigma}{n+2}(t-t_1)}\quad\text{for all}\quad t\ge t_1,\end{aligned}\tag{4.24}$$

where $\mu$ satisfies (4.9), $K_1$ satisfies (4.6). The proof of the lemma is complete. □

*Proof of Theorem* 1.2. Theorem 1.2 is a direct result of Lemma 4.2. □

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