

Decay of the Compressible Navier-Stokes Equations with Hyperbolic Heat Conduction

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Abstract. The global solution to the Cauchy problem of the compressible Navier-Stokes equations with hyperbolic heat conduction in dimension three is constructed when the initial data in H^3 norm is small. By using several elaborate energy functionals together with the interpolation trick, we simultaneously obtain the optimal L^2 -decay estimate of the solution and its derivatives when the initial data is bounded in negative Sobolev (Besov) space or $L^1(\mathbb{R}^3)$. Specially speaking, the fluid density, the fluid velocity and the fluid temperature in L^2 -norm have the same decay rate as the Navier-Stokes-Fourier equations, while the flux q has faster L^2 -decay rate as $(1+t)^{-2}$. Our proof is based on a family of scaled energy estimates with minimum derivative counts and interpolations among them without linear decay analysis for a 8×8 Green matrix of the system. To the best of our knowledge, it is the first result on the large time behavior of this system.

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Key words: Decay rate, Navier-Stokes equations, hyperbolic heat conduction, energy method.

1 Introduction

The compressible Navier-Stokes equations with hyperbolic heat conduction [13] takes the following form:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \quad (1.1a)$$

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$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla P = \operatorname{div} S, \quad (1.1b)$$

$$\partial_t \left(\rho \left(e + \frac{1}{2} u^2 \right) \right) + \operatorname{div} \left(\rho u \left(e + \frac{1}{2} u^2 \right) + u P \right) + \operatorname{div} q = \operatorname{div}(u S), \quad (1.1c)$$

$$\tau \partial_t q + q + \kappa \nabla \theta = 0, \quad (1.1d)$$

where the unknown functions $\rho, u = (u_1, u_2, \dots, u_n), P, S, e, q$ represent fluid density, velocity, pressure, stress tensor, specific internal energy per unit mass and flux, respectively. The Eq. (1.1d) represents Cattaneo law (Maxwell law, etc.), and $\tau > 0$ is the constant relaxation time and $\kappa > 0$ is the constant heat conductivity. Here we assume the fluid to be a Newtonian fluid, that is, $S = \mu(\nabla u + (\nabla u)^T) + \mu' \operatorname{div} u I$, where μ and μ' are the coefficient of viscosity and the second coefficient of viscosity, respectively, satisfying $\mu > 0, \mu' + 2\mu/n \geq 0$.

In this paper, we will study the global existence and large time behavior of the smooth solutions for the system (1.1) with the following initial data:

$$\rho(x, 0) = \rho_0(x) > 0, \quad u(x, 0) = u_0(x), \quad \theta(x, 0) = \theta_0(x) > 0, \quad q(x, 0) = q_0(x). \quad (1.2)$$

Here we consider the general equations of state and assume that the pressure $P(\rho, \theta)$ and $e = e(\rho, \theta)$ are smooth function of (ρ, θ) satisfying

$$\rho^2 e_\rho(\rho, \theta) = P(\rho, \theta) - \theta P_\theta(\rho, \theta), \quad (1.3)$$

where θ is the absolute temperature. Obviously, our assumption includes the case of a polytropic gas $P = R\rho\theta, e = c_v\theta$.

When $\tau = 0$, the system (1.1) is the classical full compressible Navier-Stokes equations, in which the relation between the heat flux and the temperature represents Fourier law, $q = -\kappa \nabla \theta$. Due to its importance for both physical and mathematical applications, the well-posed theory has been widely studied for the system combined with Fourier law, or the isentropic case, see [1, 7, 8, 10, 14–18] and references therein.

In the following, we mainly review some results on the decay rate of the closely related models. A lot of works have been done on the existence, stability and L^p -decay rates with $p \geq 2$ for either isentropic or non-isentropic (heat-conductive) cases, cf. [5, 6, 21–23] in various settings by using (weighted) energy method together with spectrum analysis. Recently, Danchin and Xu [2] developed optimal decay rate in general critical spaces and any dimension $n \geq 2$ under a mild additional decay assumption that is satisfied if the low frequencies of the initial data. On the other hand, Liu and Zeng [20] first studied the pointwise estimates of solution to general hyperbolic-parabolic systems in dimension one by using the method of Green function. Hoff and Zumbrun [11] investi-

gated the $L^p(\mathbb{R}^n)$ ($p \geq 1$) estimates for the linearized isentropic Navier-Stokes system in multi-dimensions. Liu and Wang [19] gave many delicate nonlinear estimates between different waves to deduce the pointwise estimates. Later on, there are a series of works on other fluid models, include the Navier-Stokes equation for non-isentropic case in [3], unipolar and bipolar Euler equation with damping in [31], unipolar and bipolar Navier-Stokes-Poisson equation in [30, 33, 35], Navier-Stokes-Maxwell equation in [4]. However, for the system (1.1), its 8×8 Green matrix is not easy to be derived. This is also an interesting problem and will be considered in a forthcoming paper.

When $\tau > 0$ in the system (1.1), it represents Cattaneo law, which was one of the physical laws describing the finite speed of heat conduction. It has been widely used in thermoelasticity which results in the second sound phenomenon, see [12, 24–26] and the references therein. Recently, for the Cauchy problem of the system (1.1), Hu and Racke [13] used Kawashima condition to deduce the global existence of the solution in H^3 -norm when the initial perturbation is sufficiently small. In the present paper, we further study the decay rate without the linear decay analysis by using the method established in Guo and Wang [9] for the estimates in the negative Sobolev space. The proof in [9] is based on a family of energy estimates with minimum derivative counts and interpolations among them without linear decay analysis. By using this method of energy estimates, Wang [32] considered the Navier-Stokes-Poisson equations, Tan and Wang [29] discussed the Euler equations with damping in \mathbb{R}^3 , where they also gave the estimates in the negative Besov space, and Wu and Wang [34] considered the corresponding bipolar case. The method mainly relies on the following two kinds of estimates: 1) closing the energy estimates at each l -th level (referring to the order of the spatial derivatives of the solution); 2) deriving a novel negative Sobolev estimates (or negative Besov estimates) for nonlinear equations which requires $0 \leq s < 3/2$ (or $0 < s \leq 3/2$).

The main difficulty here is to derive the energy estimates at each l -th level and construct several suitable energy functionals, especially for the new variable q . For example, when we treat the term $D^k(\rho \operatorname{div} q) D^k \theta$, to close the energy estimates at each k -th level, we find the relation between $\nabla \theta$ and q_t, q in the Eq. (1.1d) to reduce the power of the derivatives of the solution on the variable x . Besides, to close the energy estimates at each k -th level, we should make suitable combinations of the energy estimates of k -th and $(k+1)$ -th level. To this end, sometimes we need to provide the different estimates with different power of the derivative of the solution for some nonlinear terms. Then, we have for $0 \leq k \leq 2$ that

$$\frac{d}{dt} \left(\|(D^k \rho, D^k u, D^k \theta, D^k q)\|_{H^1}^2 \right)$$

$$+C \left(\|D^{k+1}\rho\|^2 + \|D^{k+1}u\|_{H^1}^2 + \|D^{k+1}\theta\|^2 + \|D^{k+1}q\|_{H^1}^2 \right) \leq 0, \tag{1.4}$$

which together with the inequality

$$\|D^{k+1}f\|_{L^2} \geq C \|f\|_{\dot{H}^{-s}}^{-\frac{1}{k+s}} \|D^k f\|_{L^2}^{1+\frac{1}{k+s}}, \quad k=0,1,2, \quad s \geq 0, \tag{1.5}$$

and the convexity of the function $f(t) = t^{1+1/(k+s)}$ and the smallness of the solution yields another differential equation for $k=0,1,2$ that

$$\begin{aligned} & \frac{d}{dt} \left(\|D^k(\rho, u, \theta, q)\|_{H^1}^2 \right) + C \left(\|(\rho, u, \theta, q)\|_{\dot{H}^{-s}} \right)^{-\frac{1}{k+s}} \\ & \times \left(\|D^k(\rho, u, \theta, q)\|_{H^1}^2 \right)^{1+\frac{1}{k+s}} \leq 0. \end{aligned} \tag{1.6}$$

From this differential equation and the boundedness of $\|(\rho, u, \theta, q)\|_{\dot{H}^{-s}}$ obtained when the initial data are bounded in \dot{H}^{-s} space, we find that

$$\|(D^k \rho, D^k u, D^k \theta, D^k q)(\cdot, t)\|_{H^1} \lesssim (1+t)^{-\frac{k+s}{2}}, \quad k=0,1,2. \tag{1.7}$$

Finally, to improve the decay rate of the flux q , we will use the Eq. (1.1d) to derive that

$$\tau \frac{d}{dt} \|D^k q\|_{L^2}^2 + \|D^k q\|_{L^2}^2 \lesssim \|D^k \nabla \theta\|_{L^2}^2, \quad k=0,1. \tag{1.8}$$

Integrating (1.8) with respect to the time t over $[0, t]$, and using the decay rate of $\|\nabla \theta\|_{L^2}$ in (1.7), we have $\|q\|_{L^2} \lesssim (1+t)^{-(k+s+1)/2}$. Then in virtue of Hardy-Littlewood-Sobolev theorem, we obtain the $L^p - L^2$ decay rates of the solution and its derivatives.

The following two assumptions are given in [13], which are essential to deduce the local existence and global existence by using the Kawashima condition.

Assumption 1.1. The initial data satisfy

$$\{(\rho_0, u_0, \theta_0, q_0)(x), x \in \mathbb{R}^n\} \subset [\rho_*, \rho^*] \times [-C_1, C_1] \times [\theta_*, \theta^*] \times [-C_1, C_1]^n := G_0,$$

where $C_1 > 0$, $0 < \rho_* < 1 < \rho^* < \infty$ and $0 < \theta_* < 1 < \theta^* < \infty$ are constants.

Assumption 1.2. For each given G_1 satisfying

$$G_0 \subset \subset G_1 \subset \subset \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{R}^n, \quad \forall (\rho, u, \theta, q) \in G_1,$$

the pressure P and the internal energy e satisfy $P(\rho, \theta), P_\theta(\rho, \theta), e_\theta(\rho, \theta) > C(G_1) > 0$, where $C(G_1)$ is a positive constants depending on G_1 .

Our global existence results including the temporal decay estimates are stated in the following theorem.

Theorem 1.1. *Under the Assumptions 1.1 and 1.2, and assume that*

$$(\rho_0 - 1, u_0, \theta_0 - 1, q_0) \in H^3(\mathbb{R}^3) \quad \text{and} \quad \epsilon_0 =: \|(\rho_0 - 1, u_0, \theta_0 - 1, q_0)\|_{H^3(\mathbb{R}^3)}$$

small. Then there exists a unique, global, classical solution $(\rho - 1, u, \theta - 1, q)$ in $H^3(\mathbb{R}^3)$. If further

$$\|(\rho_0 - 1, u_0, \theta_0 - 1, q_0)(\cdot, t)\|_{\dot{H}^{-s}} \leq C_0, \quad s \in [0, 3/2], \tag{1.9}$$

or

$$\|(\rho_0 - 1, u_0, \theta_0 - 1, q_0)(\cdot, t)\|_{\dot{B}_{2,\infty}^{-s}} \leq C_0, \quad s \in (0, 3/2], \tag{1.10}$$

then

$$\|(\rho - 1, u, \theta - 1, q)(\cdot, t)\|_{H^{-s}} \leq C_0, \quad s \in [0, 3/2], \tag{1.11}$$

or

$$\|(\rho - 1, u, \theta - 1, q)(\cdot, t)\|_{\dot{B}_{2,\infty}^{-s}} \leq C_0, \quad s \in (0, 3/2], \tag{1.12}$$

and

$$\begin{aligned} \|D^k(\rho - 1, u, \theta - 1, q)(\cdot, t)\|_{H^{3-k}} &\leq C_0(1+t)^{-\frac{k+s}{2}}, \quad k=0,1,2, \\ \|D^k q(\cdot, t)\|_{H^{3-k}} &\leq C_0(1+t)^{-\frac{k+s+1}{2}}, \quad k=0,1. \end{aligned} \tag{1.13}$$

Note that Lemma A.3 (the Hardy-Littlewood-Sobolev theorem) implies that for $p \in (1, 2], L^p \subset \dot{H}^{-s}$ with $s = 3(1/p - 1/2)$ and Lemma A.5 implies that for $p \in [1, 2), L^p \subset \dot{B}_{2,\infty}^{-s}$ with $s = 3(1/p - 1/2)$. Then Theorem 1.1 yields the following decay results of $L^p - L^2$ type.

Corollary 1.1. *Under the assumptions of Theorem 1.1 except that we replace the \dot{H}^{-s} or $\dot{B}_{2,\infty}^{-s}$ assumption by that $(\rho_0 - 1, u_0, \theta_0 - 1, q_0) \in L^p(\mathbb{R}^3)$ for some $p \in [1, 2]$, then*

$$\|D^k(\rho - 1, u, \theta - 1)(t)\|_{H^{3-k}} \leq C_0(1+t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k}{2}} \quad \text{for } k=0,1,2, \tag{1.14}$$

$$\|D^k q(t)\|_{L^2} \leq C_0(1+t)^{-\frac{3}{2}(\frac{1}{p} - \frac{1}{2}) - \frac{k+1}{2}} \quad \text{for } k=0,1. \tag{1.15}$$

Remark 1.1. We need not the initial data is small in $L^1(\mathbb{R}^3)$. In fact, the fluid density, the fluid velocity and the fluid temperature in L^2 -norm have the same decay rate as the classical Navier-Stokes equations, while the flux q has faster L^2 -decay rate as $(1+t)^{-2}$ due to the damping term in the Eq. (1.1d).

Remark 1.2. Our results include the case $\tau = 0$ ($q = -\kappa \nabla \theta$), that is, the nonisotropic Navier-Stokes equations. The isentropic case has been investigated in [9]. Besides, the method in this paper can be also applied to the damped Euler equations with hyperbolic heat conduction.

Remark 1.3. It is also interesting to derive the L^p -decay rate of the solutions with $1 \leq p < 2$ by using some linear decay analysis on the Green's function of the Cauchy problem (1.1)-(1.2), which will be studied in future.

Notations. In this paper, D^k with an integer $k \geq 0$ stands for the usual any spatial derivatives of order k . For $1 \leq p \leq \infty$ and an integer $m \geq 0$, we use L^p and $W^{m,p}$ to denote the usual Lebesgue space $L^p(\mathbb{R}^n)$ and Sobolev spaces $W^{m,p}(\mathbb{R}^n)$ with norms $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$, respectively, and set $H^m = W^{m,2}$ with norm $\|\cdot\|_{H^m}$ when $p = 2$. In addition, for $s \in \mathbb{R}$, we define a pseudo-differential operator Λ^s by

$$\Lambda^s g(x) = \int_{\mathbb{R}^n} |\xi|^s \hat{g}(\xi) e^{2\pi i x \cdot \xi} d\xi,$$

where \hat{g} denotes the Fourier transform of g . We define the homogeneous Sobolev space \dot{H}^s of all g for which $\|g\|_{\dot{H}^s}$ is finite, where $\|g\|_{\dot{H}^s} := \|\Lambda^s g\|_{L^2} = \| |\xi|^s \hat{g} \|_{L^2}$.

Let $\eta \in C_0^\infty(\mathbb{R}_\xi^3)$ be such that $\eta(\xi) = 1$ when $|\xi| \leq 1$ and $\eta(\xi) = 0$ when $|\xi| \geq 2$. We define the homogeneous Besov spaces $\dot{B}_{2,\infty}^{-s}(\mathbb{R}^3)$ with norm $\|\cdot\|_{\dot{B}_{p,r}^{-s}}$ defined by

$$\|f\|_{\dot{B}_{p,r}^{-s}} := \left(\sum_{j \in \mathbb{Z}} 2^{rsj} \|\dot{\Delta}_j f\|_{L^p}^r \right)^{\frac{1}{r}},$$

where $\dot{\Delta}_j f := F^{-1}(\varphi_j) * f$, $\varphi(\xi) = \eta(\xi) - \eta(2\xi)$ and $\varphi_j(\xi) = \varphi(2^{-j}\xi)$.

Throughout this paper, we will use a non-positive index s . For convenience, we will change the index to “ $-s$ ” with $s \geq 0$. C and C_i denote positive generic (generally large) constants that may vary at different places. For simplicity, we write $\int f := \int_{\mathbb{R}^3} f dx$ and $\|f\|_{L^2} := \|f\|$. Moreover, we use $\langle \cdot, \cdot \rangle$ to denote the inner product in $L^2(\mathbb{R}^3)$.

The rest of the paper is arranged as follows. In Section 2, we give energy estimates in H^3 norm and some estimates in \dot{H}^{-s} and $\dot{B}_{2,\infty}^{-s}$. The proof of global existence and temporal decay results of the solutions will be derived in Section 3. Finally, some useful Sobolev inequality and Besov inequality are stated in Appendix A.

2 Nonlinear energy estimates

2.1 Energy functionals in each k -level in H^3 -norm

First, by using (1.1a) and (1.3), we can reformulate the nonlinear system (1.1) for (ρ, u, θ, q) around the equilibrium state $(1, 0, 1, 0)$ as

$$\begin{aligned}
 & \partial_t \rho + \operatorname{div} u = -\operatorname{div}(\rho u), \\
 & \partial_t u + p_\rho(1, 1) \nabla \rho + p_\theta(1, 1) \nabla \theta - \mu \Delta u - (\mu + \mu') \nabla \operatorname{div} u \\
 & = -u \cdot \nabla u - \frac{1}{1 + \rho} (p_\theta(1, 1) \rho \nabla \theta + p_\rho(1, 1) \rho \nabla \rho - \mu \rho \Delta u - (\mu + \mu') \rho \nabla \operatorname{div} u), \\
 & \partial_t \theta + \frac{p_\theta(1, 1)}{e_\theta(1, 1)} \operatorname{div} u + \frac{1}{e_\theta(1, 1)} \operatorname{div} q \\
 & = -u \cdot \nabla \theta - \left(\frac{(1 + \theta) p_\theta(1 + \rho, 1 + \theta)}{(1 + \rho) e_\theta(1 + \rho, 1 + \theta)} - \frac{p_\theta(1, 1)}{e_\theta(1, 1)} \right) \operatorname{div} u \\
 & \quad - \left(\frac{1}{(1 + \rho) e_\theta(1 + \rho, 1 + \theta)} - \frac{1}{e_\theta(1, 1)} \right) \operatorname{div} q \\
 & \quad + \frac{1}{(1 + \rho) e_\theta(1 + \rho, 1 + \theta)} \left(\frac{1}{2} |\nabla u + (\nabla u)^T|^2 + |\operatorname{div} u|^2 \right), \\
 & \partial_t q + q + \nabla \theta = 0, \\
 & (\rho, u, \theta, q)(x, 0) = (\rho_0, u_0, \theta_0, q_0)(x).
 \end{aligned} \tag{2.1}$$

Hereinafter, for brevity we still denote the new variables as (ρ, u, θ, q) without confusion.

Without loss of generality, in the following, we mainly consider the case of a polytropic gas $P = R\rho\theta$, $e = c_v\theta$. In fact, the unique difference from the general case above is that there exists an additional nonlinear term as $(1/(1 + \rho))\theta \operatorname{div} q$ in the third equation below, and this term can be similarly estimated as the term $(1/(1 + \rho))\rho \operatorname{div} q$. In addition, since the specific values of all the positive parameters in (2.1), $P_\rho(1, 1)$ and $P_\theta(1, 1)$ are not essential when deducing the energy estimates, we set all of these be 1 for the convenience of narration. Then the Cauchy problem for the new variables (ρ, u, θ, q) is given by

$$\partial_t \rho + \operatorname{div} u = -\operatorname{div}(\rho u), \tag{2.2a}$$

$$\begin{aligned}
 & \partial_t u + \nabla \rho + \nabla \theta - \Delta u - \nabla \operatorname{div} u \\
 & = -u \cdot \nabla u - \frac{1}{1 + \rho} (\rho \nabla \theta + \rho \nabla \rho - \rho \Delta u - \rho \nabla \operatorname{div} u),
 \end{aligned} \tag{2.2b}$$

$$\partial_t \theta + \operatorname{div} u + \operatorname{div} q$$

$$= -u \cdot \nabla \theta - \frac{1}{1+\rho} \left(\rho \operatorname{div} u + \theta \operatorname{div} u + \rho \operatorname{div} q - \frac{1}{2} |\nabla u + (\nabla u)^T|^2 - |\operatorname{div} u|^2 \right), \quad (2.2c)$$

$$\partial_t q + q + \nabla \theta = 0, \quad (2.2d)$$

$$(\rho, u, \theta, q)(x, 0) = (\rho_0, u_0, \theta_0, q_0)(x). \quad (2.2e)$$

In this subsection, we intend to derive three energy functionals, which will be used to deduce the decay rate in Section 3. First, for the equivalent system (2.1) (or the system (2.2)), when $(\rho_0, u_0, \theta_0, q_0) \in H^3(\mathbb{R}^3) \leq \epsilon_0 \ll 1$, Hu and Racke [13] gave the following estimate for some constant $C > 0$:

$$\|\rho(\cdot, t)\|_{H^3} + \|u(\cdot, t)\|_{H^3} + \|\theta(\cdot, t)\|_{H^3} + \|q(\cdot, t)\|_{H^3} \leq C\epsilon_0, \quad (2.3)$$

where they used the Kawashima condition to prove global existence theorem. Here our energy estimates together with the local existence given in [13] can also derive the same global existence.

Basing on the estimate (2.3), we can obtain the following energy estimates in their own level, which will be used to derive three essential energy functionals.

Lemma 2.1. *If (2.3) holds, then we have*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|\rho\|^2 + \|u\|^2 + \|\theta\|^2 + \|q\|^2 \right) + \left(\|\nabla u\|^2 + \|\operatorname{div} u\|^2 + \|q\|^2 \right) \\ & \lesssim \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 + \|D\theta\|^2 + \|Dq\|^2 \right), \end{aligned} \quad (2.4)$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|D\rho\|^2 + \|Du\|^2 + \|D\theta\|^2 + \|Dq\|^2 \right) + \left(\|D\nabla u\|^2 + \|D\operatorname{div} u\|^2 + \|Dq\|^2 \right) \\ & \lesssim \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 + \|D^2 u\|^2 + \|D\theta\|^2 + \|Dq\|^2 \right). \end{aligned} \quad (2.5)$$

Proof. For $0 \leq k \leq 2$, applying D^k to (2.2) and then multiplying the resulting equations by $D^k \rho, D^k u, D^k \theta, D^k q$ respectively, summing up and integrating over \mathbb{R}^3 , one has

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|D^k(\rho, u, \theta, q)\|^2 + \left(\|D^k \nabla u\|^2 + \|D^k \operatorname{div} u\|^2 + \|D^k q\|^2 \right) \\ & = - \left\langle D^k(-\rho \operatorname{div} u - u \cdot \nabla \rho), D^k \rho \right\rangle \\ & \quad + \left\langle D^k \left(-u \cdot \nabla u + \frac{\rho}{1+\rho} \Delta u + \frac{\rho}{1+\rho} \nabla \operatorname{div} u - \frac{\rho}{1+\rho} \nabla \theta - \frac{\rho}{1+\rho} \nabla \rho \right), D^k u \right\rangle \\ & \quad + \left\langle D^k \left(-u \cdot \nabla \theta - \frac{\rho \operatorname{div} u}{1+\rho} + \frac{\theta \operatorname{div} u}{1+\rho} + \frac{\rho \operatorname{div} q}{1+\rho} + \frac{|\nabla u + (\nabla u)^T|^2}{1+\rho} + \frac{|\operatorname{div} u|^2}{1+\rho} \right), D^k \theta \right\rangle. \end{aligned} \quad (2.6)$$

When $k = 0$, by using the Hölder inequality, the Sobolev inequality $\|f\|_{L^6} \lesssim \|Df\|_{L^2}$ when $f \in H^1(\mathbb{R}^3)$ and the Cauchy inequality, we have the following estimates for the above each term:

$$\begin{aligned}
 & |\langle (-\rho \operatorname{div} u - u \cdot \nabla \rho), \rho \rangle| \\
 & \lesssim \|\rho\|_{L^6} \|Du\|_{L^2} \|\rho\|_{L^3} + \|u\|_{L^6} \|\nabla \rho\|_{L^2} \|\rho\|_{L^3} \\
 & \lesssim \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 \right), \\
 & |\langle u \cdot \nabla u, u \rangle| \lesssim \|u\|_{L^6} \|Du\| \|u\|_{L^3} \lesssim \epsilon_0 \|Du\|^2, \\
 & \left| \left\langle \left(\frac{\rho}{1+\rho} \Delta u \right), u \right\rangle \right| = \left| \left\langle \operatorname{div} \left(\frac{\rho}{1+\rho} \nabla u \right) u - \nabla \frac{\rho}{1+\rho} \nabla u, u \right\rangle \right| \\
 & = \left| - \left\langle \frac{\rho}{1+\rho} \nabla u \nabla u - \nabla \frac{\rho}{1+\rho} \nabla u, u \right\rangle \right| \lesssim \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 \right), \\
 & \left| \left\langle \left(\frac{\rho}{1+\rho} \nabla \operatorname{div} u \right), u \right\rangle \right| \lesssim \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 \right), \\
 & \left| \left\langle \frac{\rho}{1+\rho} \nabla \theta, u \right\rangle \right| \lesssim \epsilon_0 \left(\|D\theta\|^2 + \|Du\|^2 \right), \\
 & \left| \left\langle \frac{\rho}{1+\rho} \nabla \rho, u \right\rangle \right| \lesssim \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 \right), \\
 & |\langle u \cdot \nabla \theta \theta \rangle| \lesssim \epsilon_0 \|D\theta\|^2, \\
 & \left| \left\langle \frac{\rho}{1+\rho} \operatorname{div} u, \theta \right\rangle \right| + \left| \left\langle \frac{\theta}{1+\rho} \operatorname{div} u, \theta \right\rangle \right| + \left| \left\langle \frac{\rho}{1+\rho} \operatorname{div} q, \theta \right\rangle \right| \\
 & \lesssim \epsilon_0 \left(\|D\theta\|^2 + \|Du\|^2 + \|Dq\|^2 \right), \\
 & \left| \left\langle \frac{1}{1+\rho} (|\nabla u + (\nabla u)^T|^2 + |\operatorname{div} u|^2) \theta \right\rangle \right| \lesssim \epsilon_0 \|Du\|^2.
 \end{aligned}$$

Inserting these estimates into (2.6) with $k = 0$, we obtain the estimate (2.4).

When $k = 1$ in (2.6), we estimate each terms as

$$\begin{aligned}
 & |\langle D(-\rho \operatorname{div} u - u \cdot \nabla \rho), D\rho \rangle| \\
 & \lesssim |\langle D\rho Du, D\rho \rangle| + |\langle \rho D^2 u, D\rho \rangle| + |\langle u D^2 \rho, D\rho \rangle| \\
 & \lesssim \|Du\|_{L^\infty} \|D\rho\|^2 + \|\rho\|_{L^\infty} \|D^2 u\| \|D\rho\| + \|u\|_{L^6} \|D^2 \rho\|_{L^3} \|D\rho\| \\
 & \lesssim \epsilon_0 \left(\|D\rho\|^2 + \|D^2 u\|^2 + \|Du\|^2 \right),
 \end{aligned}$$

$$\begin{aligned}
& \|\langle D(u \cdot \nabla u), Du \rangle\| \lesssim \epsilon_0 \|D^2 u\|^2, \\
& \left| \left\langle D \left(\frac{\rho}{1+\rho} \Delta u \right), Du \right\rangle \right| = \left| \left\langle \frac{\rho}{1+\rho} \Delta u, D^2 u \right\rangle \right| \lesssim \epsilon_0 \|D^2 u\|^2, \\
& \left| \left\langle D \left(\frac{\rho}{1+\rho} \nabla \operatorname{div} u \right), Du \right\rangle \right| \lesssim \epsilon_0 \|D^2 u\|^2, \\
& \left| \left\langle D \left(\frac{\rho}{1+\rho} \nabla \theta \right), Du \right\rangle \right| \lesssim \epsilon_0 (\|D\theta\|^2 + \|D^2 u\|^2), \\
& |\langle D(u \cdot \nabla \theta), D\theta \rangle| = \left| \langle Du \cdot \nabla \theta, \nabla \theta \rangle - \frac{1}{2} \langle \operatorname{div} u, |D\theta|^2 \rangle \right| \lesssim \epsilon_0 (\|D\theta\|^2 + \|D^2 u\|^2), \\
& \left| \left\langle D \left(\frac{\rho}{1+\rho} \nabla \rho \right), Du \right\rangle \right| \lesssim \epsilon_0 (\|D\rho\|^2 + \|D^2 u\|^2), \\
& \left| \left\langle D \left(\frac{\rho}{1+\rho} \operatorname{div} u \right), D\theta \right\rangle \right| + \left| \left\langle D \left(\frac{\theta}{1+\rho} \operatorname{div} u \right), D\theta \right\rangle \right| \lesssim \epsilon_0 (\|D\theta\|^2 + \|D^2 u\|^2), \\
& \left| \left\langle D \left(\frac{\rho}{1+\rho} \operatorname{div} q \right), D\theta \right\rangle \right| \lesssim \left| \left\langle \frac{\rho}{1+\rho} D \operatorname{div} q, D\theta \right\rangle \right| + \left| \left\langle D \frac{\rho}{1+\rho} \operatorname{div} q, D\theta \right\rangle \right| \\
& \lesssim |\langle \rho D^2 q, D\theta \rangle| + \|D\rho\|_{L^\infty} \|Dq\| \|D\theta\| \\
& \lesssim \|\rho\|_{L^6} \|D^2 q\|_{L^3} \|D\theta\| + \|D\rho\|_{L^\infty} \|Dq\| \|D\theta\| \\
& \lesssim \epsilon_0 (\|D\theta\|^2 + \|D\rho\|^2), \\
& \left| \left\langle \frac{1}{1+\rho} (\nabla u + (\nabla u)^T)^2 + |\operatorname{div} u|^2, D\theta \right\rangle \right| \lesssim \epsilon_0 \|Du\|^2.
\end{aligned}$$

Inserting these estimates into (2.6) with $k=1$, we obtain the estimate (2.5). \square

Lemma 2.2. *If (2.3) holds, then we have*

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \|D(\rho, u, \theta, q)\|^2 + \left(\|D(\nabla u, \operatorname{div} u)\|^2 + \|Dq\|^2 \right) \\
& \lesssim \epsilon_0 (\|D^2 \rho\|^2 + \|D^2 u\|^2 + \|D^2 \theta\|^2 + \|D^2 q\|^2), \tag{2.7}
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} \frac{d}{dt} \left(\|D^2(\rho, u, \theta, q)\|^2 + \int (1+\rho) |D^2 q|^2 \right) + \left(\|D^2(\nabla u, \operatorname{div} u)\|^2 + \|D^2 q\|^2 \right) \\
& \lesssim \epsilon_0 (\|D^2 \rho\|^2 + \|D^3 u\|^2 + \|D^2 \theta\|^2 + \|D^2 q\|^2). \tag{2.8}
\end{aligned}$$

Proof. When $k=1$ in (2.6), we estimate each term as

$$|\langle D(-\rho \operatorname{div} u - u \cdot \nabla \rho), D\rho \rangle| = |\langle (-\rho \operatorname{div} u - u \cdot \nabla \rho), D^2 \rho \rangle|$$

$$\begin{aligned}
 &\lesssim \|\rho\|_{L^3} \|\operatorname{div} u\|_{L^6} \|D^2 \rho\| + \|u\|_{L^3} \|\nabla \rho\|_{L^6} \|D^2 \rho\| \\
 &\lesssim \epsilon_0 \left(\|D^2 \rho\|^2 + \|D^2 u\|^2 \right), \\
 &\quad |\langle D(u \cdot \nabla u), Du \rangle| \lesssim \epsilon_0 \|D^2 u\|^2, \\
 &\quad \left| \left\langle D \left(\frac{\rho}{1+\rho} \Delta u \right), Du \right\rangle \right| + \left| \left\langle D \left(\frac{\rho}{1+\rho} \nabla \operatorname{div} u \right), Du \right\rangle \right| \\
 &= \left| \left\langle \frac{\rho}{1+\rho} \Delta u, D^2 u \right\rangle \right| \lesssim \epsilon_0 \|D^2 u\|^2, \\
 &\quad \left| \left\langle D \left(\frac{\rho}{1+\rho} \nabla \theta \right), Du \right\rangle \right| = \left| \left\langle \frac{\rho}{1+\rho} \nabla \theta, D^2 u \right\rangle \right| \\
 &\lesssim \left\| \frac{\rho}{1+\rho} \right\|_{L^3} \|\nabla \theta\|_{L^6} \|D^2 u\| \lesssim \epsilon_0 \left(\|D^2 \theta\|^2 + \|D^2 u\|^2 \right), \\
 &\quad \left| \left\langle D \left(\frac{\rho}{1+\rho} \nabla \rho \right), Du \right\rangle \right| = \left| \left\langle \frac{\rho}{1+\rho} \nabla \rho, D^2 u \right\rangle \right| \lesssim \epsilon_0 \left(\|D^2 \rho\|^2 + \|D^2 u\|^2 \right), \\
 &\quad |\langle D(u \cdot \nabla \theta), D\theta \rangle| = |\langle u \cdot \nabla \theta, D^2 \theta \rangle| \lesssim \epsilon_0 \|D^2 \theta\|^2, \\
 &\quad \left| \left\langle D \left(\frac{\rho}{1+\rho} \operatorname{div} u \right), D\theta \right\rangle \right| + \left| \left\langle D \left(\frac{\theta}{1+\rho} \operatorname{div} u \right), D\theta \right\rangle \right| \lesssim \epsilon_0 \left(\|D^2 \theta\|^2 + \|D^2 u\|^2 \right), \\
 &\quad \left| \left\langle D \left(\frac{\rho}{1+\rho} \operatorname{div} q \right), D\theta \right\rangle \right| = \left| \left\langle \frac{\rho}{1+\rho} \operatorname{div} q, D^2 \theta \right\rangle \right| \lesssim \epsilon_0 \left(\|D^2 \theta\|^2 + \|D^2 q\|^2 \right), \\
 &\quad \left| \left\langle D \left(\frac{1}{1+\rho} |\nabla u + (\nabla u)^T|^2 + |\operatorname{div} u|^2 \right), D\theta \right\rangle \right| \\
 &= \left| \left\langle \frac{1}{1+\rho} |\nabla u + (\nabla u)^T|^2 + |\operatorname{div} u|^2, D^2 \theta \right\rangle \right| \lesssim \epsilon_0 \left(\|D^2 u\|^2 + \|D^2 \theta\|^2 \right).
 \end{aligned}$$

Inserting these estimates into (2.6) with $k = 1$, we obtain the estimate (2.7).

When $k = 2$ in (2.6), the most difficult term is $\langle D^2(\rho/(1+\rho)\operatorname{div}q), D^2\theta \rangle$. Since we want to finally derive the energy functionals in each level of the derivatives of the solution, we have to control this term without using the third derivatives of the solution. It will be estimated by using integral by parts and the Eq. (2.2d): $\nabla\theta = -q_t - q$. In fact, we have

$$\begin{aligned}
 &\left\langle D^2 \left(\frac{\rho}{1+\rho} \operatorname{div} q \right), D^2 \theta \right\rangle \\
 &= \langle D^2 \rho \operatorname{div} q, D^2 \theta \rangle + \langle D \rho D \operatorname{div} q, D^2 \theta \rangle + \langle \rho D^2 \operatorname{div} q, D^2 \theta \rangle, \tag{2.9}
 \end{aligned}$$

and the first two terms can be estimated as

$$|\langle D^2 \rho \operatorname{div} q, D^2 \theta \rangle + \langle D \rho D \operatorname{div} q, D^2 \theta \rangle| \lesssim \epsilon_0 \left(\|D^2 \rho\|^2 + \|D^2 \theta\|^2 + \|D^2 q\|^2 \right). \quad (2.10)$$

The last term in (2.9) can be rewritten as

$$\begin{aligned} \langle \rho D^2 \operatorname{div} q, D^2 \theta \rangle &= \langle \operatorname{div}(\rho D^2 q), D^2 \theta \rangle - \langle \nabla \rho D^2 q, D^2 \theta \rangle \\ &= -\langle \rho D^2 q, D^2 \nabla \theta \rangle - \langle \nabla \rho D^2 q, D^2 \theta \rangle \\ &= \langle \rho D^2 q, (D^2 q_t + D^2 q) \rangle - \langle \nabla \rho D^2 q, D^2 \theta \rangle \\ &= \left\langle \rho, \frac{d}{dt} (D^2 q)^2 \right\rangle + \left\langle \rho, D^2 q \frac{d}{dt} (D^2 q) \right\rangle - \langle \nabla \rho D^2 q, D^2 \theta \rangle \\ &= \frac{d}{dt} \langle \rho, (D^2 q)^2 \rangle - \langle \rho_t, (D^2 q)^2 \rangle - \langle \nabla \rho D^2 q, D^2 \theta \rangle \\ &= \frac{d}{dt} \langle \rho, (D^2 q)^2 \rangle + \langle (\operatorname{div} u + \rho \operatorname{div} u + \nabla \rho \cdot u), (D^2 q)^2 \rangle - \langle \nabla \rho D^2 q, D^2 \theta \rangle, \end{aligned} \quad (2.11)$$

and the last two terms can be bounded as

$$|\langle (\operatorname{div} u + \rho \operatorname{div} u + \nabla \rho \cdot u), (D^2 q)^2 \rangle - \langle \nabla \rho D^2 q, D^2 \theta \rangle| \lesssim \left(\|D^2 q\|^2 + \|D^2 \theta\|^2 \right). \quad (2.12)$$

The other terms in (2.6) with $k=2$ can be estimated as follows:

$$\begin{aligned} &|\langle D^2(-\rho \operatorname{div} u - u \cdot \nabla \rho), D^2 \rho \rangle| \\ &= |\langle D^2 \rho \operatorname{div} u, D^2 \rho \rangle + \langle D \rho D \operatorname{div} u, D^2 \rho \rangle + \langle \rho D^2 \operatorname{div} u, D^2 \rho \rangle| \\ &\lesssim \epsilon_0 \left(\|D^2 \rho\|^2 + \|D^3 u\|^2 \right), \\ &|\langle D^2(u \cdot \nabla u), D^2 u \rangle| \\ &\lesssim \|Du D^2 u\|_{L^{\frac{6}{5}}} \|D^2 u\|_{L^6} + \|u\|_{L^3} \|D^2 u\|_{L^6} \|D^3 u\| \\ &\lesssim \|Du\|_{L^3} \|D^2 u\| \|D^3 u\| + \|u\|_{L^3} \|D^3 u\|^2 \\ &\lesssim \|D^{\frac{3}{4}} u\|^{\frac{2}{3}} \|D^3 u\|^{\frac{1}{3}} \|u\|^{\frac{1}{3}} \|D^3 u\|^{\frac{2}{3}} \|D^3 u\| + \|u\|_{L^3} \|D^3 u\|^2 \lesssim \epsilon_0 \|D^3 u\|^2, \\ &\left| \left\langle D^2 \left(\frac{\rho}{1+\rho} \Delta u \right), D^2 u \right\rangle \right| + \left| \left\langle D^2 \left(\frac{\rho}{1+\rho} \nabla \operatorname{div} u \right), D^2 u \right\rangle \right| \\ &\lesssim \left| \left\langle D \left(\frac{\rho}{1+\rho} \Delta u \right), D^3 u \right\rangle \right| \lesssim \epsilon_0 \|D^3 u\|^2, \\ &\left| \left\langle D^2 \left(\frac{\rho}{1+\rho} \nabla \theta \right), D^2 u \right\rangle \right| = \left| \left\langle D \left(\frac{\rho}{1+\rho} \nabla \theta \right), D^3 u \right\rangle \right| \lesssim \epsilon_0 \left(\|D^2 \theta\|^2 + \|D^3 u\|^2 \right), \end{aligned}$$

$$\begin{aligned}
 & \left| \left\langle D^2 \left(\frac{\rho}{1+\rho} \nabla \rho \right), D^2 u \right\rangle \right| = \left| \left\langle D \left(\frac{\rho}{1+\rho} \nabla \rho \right), D^3 u \right\rangle \right| \\
 & = \left| \left\langle D \left(\frac{\rho}{1+\rho} \right) \nabla \rho, D^3 u \right\rangle + \left\langle \frac{\rho}{1+\rho} D \nabla \rho, D^3 u \right\rangle \right| \\
 & \lesssim \left\| D \left(\frac{\rho}{1+\rho} \right) \right\|_{L^3} \|\nabla \rho\|_{L^6} \|D^3 u\| \lesssim \epsilon_0 \left(\|D^2 \rho\|^2 + \|D^3 u\|^2 \right), \\
 & \left| \langle D^2(u \cdot \nabla \theta), D^2 \theta \rangle \right| = \left| \langle D^2 u \cdot \nabla \theta, D^2 \theta \rangle + \langle Du \cdot D \nabla \theta, D^2 \theta \rangle + \langle u \cdot \nabla(D^2 \theta), (D^2 \theta) \rangle \right| \\
 & = \left| \langle D^2 u \cdot \nabla \theta, D^2 \theta \rangle + \langle Du \cdot D \nabla \theta, D^2 \theta \rangle - \frac{1}{2} \langle \operatorname{div} u, (D^2 \theta)^2 \rangle \right| \lesssim \epsilon_0 \|D^2 \theta\|^2, \\
 & \left| \left\langle D^2 \left(\frac{\rho}{1+\rho} \operatorname{div} u \right), D^2 \theta \right\rangle \right| + \left| \left\langle D^2 \left(\frac{\theta}{1+\rho} \operatorname{div} u \right), D^2 \theta \right\rangle \right| \\
 & \lesssim \epsilon_0 \left(\|D^2 \theta\|^2 + \|D^3 u\|^2 \right), \\
 & \left| \left\langle D^2 \left(\frac{1}{1+\rho} (|\nabla u + (\nabla u)^T|^2 + |\operatorname{div} u|^2) \right), D^2 \theta \right\rangle \right| \lesssim \epsilon_0 \left(\|D^3 u\|^2 + \|D^2 \theta\|^2 \right).
 \end{aligned}$$

Inserting these estimates and (2.9)-(2.12) into (2.6) with $k=2$, we can immediately obtain (2.8). □

Next, we want to estimate the dissipative term of $\nabla \rho, D \nabla \rho$ from the Eq. (2.2b).

Lemma 2.3. *If (2.3) holds, we have for $k=0,1$ that*

$$\frac{d}{dt} \langle D^k u, D^k \nabla \rho \rangle + \frac{1}{2} \|D^k \nabla \rho\|^2 \lesssim \|D^{k+1} u\|^2 + \|D^{k+2} u\|^2 + \|D^{k+1} \theta\|^2. \tag{2.13}$$

Proof. First, applying D^k on the first three equations in (2.2b), and then multiplying the resulting equation by $D^k \nabla \rho$ respectively, summing up and integrating over \mathbb{R}^3 , then we have

$$\begin{aligned}
 & \langle u_t, \nabla \rho \rangle + \|\nabla \rho\|^2 \\
 & = - \left\langle \left(\nabla \theta + \Delta u + \nabla \operatorname{div} u - u \cdot \nabla u - \frac{\rho}{1+\rho} (\nabla \rho + \Delta u + \nabla \operatorname{div} u) \right), \nabla \rho \right\rangle \\
 & \lesssim \|D \theta\|^2 + \frac{1}{2} \|D \rho\|^2 + \|D^2 u\|^2, \tag{2.14}
 \end{aligned}$$

$$\begin{aligned}
 \langle u_t, \nabla \rho \rangle & = \frac{d}{dt} \langle u, \nabla \rho \rangle - \langle u, \nabla \rho_t \rangle = \frac{d}{dt} \langle u, \nabla \rho \rangle + \langle \operatorname{div} u, \rho_t \rangle \\
 & = \frac{d}{dt} \langle u, \nabla \rho \rangle + \langle \operatorname{div} u, (-\operatorname{div} u - \rho \operatorname{div} u - u \cdot \nabla \rho) \rangle \lesssim \frac{d}{dt} \langle u, \nabla \rho \rangle + \|Du\|^2. \tag{2.15}
 \end{aligned}$$

Inserting (2.15) into (2.14), we get the estimate (2.13) when $k=0$.

When $k=1$, we also have

$$\begin{aligned} \langle Du_t, D\nabla\rho \rangle + \|D\nabla\rho\|^2 &= - \int \left(D\nabla\theta D\nabla\rho + D\Delta u D\nabla\rho + D\nabla\operatorname{div}u D\nabla\rho \right. \\ &\quad \left. - D(u \cdot \nabla u) D\nabla\rho - D\left(\frac{\rho}{1+\rho}\nabla\rho\right) D\nabla\rho \right. \\ &\quad \left. + D\left(\frac{\rho}{1+\rho}\Delta u\right) D\nabla\rho + D\left(\frac{\rho}{1+\rho}\nabla\operatorname{div}u\right) D\nabla\rho \right) \\ &\lesssim \|D^2\theta\|^2 + \frac{1}{2}\|D^2\rho\|^2 + \|D^3u\|^2 + \epsilon_0\|D^2u\|^2, \end{aligned} \quad (2.16)$$

and

$$\begin{aligned} \langle Du_t, D\nabla\rho \rangle &= \frac{d}{dt} \langle Du, D\nabla\rho \rangle - \langle Du, D\nabla\rho_t \rangle \\ &= \frac{d}{dt} \langle Du, D\nabla\rho \rangle + \langle \operatorname{div}Du, D\rho_t \rangle \\ &= \frac{d}{dt} \langle Du, D\nabla\rho \rangle + \langle (-\operatorname{div}Du, \operatorname{div}Du) - \langle D\rho, \operatorname{div}Du \rangle - \langle Du \cdot \nabla D\rho \rangle \\ &\lesssim \frac{d}{dt} \langle Du, D\nabla\rho \rangle + \|D^2u\|^2. \end{aligned} \quad (2.17)$$

Inserting (2.17) into (2.16), we also get the estimate (2.13). \square

The following lemma is to derive the dissipation of $\nabla\theta$ and $D\nabla\theta$ by using the Eq. (2.2d).

Lemma 2.4. *If (2.3) holds, we have*

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|\rho\|^2 + \|u\|^2 + \|\theta\|^2 + \langle q, \nabla\theta \rangle \right) + \left[\|\nabla u\|^2 + \|\operatorname{div}u\|^2 + \|\nabla\theta\|^2 \right] \\ &\lesssim \|Dq\|^2 + \frac{1}{2}\|Du\|^2 + \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 + \|D\theta\|^2 \right), \end{aligned} \quad (2.18)$$

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \left(\|D\rho\|^2 + \|Du\|^2 + \|D\theta\|^2 + \langle Dq, D\nabla\theta \rangle \right) \\ &\quad + \left[\|D\nabla u\|^2 + \|D\operatorname{div}u\|^2 + \|D\nabla\theta\|^2 \right] \\ &\lesssim \|D^2q\|^2 + \frac{1}{2}\|D^2u\|^2 + \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 + \|D^2u\|^2 + \|D\theta\|^2 + \|D^2\theta\|^2 \right). \end{aligned} \quad (2.19)$$

Proof. First, apply D^k on the Eqs. (2.2a)-(2.2c), multiply the resulting equations by $D^k\rho, D^ku, D^k\theta$ respectively and sum up, then integrate over \mathbb{R}^3 . A similar process as in the estimates (2.4) and (2.5), one has

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|(\rho, u, \theta)\|^2 + \langle \theta, \operatorname{div} q \rangle \right) + \|\nabla u\|^2 + \|\operatorname{div} u\|^2 \\ & \lesssim \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 + \|D\theta\|^2 \right), \end{aligned} \tag{2.20}$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|D(\rho, u, \theta)\|^2 + \langle D\theta, D\operatorname{div} q \rangle \right) + \|D\nabla u\|^2 + \|D\operatorname{div} u\|^2 \\ & \lesssim \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 + \|D^2u\|^2 + \|D\theta\|^2 \right). \end{aligned} \tag{2.21}$$

The above underlined terms can be canceled by using the following two estimates. In fact, applying D^k on the Eq. (2.2d), and multiplying the resulting equation by $D^k\nabla\theta$, one has

$$\begin{aligned} 0 &= \langle D^k(q_t + q + \nabla\theta), \nabla\theta \rangle \\ &= \|D^k\nabla\theta\|^2 + \langle \partial_t D^k q, D^k\nabla\theta \rangle + \langle D^k q, D^k\nabla\theta \rangle \\ &= \|D^k\nabla\theta\|^2 + \partial_t \langle D^k q, D^k\nabla\theta \rangle - \langle D^k q, D^k\nabla\theta_t \rangle + \langle D^k q, D^k\nabla\theta \rangle \\ &= \|D^k\nabla\theta\|^2 + \partial_t \langle D^k q, D^k\nabla\theta \rangle + \langle D^k \operatorname{div} q, D^k\theta_t \rangle + \langle D^k q, D^k\nabla\theta \rangle. \end{aligned} \tag{2.22}$$

The third term above can be estimated by using the Eq. (2.2c) as

$$\begin{aligned} |\langle \operatorname{div} q, \theta_t \rangle| &= \left| \left\langle \operatorname{div} q, \left(-\operatorname{div} u - \operatorname{div} q - u \cdot \nabla\theta - \frac{1}{1+\rho} \right. \right. \right. \\ & \quad \left. \left. \left. \times \left(\rho \operatorname{div} u + \theta \operatorname{div} u + \rho \operatorname{div} q - \frac{1}{2} |\nabla u + (\nabla u)^T|^2 - |\operatorname{div} u|^2 \right) \right) \right\rangle \right| \\ & \lesssim \|Dq\|^2 + \|Du\|^2, \end{aligned} \tag{2.23}$$

$$\begin{aligned} |\langle D\operatorname{div} q, D\theta_t \rangle| &= \left| \left\langle D\operatorname{div} q, \left[-D\operatorname{div} u - D\operatorname{div} q - D(u \cdot \nabla\theta) - D \left(\frac{\rho}{1+\rho} \operatorname{div} u \right) \right. \right. \right. \\ & \quad \left. \left. \left. + D(\theta \operatorname{div} u) + D \left(\rho \operatorname{div} q - \frac{1}{2} |\nabla u + (\nabla u)^T|^2 \right) - D(|\operatorname{div} u|^2) \right] \right\rangle \right| \\ & \lesssim \|D^2q\|^2 + \|D^2u\|^2 + \epsilon_0 \|D^2\theta\|^2, \end{aligned} \tag{2.24}$$

where we have used the estimate (2.3), the Hölder inequality and the Cauchy

inequality. Summing up (2.23) and (2.24) into (2.22), we have

$$\|\nabla\theta\|^2 + \partial_t \langle q, \nabla\theta \rangle + \langle q, \nabla\theta \rangle \lesssim \|Dq\|^2 + \|Du\|^2, \quad (2.25)$$

$$\|D\nabla\theta\|^2 + \partial_t \langle Dq, D\nabla\theta \rangle + \langle Dq, D\nabla\theta \rangle \lesssim \|D^2q\|^2 + \|D^2u\|^2 + \epsilon_0 \|D^2\theta\|^2. \quad (2.26)$$

Then summing up (2.25) and (2.20), and summing up (2.26) and (2.21), we have

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left(|\rho|^2 + |u|^2 + |\theta|^2 + q\nabla\theta \right) + \left[\|\nabla u\|^2 + \|\operatorname{div}u\|^2 + \|\nabla\theta\|^2 \right] \\ & \lesssim \|Dq\|^2 + \|Du\|^2 + \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 + \|D\theta\|^2 \right), \end{aligned} \quad (2.27)$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int \left(|D\rho|^2 + |Du|^2 + |D\theta|^2 + DqD\nabla\theta \right) + \left[\|D\nabla u\|^2 + \|D\operatorname{div}u\|^2 + \|D\nabla\theta\|^2 \right] \\ & \lesssim \|D^2q\|^2 + \|D^2u\|^2 + \epsilon_0 \left(\|D\rho\|^2 + \|Du\|^2 + \|D^2u\|^2 + \|D\theta\|^2 + \|D^2\theta\|^2 \right). \end{aligned} \quad (2.28)$$

This completes the proof of this lemma. \square

Next, in the same way as in Lemma 2.2-Lemma 2.4, one can easily deduce the following energy estimate for the second-order and the third-order of derivatives of the solution.

Lemma 2.5. *If (2.3) holds, then we have*

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|D^2(\rho, u, \theta, q)\|^2 \right) + \|D^2(\nabla u, \operatorname{div}u, q)\|^2 \\ & \lesssim \epsilon_0 \left(\|D^3(\rho, u, \theta)\|^2 + \|D^2q\|^2 \right), \end{aligned} \quad (2.29)$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|D^3(\rho, u, \theta)\|^2 + \int (1+\rho) |D^3q|^2 \right) + \left[\|D^3(\nabla u, \operatorname{div}u)\|^2 + \|D^3q\|^2 \right] \\ & \lesssim \epsilon_0 \left(\|D^3\rho\|^2 + \|D^4u\|^2 + \|D^3\theta\|^2 + \|D^3q\|^2 \right), \end{aligned} \quad (2.30)$$

$$\frac{d}{dt} \langle D^2u, D^2\nabla\rho \rangle + \frac{1}{2} \|D^2\nabla\rho\|^2 \lesssim \|D^3u\|^2 + \|D^4u\|^2 + \|D^3\theta\|^2, \quad (2.31)$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \left(\|D^2(\rho, u, \theta)\|^2 + \langle D^2q, D^2\nabla\theta \rangle \right) + \left[\|D^2(\nabla u, \operatorname{div}u)\|^2 + \|D^2\nabla\theta\|^2 \right] \\ & \lesssim \|D^3q\|^2 + \frac{1}{2} \|D^3u\|^2 + \epsilon_0 \left(\|D^2\rho\|^2 + \|D^2u\|^2 + \|D^3u\|^2 + \|D^2\theta\|^2 + \|D^3\theta\|^2 \right). \end{aligned} \quad (2.32)$$

From these delicate energy estimates, we can obtain the following energy functionals.

Proposition 2.1. *If (2.3) holds, we have for some constant $C > 0$ that*

$$\begin{aligned} & \frac{d}{dt} \left(\|(\rho, u, \theta, q)\|_{H^1}^2 \right) \\ & + C \left(\|D\rho\|^2 + \|Du\|_{H^1}^2 + \|D\theta\|^2 + \|q\|_{H^1}^2 \right) \leq 0, \end{aligned} \tag{2.33}$$

$$\begin{aligned} & \frac{d}{dt} \left(\|(D\rho, Du, D\theta, Dq)\|_{H^1}^2 \right) \\ & + C \left(\|D^2\rho\|^2 + \|D^2u\|_{H^1}^2 + \|D^2\theta\|^2 + \|Dq\|_{H^1}^2 \right) \leq 0, \end{aligned} \tag{2.34}$$

$$\begin{aligned} & \frac{d}{dt} \left(\|(D^2\rho, D^2u, D^2\theta, D^2q)\|_{H^1}^2 \right) \\ & + C \left(\|D^3\rho\|^2 + \|D^3u\|_{H^1}^2 + \|D^3\theta\|^2 + \|D^2q\|_{H^1}^2 \right) \leq 0. \end{aligned} \tag{2.35}$$

Proof. Summing up (2.4), (2.5), $(1/2) \times (2.13)$ and $(1/2) \times (2.18)$, we can immediately get

$$\begin{aligned} & \frac{d}{dt} \left(\|(\rho, u, \theta, q)\|_{H^1}^2 + \langle u, \nabla \rho \rangle + \langle q, \nabla \theta \rangle \right) \\ & + C \left(\|D\rho\|^2 + \|Du\|_{H^1}^2 + \|D\theta\|^2 + \|q\|_{H^1}^2 \right) \leq 0, \end{aligned} \tag{2.36}$$

which is equivalent to (2.33), since

$$|\langle Du, D\nabla \rho \rangle + \langle Dq, D\nabla \theta \rangle| \leq \frac{1}{2} \left(\|Du\|^2 + \|D^2\rho\|^2 + \|Dq\|^2 + \|D^2\theta\|^2 \right).$$

In the same way, summing up (2.7), (2.8), $(1/2) \times (2.13)$ and $(1/2) \times (2.19)$, we can obtain (2.34). At last, (2.35) can be deduced by using Lemma 2.5. \square

2.2 Estimates in $\dot{H}^{-s}(\mathbb{R}^3)$

The following lemma plays a key role in the proof of Theorem 1.1. It shows an energy estimate of the solutions in the negative Sobolev space $\dot{H}^{-s}(\mathbb{R}^3)$. Namely, we have

Lemma 2.6. *If (2.3) holds, for $s \in (0, 1/2]$, we have*

$$\begin{aligned} & \frac{d}{dt} \left(\|\Lambda^{-s}(\rho, u, \theta, q)\|^2 + \|\nabla \Lambda^{-s}u\|^2 + \|\operatorname{div} \Lambda^{-s}u\|^2 + \|\Lambda^{-s}q\|^2 \right) \\ & \lesssim \left(\|D\rho\|_{H^1}^2 + \|Du\|_{H^2}^2 + \|D\theta\|^2 + \|Dq\|^2 \right) \|\Lambda^{-s}(\rho, u, \theta, q)\|, \end{aligned} \tag{2.37}$$

and for $s \in (1/2, 3/2)$, we have

$$\begin{aligned} & \frac{d}{dt} \|\Lambda^{-s}(\rho, u, \theta, q)\|^2 + \|\nabla \Lambda^{-s}u\|^2 + \|\operatorname{div} \Lambda^{-s}u\|^2 + \|\Lambda^{-s}q\|^2 \\ & \lesssim \|(\rho, u, \theta)\|^{s-\frac{1}{2}} \left(\|D(\rho, \theta, q)\|^{\frac{5}{2}-s} + \|D^2u\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s}(\rho, u, \theta, q)\|. \end{aligned} \tag{2.38}$$

Proof. Applying Λ^{-s} to the Eq. (2.2), and multiplying the resulting identity by $\Lambda^{-s}\rho, \Lambda^{-s}u, \Lambda^{-s}\theta$ and $\Lambda^{-s}q$, respectively, and integrating over \mathbb{R}^3 by parts, we get

$$\begin{aligned} & \frac{d}{dt} \|\Lambda^{-s}(\rho, u, \theta, q)\|^2 + \|\nabla \Lambda^{-s}u\|^2 + \|\operatorname{div} \Lambda^{-s}u\|^2 + \|\Lambda^{-s}q\|^2 \\ & = \langle \Lambda^{-s}(-\rho \operatorname{div} u - u \cdot \nabla \rho), \Lambda^{-s}\rho \rangle \\ & \quad + \left\langle \Lambda^{-s} \left(-u \cdot \nabla u + \frac{\rho}{1+\rho} \Delta u + \frac{\rho}{1+\rho} \nabla \operatorname{div} u - \frac{\rho}{1+\rho} \nabla \theta - \frac{\rho}{1+\rho} \nabla \rho \right), \Lambda^{-s}u \right\rangle \\ & \quad + \left\langle \Lambda^{-s} \left(-u \cdot \nabla \theta - \frac{\rho \operatorname{div} u}{1+\rho} - \frac{\theta \operatorname{div} u}{1+\rho} - \frac{\rho \operatorname{div} q}{1+\rho} + \frac{|\nabla u + (\nabla u)^T|^2}{1+\rho} \right. \right. \\ & \quad \left. \left. + \frac{|\operatorname{div} u|^2}{1+\rho} \right), \Lambda^{-s}\theta \right\rangle. \end{aligned} \tag{2.39}$$

If $s \in (0, 1/2]$, then by Lemmas A.1, A.3 and Young's inequality, the first term in the right-hand side of (2.37) can be estimated as follows:

$$\begin{aligned} & \|\langle \Lambda^{-s}(\rho \operatorname{div} u), \Lambda^{-s}\rho \rangle\| \\ & \lesssim \|\Lambda^{-s}(\rho \operatorname{div} u)\| \|(\rho \operatorname{div} u)\rho\| \lesssim \|\rho \operatorname{div} u\|_{L^{\frac{1}{1/2+s/3}}} \|\Lambda^{-s}\rho\| \\ & \lesssim \|\rho\|_{L^{\frac{3}{s}}} \|Du\| \|\Lambda^{-s}\rho\| \lesssim \|D\rho\|^{\frac{1}{2}-s} \|D^2\rho\|^{\frac{1}{2}+s} \|Du\| \|\Lambda^{-s}\rho\| \\ & \lesssim \left(\|D\rho\|_{H^1}^2 + \|Du\|^2 \right) \|\Lambda^{-s}\rho\|, \end{aligned} \tag{2.40}$$

where we used the facts $1/2 + s/3 < 1$ and $3/s \geq 6$. Similarly, it holds that

$$|\langle \Lambda^{-s}(u \cdot \nabla \rho), \Lambda^{-s}\rho \rangle| \lesssim \left(\|Du\|_{H^1}^2 + \|D\rho\|^2 \right) \|\Lambda^{-s}\rho\|, \tag{2.41a}$$

$$\begin{aligned} & \left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \Delta u \right), \Lambda^{-s}u \right\rangle \right| + \left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \nabla \operatorname{div} u \right), \Lambda^{-s}u \right\rangle \right| \\ & \lesssim \|Du\|_{H^1}^2 \|\Lambda^{-s}u\|, \end{aligned} \tag{2.41b}$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \nabla \theta \right), \Lambda^{-s}u \right\rangle \right| \lesssim \left(\|D\rho\|_{H^1}^2 + \|D\theta\|^2 \right) \|\Lambda^{-s}u\|, \tag{2.41c}$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \nabla \rho \right), \Lambda^{-s} u \right\rangle \right| \lesssim \left(\|D\rho\|_{H^1}^2 + \|D\rho\|^2 \right) \|\Lambda^{-s} u\|, \tag{2.41d}$$

$$|\langle \Lambda^{-s}(u \cdot \nabla \theta), \Lambda^{-s} \theta \rangle| \lesssim \left(\|Du\|_{H^1}^2 + \|D\theta\|^2 \right) \|\Lambda^{-s} \theta\|, \tag{2.41e}$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \operatorname{div} u \right), \Lambda^{-s} \theta \right\rangle \right| \lesssim \left(\|D\rho\|_{H^1}^2 + \|Du\|^2 \right) \|\Lambda^{-s} \theta\|, \tag{2.41f}$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\theta}{1+\rho} \operatorname{div} u \right), \Lambda^{-s} \theta \right\rangle \right| \lesssim \left(\|D\theta\|_{H^1}^2 + \|Du\|^2 \right) \|\Lambda^{-s} \theta\|, \tag{2.41g}$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \operatorname{div} q \right), \Lambda^{-s} \theta \right\rangle \right| \lesssim \left(\|D\rho\|_{H^1}^2 + \|Dq\|^2 \right) \|\Lambda^{-s} \theta\|, \tag{2.41h}$$

$$\begin{aligned} & \left| \left\langle \Lambda^{-s} \left(\frac{|\nabla u + (\nabla u)^T|^2}{1+\rho} + \frac{|\operatorname{div} u|^2}{1+\rho} \right), \Lambda^{-s} \theta \right\rangle \right| \\ & \lesssim \left(\|D^2 u\|_{H^1}^2 + \|Du\|^2 \right) \|\Lambda^{-s} \theta\|. \end{aligned} \tag{2.41i}$$

Combining (2.39)-(2.41), we can immediately obtain (2.37).

Now if $s \in (1/2, 3/2)$, then $1/2 + s/3 < 1$ and $2 < 3/s < 6$. We shall estimate the right-hand side of (2.39) in a different way. Using the Sobolev inequality and the Cauchy inequality, we have

$$\begin{aligned} & |\langle \Lambda^{-s}(\rho \operatorname{div} u), \Lambda^{-s} \rho \rangle| \\ & \lesssim \|\Lambda^{-s}(\rho \operatorname{div} u)\| \|\Lambda^{-s} \rho\| \lesssim \|\rho \operatorname{div} u\|_{L^{\frac{1}{1/2+s/3}}} \|\Lambda^{-s} \rho\| \\ & \lesssim \|\rho\|_{L^{\frac{3}{s}}} \|Du\| \|\Lambda^{-s} \rho\| \lesssim \|\rho\|^{s-\frac{1}{2}} \|D\rho\|^{\frac{3}{2}-s} \|Du\| \|\Lambda^{-s} \rho\| \\ & \lesssim \|\rho\|^{s-\frac{1}{2}} \left(\|D\rho\|^{\frac{5}{2}-s} + \|Du\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s} \rho\|, \end{aligned} \tag{2.42}$$

where we have used the facts $1/2 + s/3 < 1$ and $3/s \geq 6$. Similarly, it holds for $s \in (1/2, 3/2)$ that

$$|\langle \Lambda^{-s}(u \cdot \nabla \rho), \Lambda^{-s} \rho \rangle| \lesssim \|u\|^{s-\frac{1}{2}} \|D\rho\|^{\frac{5}{2}-s} \|\Lambda^{-s} \rho\|, \tag{2.43a}$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \Delta u \right), \Lambda^{-s} u \right\rangle \right| \lesssim \|u\|^{s-\frac{1}{2}} \|Du\|^{\frac{5}{2}-s} \|\Lambda^{-s} u\|, \tag{2.43b}$$

$$\begin{aligned} & \left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \nabla \operatorname{div} u \right), \Lambda^{-s} u \right\rangle \right| \\ & \lesssim \|\rho\|^{s-\frac{1}{2}} \left(\|D\rho\|^{\frac{5}{2}-s} + \|D^2 u\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s} u\|, \end{aligned} \tag{2.43c}$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \nabla \theta \right), \Lambda^{-s} u \right\rangle \right| \lesssim \|\rho\|^{s-\frac{1}{2}} \left(\|D\rho\|^{\frac{5}{2}-s} + \|D\theta\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s} u\|, \tag{2.43d}$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \nabla \rho \right), \Lambda^{-s} u \right\rangle \right| \lesssim \|\rho\|^{s-\frac{1}{2}} \|D\rho\|^{\frac{5}{2}-s} \|\Lambda^{-s} u\|, \quad (2.43e)$$

$$|\langle \Lambda^{-s} (u \cdot \nabla \theta), \Lambda^{-s} \theta \rangle| \lesssim \|u\|^{s-\frac{1}{2}} \left(\|D\theta\|^{\frac{5}{2}-s} + \|Du\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s} \theta\|, \quad (2.43f)$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \operatorname{div} u \right), \Lambda^{-s} \theta \right\rangle \right| \lesssim \|\rho\|^{s-\frac{1}{2}} \left(\|D\rho\|^{\frac{5}{2}-s} + \|Du\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s} \theta\|, \quad (2.43g)$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\theta}{1+\rho} \operatorname{div} u \right), \Lambda^{-s} \theta \right\rangle \right| \lesssim \|\theta\|^{s-\frac{1}{2}} \left(\|D\rho\|^{\frac{5}{2}-s} + \|Du\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s} \theta\|, \quad (2.43h)$$

$$\left| \left\langle \Lambda^{-s} \left(\frac{\rho}{1+\rho} \operatorname{div} q \right), \Lambda^{-s} \theta \right\rangle \right| \lesssim \|\rho\|^{s-\frac{1}{2}} \left(\|D\rho\|^{\frac{5}{2}-s} + \|Dq\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s} \theta\|, \quad (2.43i)$$

$$\begin{aligned} & \left| \left\langle \Lambda^{-s} \left(\frac{|\nabla u + (\nabla u)^T|^2}{1+\rho} + \frac{|\operatorname{div} u|^2}{1+\rho} \right), \Lambda^{-s} \theta \right\rangle \right| \\ & \lesssim \|Du\|^{s-\frac{1}{2}} \left(\|D^2 u\|^{\frac{5}{2}-s} + \|Du\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s} \theta\|. \end{aligned} \quad (2.43j)$$

Combining (2.39), (2.42) and (2.43), we can immediately obtain (2.38). \square

2.3 Estimates in $\dot{B}_{2,\infty}^{-s}(\mathbb{R}^3)$

In this subsection, we will derive the evolution of the negative Besov norms of the solutions.

Lemma 2.7. *If (2.3) holds, for $s \in (0, 1/2]$, we have*

$$\begin{aligned} & \frac{d}{dt} \|\dot{\Delta}_j(\rho, u, \theta, q)\|^2 + \left(\|\nabla \dot{\Delta}_j u\|^2 + \|\operatorname{div} \dot{\Delta}_j u\|^2 + \|\dot{\Delta}_j q\|^2 \right) \\ & \lesssim \left(\|D\rho\|_{H^1}^2 + \|Du\|_{H^2}^2 + \|D\theta\|^2 + \|Dq\|^2 \right) \|\dot{\Delta}_j(\rho, u, \theta, q)\|, \end{aligned} \quad (2.44)$$

and for $s \in (1/2, 3/2]$, we have

$$\begin{aligned} & \frac{d}{dt} \|\dot{\Delta}_j(\rho, u, \theta, q)\|^2 + \left(\|\nabla \dot{\Delta}_j u\|^2 + \|\operatorname{div} \dot{\Delta}_j u\|^2 + \|\dot{\Delta}_j q\|^2 \right) \\ & \lesssim \|(\rho, u, \theta)\|^{s-\frac{1}{2}} \left(\|D(\rho, \theta, q)\|^{\frac{5}{2}-s} + \|D^2 u\|^{\frac{5}{2}-s} \right) \|\dot{\Delta}_j(\rho, u, \theta, q)\|. \end{aligned} \quad (2.45)$$

Proof. Apply $\dot{\Delta}_j$ to the Eq. (2.2), and multiply the resulting identity by $\dot{\Delta}_j \rho$, $\dot{\Delta}_j u$, $\dot{\Delta}_j \theta$ and $\dot{\Delta}_j q$, respectively, and integrate over \mathbb{R}^3 by parts. Then, as the proof of Lemma 2.6, applying Lemma A.5 instead to estimate the $\dot{B}_{2,\infty}^{-s}$ norm, we complete the proof of Lemma 2.7. \square

3 Proof of Theorem 1.1

The global existence of the solution to the Cauchy problem (1.1)-(1.2) with small initial data in H^3 -norm can be deduced from the local existence in [13] and our energy estimates in Section 2. Since Hu and Racke [13] have obtained the global existence by using Kawashima condition, we omit the proof here. We mainly prove the optimal time decay rates of the unique global solution to the Cauchy problem (1.1)-(1.2) in Theorem 1.1 in this section.

First, from Lemma 2.6, we need to distinct the arguments by the value of s . For $s \in [0, 1/2]$, integrating (2.37) in time, and by using the energy functionals in Proposition 2.1, we have

$$\begin{aligned} \|(\rho, u, \theta, q)\|_{\dot{H}^{-s}}^2 &\leq \|(\rho_0, u_0, \theta_0, q_0)\|_{\dot{H}^{-s}}^2 + C \int_0^t \|D(\rho, u, \theta, q)\|_{H^1}^2 \|(\rho, u, \theta, q)\|_{\dot{H}^{-s}} d\tau \\ &\leq C_0 \left(1 + \sup_{0 \leq \tau \leq t} \|(\rho, u, \theta, q)\|_{\dot{H}^{-s}} \right), \end{aligned} \tag{3.1}$$

which yields

$$\|(\rho, u, \theta, q)\|_{\dot{H}^{-s}} \leq C_0 \quad \text{for } s \in [0, 1/2]. \tag{3.2}$$

Using Lemma 2.7, we similarly have

$$\|(\rho, u, \theta, q)\|_{\dot{B}_{2,\infty}^{-s}} \leq C_0 \quad \text{for } s \in (0, 1/2]. \tag{3.3}$$

If $0 \leq l \leq 2$, we may use Lemma A.2 to have

$$\|D^{l+1}f\|_{L^2} \geq C \|f\|_{\dot{H}^{-s}}^{-\frac{1}{l+s}} \|D^l f\|_{L^2}^{1+\frac{1}{l+s}}. \tag{3.4}$$

Recall the first energy functional in Proposition 2.1,

$$\frac{d}{dt} \left(\|(\rho, u, \theta, q)\|_{H^1}^2 \right) + C \left(\|D\rho\|^2 + \|Du\|_{H^1}^2 + \|D\theta\|^2 + \|q\|_{H^1}^2 \right) \leq 0. \tag{3.5}$$

We want to use (3.4) and (3.5) to deduce the decay rate when $s \in [0, 1/2]$. In fact, from the convexity of the function $g(a) = a^{1+1/(l+s)}$ with $a > 0$, we have the fact that

$$a^{1+\frac{1}{l+s}} + b^{1+\frac{1}{l+s}} \geq \frac{1}{2}(a+b)^{1+\frac{1}{l+s}}, \quad a > 0, \quad b > 0. \tag{3.6}$$

Then we can claim that

$$\left(\|D\rho\|^2 + \|Du\|_{H^1}^2 + \|D\theta\|^2 + \|q\|_{H^1}^2 \right) \geq C \left(\|(\rho, u, \theta, q)\|_{H^1}^2 \right)^{1+\frac{1}{l+s}}. \tag{3.7}$$

In fact, from (3.4) and (3.6) and the smallness of the solution in $H^3(\mathbb{R}^3)$, we have

$$\begin{aligned} \|Du\|_{H^1}^2 &\geq C(\|u\|_{H^1}^2)^{1+\frac{1}{s}}, \quad \|Dq\|_{H^1}^2 \geq C(\|q\|_{H^1}^2)^{1+\frac{1}{s}}, \quad \text{since } \|q\|_{H^1} \text{ is small, (3.8)} \\ \|D\rho\|^2 &= \frac{1}{2}\|D\rho\|^2 + \frac{1}{2}\|D\rho\|^2 \geq \frac{1}{2}\|D\rho\|^2 + \frac{1}{2}C_1^2(\|\rho\|^2)^{1+\frac{1}{s}} \\ &\geq \frac{1}{2}\min\{1, C_1^2\} \left[\|D\rho\|^2 + (\|\rho\|^2)^{1+\frac{1}{s}} \right] \\ &\geq \frac{1}{2}\min\{1, C_1^2\} \left[(\|D\rho\|^2)^{1+\frac{1}{s}} + (\|\rho\|^2)^{1+\frac{1}{s}} \right] \\ &\geq \frac{1}{4}\min\{1, C_1^2\} (\|\rho\|^2 + \|D\rho\|^2)^{1+\frac{1}{s}} \\ &= \frac{1}{4}\min\{1, C_1^2\} (\|\rho\|_{H^1}^2)^{1+\frac{1}{s}}. \end{aligned} \quad (3.9)$$

In the same way, we have

$$\|D\theta\|^2 \geq \frac{1}{4}\min\{1, C_1^2\} (\|\theta\|_{H^1}^2)^{1+\frac{1}{s}}. \quad (3.10)$$

Then using the fact (3.6) again, and from (3.8)-(3.10), we can get (3.7). Inserting (3.7) into (3.5), and using (3.2) and (3.3), we have a differential inequality on $\|(\rho, u, \theta, q)\|_{H^1}^2$,

$$\frac{d}{dt} \left(\|(\rho, u, \theta, q)\|_{H^1}^2 \right) + C \left(\|(\rho, u, \theta, q)\|_{H^1}^2 \right)^{1+\frac{1}{s}} \leq 0, \quad (3.11)$$

which implies for $s \in [0, 1/2]$ that

$$\|(\rho, u, \theta, q)\|_{H^1}^2 \leq C(1+t)^{-s}. \quad (3.12)$$

Similarly, from the energy functionals (2.34)-(2.35) for the first order and the second-order of derivatives of the solution, we also have

$$\begin{aligned} \|D(\rho, u, \theta, q)\|_{H^1}^2 &\leq C(1+t)^{-(1+s)} \quad \text{for } s \in [0, 1/2], \\ \|D^2(\rho, u, \theta, q)\|_{H^1}^2 &\leq C(1+t)^{-(2+s)} \quad \text{for } s \in [0, 1/2]. \end{aligned} \quad (3.13)$$

For $s \in (1/2, 3/2)$, notice that the arguments for the case $s \in [0, 1/2]$ can not be applied to this case (see Lemma 2.6). We will use (3.12)-(3.13) with $s = 1/2$ to

derive the decay rate for $s \in (1/2, 3/2)$. Integrating (2.38) for $s \in (1/2, 3/2)$ in time, we have

$$\begin{aligned}
& \|\Lambda^{-s}(\rho, u, \theta, q)\|^2 \leq \|\Lambda^{-s}(\rho_0, u_0, \theta_0, q_0)\|^2 \\
& \quad + \int_0^t \|(\rho, u, \theta)\|^{s-\frac{1}{2}} \left(\|D(\rho, \theta, q)\|^{\frac{5}{2}-s} + \|D^2 u\|^{\frac{5}{2}-s} \right) \|\Lambda^{-s}(\rho, u, \theta, q)\| d\tau \\
& \lesssim C_0 + \sup_{0 \leq \tau \leq t} \left\{ \|\Lambda^{-s}(\rho, u, \theta, q)\| \right\} \int_0^t (1+\tau)^{-\frac{1}{4}(s-\frac{1}{2})} (1+\tau)^{-\frac{3}{4}(\frac{5}{2}-s)} d\tau \\
& \lesssim C_0 + \sup_{0 \leq \tau \leq t} \left\{ \|\Lambda^{-s}(\rho, u, \theta, q)\| \right\} \int_0^t (1+\tau)^{-(\frac{7}{4}-\frac{s}{2})} d\tau \\
& \lesssim C_0 + C_1 \sup_{0 \leq \tau \leq t} \left\{ \|\Lambda^{-s}(\rho, u, \theta, q)\| \right\}, \tag{3.14}
\end{aligned}$$

which implies that

$$\|(\rho, u, \theta, q)\|_{\dot{H}^{-s}} \leq CC_0, \quad s \in [0, 3/2]. \tag{3.15}$$

On the other hand, by using Lemmas A.4-A.7, a similar argument as leading to the estimate (3.15) for the negative Sobolev space can immediately yields that in the negative Besov space

$$\|(\rho, u, \theta, q)\|_{\dot{B}_{2,\infty}^{-s}} \leq CC_0, \quad s \in (0, 3/2]. \tag{3.16}$$

Then from (3.15)-(3.16), we may repeat the arguments leading to (3.12)-(3.13) for $s \in [0, 1/2]$ to prove that it also holds for $s \in (1/2, 3/2)$ when the initial data is in negative Sobolev space, and for $s \in (1/2, 3/2]$ when the initial data is in negative Besov space, that is,

$$\|D^k(\rho, u, \theta, q)\| \lesssim (1+t)^{-\frac{k+s}{2}}, \quad k=0,1,2. \tag{3.17}$$

Finally, we further improve the decay rate for q and Dq due to the damping mechanism of the Eq. (2.2d). In fact, multiplying (2.2d) by q and integrating over \mathbb{R}^3 , we have

$$\frac{d}{dt} \|q\|^2 + \|q\|^2 = -\langle q, \nabla \theta \rangle \leq C \|\nabla \theta\|^2 + \frac{1}{2} \|q\|^2, \tag{3.18}$$

which implies that

$$\frac{d}{dt} \|q\|^2 + \frac{1}{2} \|q\|^2 \leq C \|\nabla \theta\|^2. \tag{3.19}$$

Integrating (3.19) on time t , we have

$$\begin{aligned} \|q\|^2 &\lesssim e^{-\frac{t}{2}} \|q_0\|^2 + \int_0^t e^{-\frac{t-\tau}{2}} \|D\theta\|^2(\tau) d\tau \\ &\lesssim e^{-\frac{t}{2}} + \int_0^t e^{-\frac{t-\tau}{2}} (1+\tau)^{-(1+s)} d\tau \lesssim e^{-\frac{t}{2}} + (1+t)^{-(1+s)}, \end{aligned} \quad (3.20)$$

that is,

$$\|q\| \lesssim (1+t)^{-\frac{1+s}{2}}. \quad (3.21)$$

Similarly, we also have

$$\|Dq\| \lesssim (1+t)^{-\frac{2+s}{2}}. \quad (3.22)$$

This completes the proof of Theorem 1.1. \square

Appendix A

In this section we give some Sobolev inequalities and Besov inequalities, which have been used in the above sections.

Lemma A.1 (Gagliardo-Nirenberg inequality). *Let $0 \leq m, k \leq l$, then we have*

$$\|D^k g\|_{L^p} \leq C \|D^m g\|_{L^q}^{1-\alpha} \|D^l g\|_{L^r}^\alpha,$$

where k satisfies

$$\frac{1}{p} - \frac{k}{n} = (1-\alpha) \left(\frac{1}{q} - \frac{m}{n} \right) + \alpha \left(\frac{1}{r} - \frac{l}{n} \right).$$

Lemma A.2 ([9, Lemma A.5]). *Let $s \geq 0$ and $l \geq 0$, then we have*

$$\|D^l g\|_{L^2} \leq C \|D^{l+1} g\|_{L^2}^{1-\alpha} \|g\|_{\dot{H}^{-s}}^\alpha, \quad \alpha = \frac{1}{l+s+1}.$$

Lemma A.3 ([28, Theorem 1]). *Let $0 < s < n, 1 < p < q < \infty, 1/q + s/n = 1/p$, then*

$$\|\Lambda^{-s} g\|_{L^q} \leq C \|g\|_{L^p}.$$

Next, we give some lemmas on Besov space $\dot{B}_{2,\infty}^{-s}$.

Lemma A.4 ([27, Lemma 4.5]). *Suppose $k \geq 0$ and $s > 0$, then we have*

$$\|D^k f\|_{L^2} \leq C \|D^{k+1} f\|_{L^2}^{1-\alpha} \|f\|_{\dot{B}_{2,\infty}^{-s}}^\alpha, \quad \alpha = \frac{1}{l+1+s}.$$

Lemma A.5 ([27, Lemma 4.6]). *Suppose that $s > 0$ and $1 \leq p < 2$. We have the embedding $L^p \subset \dot{B}_{q,\infty}^{-s}$ with $1/2 + s/3 = 1/p$. In particular, we have*

$$\|f\|_{\dot{B}_{2,\infty}^{-s}} \leq C \|f\|_{L^p}.$$

Lemma A.6 ([29, Lemma A.7]). *If $1 \leq r_1 \leq r_2 \leq \infty$, then*

$$\dot{B}_{2,r_1}^{-s} \subset \dot{B}_{2,r_2}^{-s}.$$

Lemma A.7 ([29, Lemma A.8]). *If $m > l \geq k$ and $1 \leq p \leq q \leq r \leq \infty$, then we have*

$$\|g\|_{\dot{B}_{2,q}^l} \leq C \|g\|_{\dot{B}_{2,r}^k}^\alpha \|g\|_{\dot{B}_{2,p}^m}^{1-\alpha},$$

where $l = k\alpha + m(1 - \alpha)$, $1/q = \alpha/r + 1 - \alpha/p$.

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