

The Euler Limit of the Relativistic Boltzmann Equation

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Abstract. In this work we prove the existence and uniqueness theorems of the solutions to the relativistic Boltzmann equation for analytic initial fluctuations on a time interval independent of the Knudsen number $\epsilon > 0$. As $\epsilon \rightarrow 0$, we prove that the solution of the relativistic Boltzmann equation tends to the local relativistic Maxwellian, whose fluid-dynamical parameters solve the relativistic Euler equations and the convergence rate is also obtained. Due to this convergence rate, the Hilbert expansion is verified in the short time interval for the relativistic Boltzmann equation. We also consider the physically important initial layer problem. As a by-product, an existence theorem for the relativistic Euler equations without the assumption of the non-vacuum fluid states is obtained.

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1 Introduction

The relativistic Boltzmann equation, which is a fundamental model describing the motion of fast moving particles in kinetic theory, takes the form of

$$P \otimes \partial_X F = -\frac{1}{\epsilon} \mathcal{C}(F, F). \quad (1.1)$$

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Here \otimes represents the Lorentz inner product $(+---)$ of 4-vectors. As is customary we write $X = (x_0, x)$ with $x \in \mathbb{R}^3$ and $x_0 = -t$, and $P = (p_0, p)$ with momentum $p \in \mathbb{R}^3$ and energy $p_0 = \sqrt{c^2 + |p|^2}$, where c denotes the speed of light. For convenience of presentation, we rewrite (1.1) as

$$\partial_t F + \hat{p} \cdot \nabla_x F = \frac{1}{\epsilon} Q(F, F) \tag{1.2}$$

with $Q(F, F) = C(F, F) / p_0$, where the unknown $F = F^\epsilon(t, x, p)$ stands for the density distribution function of time $t \geq 0$, space $x \in \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$ and the dimensionless parameter ϵ is the Knudsen number, which is the ratio of the particle mean free path to a characteristic physical length scale. Here the dot represents the standard Euclidean dot product, and the normalized velocity of a particle is denoted as

$$\hat{p} = c \frac{p}{p_0} = \frac{p}{\sqrt{1 + |p|^2/c^2}}.$$

For notational simplicity we normalize all the physical constants to be one. Then

$$p_0 = \sqrt{1 + |p|^2}, \quad \hat{p} = \frac{p}{p_0}. \tag{1.3}$$

We rewrite (1.2) supplemented with initial data as

$$\partial_t F + \hat{p} \cdot \nabla_x F = \frac{1}{\epsilon} Q(F, F), \quad F(0, x, p) = F_0(x, p). \tag{1.4}$$

To describe the relativistic Boltzmann collision term, we introduce the relative momentum g as

$$g = g(p, q) = \sqrt{2(p_0 q_0 - p \cdot q - 1)} \tag{1.5}$$

and also the quantity s as

$$s = s(p, q) = g^2 + 4 = 2(p_0 q_0 - p \cdot q + 1). \tag{1.6}$$

Note that $s = g^2 + 4$ and this may differ from that in [19] by a constant factor. The Møller velocity is given by

$$v_\phi = v_\phi(p, q) = \sqrt{\left| \frac{p}{p_0} - \frac{q}{q_0} \right|^2 - \left| \frac{p}{p_0} \times \frac{q}{q_0} \right|^2} = \frac{g\sqrt{s}}{2p_0 q_0}. \tag{1.7}$$

Then we may express the collision operator $Q(F, G)$ in the form (see [11, 14, 19])

$$Q(F, G) = \int_{\mathbb{R}^3} \int_{S^2} v_\phi \sigma(g, \theta) [F(p')G(q') - F(p)G(q)] dq d\omega, \tag{1.8}$$

where $d\omega$ is the surface measure on the unit sphere S^2 in \mathbb{R}^3 , and $\sigma(g, \theta)$ is the scattering kernel. As is standard, we abbreviate $F(t, x, p)$ by $F(p)$, etc., and use primes to represent the results of collisions. The conservation of momentum and energy is

$$p' + q' = p + q, \quad (1.9)$$

$$\sqrt{1 + |p'|^2} + \sqrt{1 + |q'|^2} = \sqrt{1 + |p|^2} + \sqrt{1 + |q|^2} \quad (1.10)$$

for any $p, q \in \mathbb{R}^3$. Finally, the scattering angle θ is defined as follows. Given 4-vectors $P = (p_0, p)$ and $Q = (q_0, q)$, with the Lorentz inner product, the angle θ is given by

$$\cos\theta = \frac{(P - Q) \otimes (P' - Q')}{(P - Q) \otimes (P - Q)}.$$

Here $q_0 = \sqrt{1 + |q|^2}$ and $q'_0 = \sqrt{1 + |q'|^2}$. As in [42], by (1.9) and (1.10), the post-collisional momentum can be written

$$\begin{cases} p' = \frac{p+q}{2} + \frac{g}{2} \left(\omega + (\varrho - 1)(p+q) \frac{(p+q) \cdot \omega}{|p+q|^2} \right), \\ q' = \frac{p+q}{2} - \frac{g}{2} \left(\omega + (\varrho - 1)(p+q) \frac{(p+q) \cdot \omega}{|p+q|^2} \right), \end{cases} \quad (1.11)$$

where $\varrho = (p_0 + q_0) / \sqrt{s}$. The energies are then

$$\begin{cases} p'_0 = \frac{p_0 + q_0}{2} + \frac{g}{2\sqrt{s}} \omega \cdot (p+q), \\ q'_0 = \frac{p_0 + q_0}{2} - \frac{g}{2\sqrt{s}} \omega \cdot (p+q). \end{cases}$$

This and (1.11) clearly satisfy (1.9) and (1.10). Given any function $n = n(t, x)$, $\theta = \theta(t, x)$ and $u = u(t, x)$, we define the corresponding Maxwellian $J = J(n, \theta, u; p)$ as follows:

$$J = J(n, \theta, u; p) =: \frac{nz}{4\pi K_2(z)} \exp\left(-z \left(\sqrt{(1 + |u|^2)(1 + |p|^2)} - u \cdot p\right)\right), \quad z = \frac{1}{\theta}. \quad (1.12)$$

Here $K_2(z)$ is the Bessel function defined in (1.18). It can be shown (see e.g. [11, Chapter 2] and [43]) that

$$\mathcal{Q}(F, F) \equiv 0 \Leftrightarrow F \text{ is a Maxwellian of the form } J(n, \theta, u; p). \quad (1.13)$$

As the classical version, if (n, θ, u) are constants in x and t , a Maxwellian J is called a global relativistic Maxwellian. If they depend on x and t , a Maxwellian J is called a local relativistic Maxwellian.

We now state the conditions on the collisional cross-section as in [14, 19, 42].

Hypothesis on the collision kernel. We assume the collision kernel $\sigma(g, \theta)$ in (1.8) satisfies the following growth/decay estimates:

$$c_1 \frac{g^{a+1}}{1+g} \sin^Y \theta \leq \sigma(g, \theta) \leq c_2 (g^a + g^{-b}) \sin^Y \theta, \tag{1.14}$$

where c_1 and c_2 are positive constants, $b \in [0, 1/2)$, $a \in [0, 2-2b)$, and either $Y \geq 0$ or

$$|Y| < \min \left\{ 2-a, \frac{1}{2}-b, \frac{1}{3}(2-2b-a) \right\}.$$

In the present paper we consider the asymptotic problem for the nonlinear relativistic Boltzmann equation (1.4) as $\epsilon \rightarrow 0$, at the level of the relativistic Euler equation. To find the corresponding reduced problem, suppose $F = F^\epsilon$ has a limit F^0 and $\epsilon(\partial_t F^\epsilon + \hat{p} \cdot \nabla_x F^\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. Then letting ϵ tend to 0, we find

$$Q(F^0, F^0) = 0. \tag{1.15}$$

By (1.12) and (1.13), we have that $F^0 = F^0(n, \theta, u; p)$ is the local relativistic Maxwellians (1.12). We define the collision invariants

$$h_0(p) = 1, \quad h_j(p) = p_j, \quad 1 \leq j \leq 3, \quad h_4(p) = p_0.$$

For a smooth function $F(p)$, the collision operator satisfies

$$\langle h_j, Q(F, F) \rangle = 0, \quad 0 \leq j \leq 4. \tag{1.16}$$

Here the sign $\langle \cdot, \cdot \rangle$ is the standard $L^2(\mathbb{R}_p^2)$ inner product. We have from (1.4) and (1.16) that

$$\frac{\partial}{\partial t} \langle h_j, F^\epsilon \rangle + \langle h_j, \hat{p} \cdot \nabla_x F^\epsilon \rangle = 0, \quad 0 \leq j \leq 4. \tag{1.17}$$

Passing to the limit as $\epsilon \rightarrow 0$ and using (1.12) and (1.17), we can arrive at the relativistic Euler equations. Then we shall deduce the relativistic Euler equations as in [3, 35, 43].

As in [11, Chapter 2], the Bessel function $K_j(z)$ is defined by

$$K_j(z) = \frac{(2j)j!}{(2j)!} \frac{1}{z^j} \int_z^\infty e^{-\lambda} (\lambda^2 - z^2)^{j-\frac{1}{2}} d\lambda, \quad j \geq 0. \tag{1.18}$$

We have the following properties for the Bessel function in [11, Chapter 2]:

$$K_{j+1}(z) = \frac{2j}{z}K_j(z) + K_{j-1}(z),$$

$$K_j(z) = \frac{2^{j-1}(j-1)!}{(2j-2)!} \frac{1}{z^j} \int_z^\infty \lambda e^{-\lambda} (\lambda^2 - z^2)^{j-\frac{3}{2}} d\lambda, \quad j \geq 1.$$

By using this, (1.12), (1.18) and the direct calculations, one has

$$\mathcal{P} = \frac{1}{3} \int_{\mathbb{R}^3} |p|^2 J(n, \theta, 0; p) \frac{dp}{p_0} = \frac{n}{z},$$

$$\mathcal{E} = \int_{\mathbb{R}^3} p_0 J(n, \theta, 0; p) dp = n \left(\frac{3}{z} + \frac{K_1(z)}{K_2(z)} \right). \quad (1.19)$$

By these we define as

$$n\Phi(z) = \mathcal{E} + \mathcal{P} = n \left(\frac{3}{z} + \frac{K_1(z)}{K_2(z)} \right) + \frac{n}{z}. \quad (1.20)$$

Thanks to the classical decomposition [3, 11, 43], we have

$$T_{ij}(t, x) = \int_{\mathbb{R}^3} p_i p_j J(n, \theta, u; p) \frac{dp}{p_0} = -\mathcal{P} g_{ij} + (\mathcal{E} + \mathcal{P}) u_i u_j, \quad 0 \leq i, j \leq 3. \quad (1.21)$$

Here $u_0 = \sqrt{1 + |u|^2}$ and g_{ij} is the metric tensor given by

$$\begin{aligned} g_{00} &= 1, \\ g_{ii} &= -1, \quad \text{if } i = 1, 2, 3, \\ g_{ij} &= 0, \quad \text{if } i \neq j. \end{aligned} \quad (1.22)$$

Thus we have from (1.19)-(1.21) and this that

$$\begin{cases} T_{00}(t, x) = \int_{\mathbb{R}^3} p_0 p_0 J(n, \theta, u; p) \frac{dp}{p_0} = -\mathcal{P} + n\Phi(z)(1 + |u|^2), \\ T_{j0}(t, x) = \int_{\mathbb{R}^3} p_j p_0 J(n, \theta, u; p) \frac{dp}{p_0} = n\Phi(z) u_j u_0, & 1 \leq j \leq 3, \\ T_{ij}(t, x) = \int_{\mathbb{R}^3} p_i p_j J(n, \theta, u; p) \frac{dp}{p_0} = -\mathcal{P} g_{ij} + n\Phi(z) u_i u_j, & 1 \leq i, j \leq 3. \end{cases} \quad (1.23)$$

We define some quantities as

$$nu_j(t, x) = \int_{\mathbb{R}^3} p_j J(n, \theta, u; p) \frac{dp}{p_0}, \quad 0 \leq j \leq 3.$$

By using this, (1.17) and (1.23), we deduce the relativistic Euler equations as

$$\begin{cases} \frac{\partial}{\partial t}(n\sqrt{1+|u|^2}) + \sum_{j=1}^3 \frac{\partial(nu_j)}{\partial x_j} = 0, \\ \frac{\partial}{\partial t}(n\Phi(z)u_i\sqrt{1+|u|^2}) + \sum_{j=1}^3 \frac{\partial}{\partial x_j}(-\mathcal{P}g_{ij} + n\Phi(z)u_ju_i) = 0, \quad 1 \leq i \leq 3, \\ \frac{\partial}{\partial t}(-\mathcal{P} + n\Phi(z)(1+|u|^2)) + \sum_{j=1}^3 \frac{\partial}{\partial x_j}(n\Phi(z)u_j\sqrt{1+|u|^2}) = 0, \end{cases} \quad (1.24)$$

supplemented by the equations

$$n\Phi(z) = n\left(\frac{3}{z} + \frac{K_1(z)}{K_2(z)}\right) + \frac{n}{z}, \quad \mathcal{P} = \frac{n}{z}, \quad z = \frac{1}{\theta}.$$

The system (1.24) may be considered as the Euler equations derived from the relativistic Boltzmann equation. This is also obtained as the first approximation to the Hilbert expansion as the classical case.

Integrating (1.17) over t , taking $\epsilon \rightarrow 0$, and putting $t = 0$, one has

$$\langle h_j, F^0(0) \rangle = \langle h_j, F_0 \rangle. \quad (1.25)$$

Denote by \mathcal{Q}^* the symmetrized bilinear form

$$\begin{aligned} & \mathcal{Q}^*(F, G)(p) \\ &= \frac{1}{2} \int_{\mathbb{R}^3} \int_{S^2} v_\phi \sigma(g, \theta) [F(p')G(q') + F(q')G(p') - F(p)G(q) - F(q)G(p)] dqd\omega. \end{aligned} \quad (1.26)$$

Let $J_0 = J_0(p) = J(1, 1, 0; p)$, which is a global relativistic Maxwellian as (1.12). We define the standard perturbation $f(t, x, p)$ to this relativistic Maxwellian as $F = J_0 + \sqrt{J_0}f$. The relativistic Boltzmann equation (1.4) for $f = f^\epsilon(t, x, p)$ is given by

$$\partial_t f + \hat{p} \cdot \nabla_x f = \frac{1}{\epsilon} Lf + \frac{1}{\epsilon} \Gamma(f, f), \quad f(0, x, p) = f_0(x, p). \quad (1.27)$$

Here the standard linearized collision operator L is (see [14, 19])

$$Lf = \frac{2}{\sqrt{J_0}} \mathcal{Q}^*(J_0, \sqrt{J_0}f) = -\nu(p)f + Kf. \quad (1.28)$$

Above the multiplication operator takes the form

$$\nu(p) = \int_{\mathbb{R}^3} \int_{S^2} v_\phi \sigma(g, \theta) J_0(q) dqd\omega. \quad (1.29)$$

Notice that $K = K_2 - K_1$ is given by [14, 19]

$$K_1 f = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} v_\phi \sigma(g, \theta) \sqrt{J_0(q)J_0(p)} f(q) dq d\omega,$$

$$K_2 f = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} v_\phi \sigma(g, \theta) \sqrt{J_0(q)} \left\{ \sqrt{J_0(q')} f(p') + \sqrt{J_0(p')} f(q') \right\} dq d\omega.$$

The nonlinear collision operator $\Gamma(f_1, f_2)$ is defined by

$$\Gamma(f_1, f_2) = \frac{1}{\sqrt{J_0}} \mathcal{Q}^*(\sqrt{J_0} f_1, \sqrt{J_0} f_2). \tag{1.30}$$

To present the results in this paper, the following notations are needed. Let L^2 be the Hilbert space of complex-valued function $f(y)$ on $L^2(\mathbb{R}_y^3)$ with the inner product

$$\langle f, g \rangle = \int_{\mathbb{R}^3} f(y) \overline{g(y)} dy,$$

and the corresponding norm $\|\cdot\|$. We also use the following L^∞ norm with the p -weight as

$$\|u\|_\beta = \sup_{p \in \mathbb{R}^3} (1 + |p|)^\beta |u(p)|, \quad \beta \in \mathbb{R}.$$

For any function $u \in S'(\mathbb{R}_x^3 \times \mathbb{R}_p^3)$ with respect to the variable x , we define as

$$\widehat{u}(k, p) = \mathcal{F}_x u(k, p) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}_x^3} e^{-ik \cdot x} u(x, p) dx, \quad k \in \mathbb{R}^3, \quad i = \sqrt{-1}.$$

Let $X_\beta^{\alpha, l}$ denote the Banach space equipped with the norm

$$\|u\|_{\alpha, l, \beta} = \sup_{k, p \in \mathbb{R}^3} e^{\alpha|k|} (1 + |k|)^l (1 + |p|)^\beta |\widehat{u}(k, p)| < \infty, \tag{1.31}$$

where $\alpha, l, \beta \in \mathbb{R}$. Then if $\alpha > 0$, $u \in X_\beta^{\alpha, l}$ are analytic in $x \in \mathbb{R}^3 + iB_\alpha$, where

$$B_\alpha = \{x \in \mathbb{R}^3 \mid |x| < \alpha\}.$$

The space $\dot{X}_\beta^{\alpha, l}$ is the closed subspace of $X_\beta^{\alpha, l}$ such that

$$u \in \dot{X}_\beta^{\alpha, l} \Leftrightarrow u \in X_\beta^{\alpha, l}, \quad \|\mathcal{F}_x^{-1}(\chi(|k| + |p| > R) \widehat{u}(k, p))\|_{\alpha, l, \beta} \rightarrow 0, \quad R \rightarrow \infty, \tag{1.32}$$

where $\chi(|k| + |p| > R)$ is the characteristic function of the domain $|k| + |p| > R$ in $\mathbb{R}_k^3 \times \mathbb{R}_p^3$. Then we define the Banach spaces

$$Y_\beta^{\alpha, \gamma, l}([0, \tau]) = \left\{ u = u(t) \mid \|u\|_{\alpha, \gamma, l, \beta, \tau} = \sup_{t \in [0, \tau]} \|u(t)\|_{\alpha - \gamma t, l, \beta} < \infty \right\}, \tag{1.33}$$

$$\dot{Y}_\beta^{\alpha, \gamma, l}(I) = \left\{ u = u(t) \mid \mathcal{F}_x^{-1}(e^{-\gamma t|k|} \hat{u}(t)) \in B^0(I; \dot{X}_\beta^{\alpha, l}) \right\}, \tag{1.34}$$

where $\gamma \in \mathbb{R}$, $I \subset \mathbb{R}$ is an interval and $B^0(I; X)$ denote the space of bounded continuous functions defined on I with values in a Banach space X . Similarly $C^0(I; X)$ will denote the space of X -valued continuous function on I . We also define the space

$$Z_{\beta, \tau}^{\alpha, \gamma, l} = B^0\left((0, \infty); \dot{Y}_\beta^{\alpha, \gamma, l}([0, \tau])\right), \quad \|u\|_{\alpha, \gamma, l, \beta, \tau} := \sup_{\epsilon > 0, t \in [0, \tau]} \|u^\epsilon(t)\|_{\alpha - \gamma t, l, \beta}. \tag{1.35}$$

With the above preparation, the main results can be stated as follows.

Theorem 1.1. *Suppose that $\alpha > 0$, $l > 3$ and $\beta > 5/2$. There exist positive numbers a_0, a_1 and for the initial data f_0 satisfying $f_0 \in \dot{X}_\beta^{\alpha, l}$ and $\|f_0\|_{\alpha, l, \beta} < a_0$, the followings hold with some positive numbers γ and τ :*

(i) *For each $\epsilon > 0$, (1.4) has a unique classical solution $F = J_0 + \sqrt{J_0}f$ on the time interval $[0, \tau]$ such that $f = f^\epsilon \in Z_{\beta, \tau}^{\alpha, \gamma, l}$,*

$$\|f^\epsilon\|_{\alpha, \gamma, l, \beta, \tau} < a_1 \|f_0\|_{\alpha, l, \beta}, \quad \partial_t f^\epsilon \in C^0\left((0, \infty); \dot{Y}_{\beta-1}^{\alpha, \gamma, l}([0, \tau])\right).$$

(ii) *As $\epsilon \rightarrow 0$, f^ϵ converges to a limit $f^0 \in \dot{Y}_\beta^{\alpha, \gamma, l}((0, \tau])$ strongly in $\dot{Y}_{\beta-1}^{\alpha, \gamma, l}([\delta, \tau])$ for any $\delta > 0$.*

(iii) *For $t \in (0, \tau]$, $F^0(t) = J_0 + \sqrt{J_0}f^0(t)$ is a local relativistic Maxwellian whose hydrodynamical quantities $(n(t, x), u(t, x), \theta(t, x))$ of $F^0(t)$ solve the relativistic Euler equations (1.24) with the initial data (1.25) in classical sense.*

In Theorem 1.1, $n(t, x) = 0$ may be allowed if a_0 can be found large enough. Then $u(t, x)$ and $\theta(t, x)$ happen to be singular at those point x where $n(t, x) = 0$ and no uniqueness results are known for such initial data.

The following result is about the convergence rate about ϵ . It is far short for the verification of the Hilbert expansion.

Theorem 1.2. *Suppose that $\alpha > 0, l > 3 + \sigma$ with $\sigma \in (0, 1)$ and $\beta > 7/2$. There exist positive numbers $a'_0 \leq a_0, a'_1$ and for the initial data f_0 satisfying $f_0 \in \dot{X}_\beta^{\alpha, l}$ and $\|f_0\|_{\alpha, l, \beta} < a'_0$. For any $t \in (0, \tau]$ and any $\epsilon > 0$ small enough, the solution f^ϵ obtained in Theorem 1.1 has the convergence rate as*

$$\|f^\epsilon(t) - f^0(t)\|_{\alpha - \gamma t, l - \sigma, \beta - 1} < a'_1 \left(\frac{\epsilon}{t^2}\right)^{\frac{\sigma}{2}}.$$

The above convergence is not uniform in t near $t = 0$ and the limit f^0 has a discontinuity at $t = 0$. This singular behavior at $\epsilon = t = 0$ describes the initial layer of the solution to the relativistic Boltzmann equation. In order for the initial layer not to appear, if the initial F_0 itself is a local relativistic Maxwellian, we have the following uniform convergence.

Theorem 1.3. *Suppose that $\alpha > 0, l > 3$ and $\beta > 7/2$. If $F_0 = J_0 + \sqrt{J_0} f_0$ is a local relativistic Maxwellian and $f_0 \in \dot{X}_{\beta-1}^{\alpha, l+1}$, for the solution f^ϵ and f^0 in Theorem 1.1, f^ϵ converges to $f^0 \in \dot{Y}_\beta^{\alpha, \gamma, l}([0, \tau])$ strongly in $\dot{Y}_{\beta-1}^{\alpha, \gamma, l}([0, \tau])$ as $\epsilon \rightarrow 0$. Moreover, the hydrodynamical quantities $(n(t, x), u(t, x), \theta(t, x))$ of $F^0(t)$ are unique classical solution to the Cauchy problem for (1.24) with the initial data (1.25).*

In the rest of the introduction, we will first review some previous works related to this paper. A brief history of relativistic kinetic theory (cf. [8, 11]) was given in [42]; interested readers may refer to that paper and references therein. Here we only recall the local-in-time solution [4], solutions and hydrodynamics for the linearized equation [13, 14], large-data solutions [15, 33] by DiPerna-Lions' renormalized theory [12], small-data solutions near vacuum [18, 43], asymptotic stability of the relativistic Maxwellian for hard potentials [17, 19, 20, 27, 31, 32, 48] and for soft potentials [42, 44], and stability of solutions with respect to initial data [28–30].

There have been numerous important contributions to the subject of fluid dynamic limits of the non-relativistic Boltzmann equation. In the context of DiPerna and Lions [12] renormalized weak solutions, we mention the fluid limits has been extensively studied in [1, 2, 21, 22, 38, 41] and the references therein. Due to length constraints, it is impossible to give a comprehensive list. For fluid limits in the context of strong solutions to the non-relativistic Euler and Boltzmann equations, we mention the work of Nishida [39], Ukai and Asano [46], Caflisch [5], Guo [23], Liu *et al.* [37], Guo and Jang [25], Guo *et al.* [26] and the references therein.

For the compressible Euler limit, there are less results about this. For a given smooth solution of the compressible Euler equations for gas dynamics, Caflisch

constructed a corresponding solution of the Boltzmann equation with a zero initial condition for the remainder and hard potentials in [5], which tends to the solution of Euler equations as the mean free path tends to zero, and the Hilbert expansion about the local Maxwellian associated with the solution of Euler equations is used in the construction of the solution to the Boltzmann equation. Under both hard and soft potentials, Guo *et al.* [26] made use of some L^2 - L^∞ interplay estimates in [24] to remove the assumption on the initial data of the remainder in [5] and the positivity of the initial datum can be guaranteed. Their results are generalized in [25] to the Vlasov-Poisson-Boltzmann system. Lachowicz also considered the initial layer and the local existence theorem for the Boltzmann equation in [36]. On the other hand, for $\epsilon > 0$, Nishida [39] constructed the local solution in the analytical function space by the abstract Cauchy-Kovalevski theorem and the spectral structure of the Boltzmann equation. Then he proved that this solution approaches the local Maxwellian distribution with the hydrodynamic quantities related to satisfy the nonlinear compressible Euler equations as $\epsilon \rightarrow 0$. Later Ukai and Asano [46] improved Nishida's results by using the classical contraction mapping principle on a space with a time-dependent norm and also considered the initial layer of the Boltzmann equation.

Much less is known for the relativistic Boltzmann equation. Formal fluid limit calculations are shown in the textbooks [8, 11]. Linearized hydrodynamics are also studied in [13, 14]. Speck and Strain [43] first constructed the local solution of the complicated relativistic Euler equations around the constant state in the Sobolev space with the two conjectures and later these two conjectures were proved in [6]. Then they made use of some new L^2 - L^∞ interplay estimates in [24], some new relativistic estimates in [42] and the Hilbert expansion about the local relativistic Maxwellian associated with the solution of relativistic Euler equations obtained in the first step to construct the local solution to the relativistic Boltzmann equation as in [5, 25, 26]. For a detailed discussion of the relativistic Euler equations, we refer the reader to Christodoulou's articles [9, 10, 43].

In this paper we prove the local existence and uniqueness of the solutions to the relativistic Boltzmann equation for analytic initial fluctuations on a time interval independent of the Knudsen number $\epsilon > 0$. As $\epsilon \rightarrow 0$, we prove that the solution of the relativistic Boltzmann equation tends to the local relativistic Maxwellian, whose fluid-dynamical parameters solve the relativistic Euler equations and the convergence rate is obtained. The Hilbert expansion is verified in the short time interval for the relativistic Boltzmann equation. We also consider the physically important initial layer problem. It seems difficult to obtain the solutions to the relativistic Euler equations with vacuum in the Sobolev space. In this case the vacuum state may be allowed if a_0 in Theorem 1.1 can be found

large enough. As a by-product, an existence result for the relativistic Euler system without the assumption of the non-vacuum fluid states is obtained in this paper. Our methods and results can be viewed as an extension of the program initiated by Nishida [39] and Ukai and Asano [46], who proved analogous results for the non-relativistic Boltzmann equation and Euler equations. We will utilize strategies from Nishida [39] and Ukai and Asano [46] to get the local existence, limit convergence as well as the related convergence rate and the initial layer of the relativistic Boltzmann equation by the spectrum structure of the linearized relativistic Boltzmann equation. Although the spectrum structure of the linearized relativistic Boltzmann equation has been studied in [13, 14], we need more detailed spectral information to prove our results. For this, we first analyze the spectral structure of the linearized relativistic Boltzmann equation by using the semigroup theory, the linear operator perturbation theory and some ideas in [45, 47] to get the complete spectral results. For the relativistic case, the decay about p of the operator K is very slow by (2.4) and we are forced to repeat to use the iterations of the operator K to get the desired estimates. When we prove the Euler limit results, such a good iteration structure is destroyed and we can not use such a iteration methods. We have to devise the convergence norm and make use of the boundedness of \hat{p} , which is unique for the relativistic case, to compensate such a loss. To get the convergence rate, it is crucially used that the boundedness of \hat{p} and successive iteration to take care of the slow decay about p of the operator K .

The rest of the paper is organized as follows. In Section 2, we employ the semigroup theory and the linear operator perturbation theory to analyze the spectrum structure of the linearized relativistic Boltzmann equation. In Section 3, we make use of the spectrum structure of the linearized relativistic Boltzmann equation and the classical contraction mapping principle on a space with a time-dependent norm to get the local solution of the relativistic Boltzmann equation. In Sections 4 and 5, we consider the Euler limit of the relativistic Boltzmann equation and the convergence rate. In the last section we consider the initial layer.

2 Spectrum analysis of the linearized equation

In this section, we will analyze the spectrum structure of the linearized relativistic Boltzmann equation, which will be used for later analysis. To this end we first quote some properties of L from [19, 20]. The operator L has the decomposition $L = -\Lambda + K$, where $\nu(p)$ is the multiplication operator

$$\Lambda = \nu(p) \in L_{loc}^{\infty}(\mathbb{R}_p^3), \quad \nu_0 p_0^{\frac{a}{2}} \leq \nu(p) \leq \nu_1 p_0^{\frac{a}{2}} \leq \nu_1(1 + |p|), \quad a \in [0, 2) \quad (2.1)$$

with some positive constants ν_0 and ν_1 . And the operator K is a self-adjoint compact operator on $L^2(\mathbb{R}_p^3)$ with a real symmetric integral kernel $K(p, q)$ which enjoys the estimates

$$\sup_p \int_{\mathbb{R}^3} |K(p, q)| dq < \infty, \quad \sup_p \int_{\mathbb{R}^3} |K(p, q)|^2 dq < \infty. \tag{2.2}$$

For any $l \in \mathbb{R}$ and any small $c > 0$, we easily see

$$(1 + |q|^2)^l e^{-c|p-q|} \leq C_l (1 + |p|^2)^l (1 + |p-q|^2)^{|l|} e^{-c|p-q|} \leq C_l (1 + |p|^2)^l.$$

By using this and the similar arguments as [19, p. 327], for any $\beta \in \mathbb{R}$ and $\eta := 1 - [3|Y| + a + 2b]/2 \in (0, 1]$, one has

$$\int_{\mathbb{R}^3} |K(p, q)| (1 + |q|^2)^{-\frac{\beta}{2}} dq < C (1 + |p|^2)^{-\frac{1}{2}(\beta + \eta)}. \tag{2.3}$$

For any $\beta \in \mathbb{R}$, we have from (2.2) and (2.3) that

$$\|Kf\|_\beta \leq C \|f\|_{\beta - \eta}, \quad \|Kf\|_0 \leq C \|f\|. \tag{2.4}$$

The linearized operator L is an unbounded, self-adjoint and non-positive operator on $L^2(\mathbb{R}_p^3)$ with the domain $D(L) = \{f \in L^2(\mathbb{R}_p^3); v(p)f \in L^2(\mathbb{R}_p^3)\}$. The null space of the operator L is the space of collision invariants

$$\mathcal{N} = \text{span} \left\{ \sqrt{J_0}, p_1 \sqrt{J_0}, p_2 \sqrt{J_0}, p_3 \sqrt{J_0}, p_0 \sqrt{J_0} \right\}.$$

By the direct calculations as [20], we arrive at an orthonormal basis for \mathcal{N} as

$$\mathcal{N} = \text{span} \{ \psi_0, \psi_1, \psi_2, \psi_3, \psi_4 \}. \tag{2.5}$$

Define P_0 as a velocity projection operator from $L^2(\mathbb{R}_p^3)$ to \mathcal{N} and denote $Q_0 = I - P_0$. Then any function $f(t, x, p)$ for any fixed (t, x) can be uniquely decomposed as the sum of the macroscopic component $P_0 f$ and microscopic component $Q_0 f$.

Define the operators as

$$A^\epsilon := -\hat{p} \cdot \nabla_x - \frac{1}{\epsilon} \Lambda, \quad B^\epsilon := A^\epsilon + \frac{1}{\epsilon} K. \tag{2.6}$$

If $f^\epsilon = f^\epsilon(t)$ is a solution to (1.27), then it should solve the integral equation

$$f^\epsilon(t) = e^{tB^\epsilon} f_0 + \int_0^t e^{(t-s)B^\epsilon} \frac{1}{\epsilon} \Gamma(f^\epsilon(s), f^\epsilon(s)) ds. \tag{2.7}$$

We make use of (2.6) and the Fourier transform to arrive at the operators as

$$\widehat{A}^\epsilon(k) := -ik \cdot \widehat{p} - \frac{1}{\epsilon} \Lambda, \quad \widehat{B}^\epsilon(k) := -ik \cdot \widehat{p} + \frac{1}{\epsilon} L. \quad (2.8)$$

Here

$$f \in D(\widehat{A}^\epsilon(k)) = \left\{ f \in L^2(\mathbb{R}_p^3); \nu(p)f \in L^2(\mathbb{R}_p^3), ik \cdot \widehat{p}f \in L^2(\mathbb{R}_p^3) \right\}$$

and $D(\widehat{A}^\epsilon(k)) = D(\widehat{B}^\epsilon(k))$.

We further denote $\widehat{B}(k) = \widehat{B}^1(k)$ and $\widehat{A}(k) = \widehat{A}^1(k)$. In what follows we will study the spectrum structure of the operator $\widehat{B}(k)$. To this end we shall use $\sigma(A)$ to denote the spectrum of the operator A . The discrete spectrum of A , $\sigma_d(A)$, is the set of all isolated eigenvalues with finite multiplicity. The essential spectrum of A , $\sigma_{ess}(A)$, is the set $\sigma(A) \setminus \sigma_d(A)$. We denote $\text{Res}(A)$ to be the resolvent set of the operator A . We also use $D(A)$ to denote the domain of definition of the operator A . By virtue of (2.1) and (2.8), we readily see that $\widehat{A}(k)$ is a maximally dissipative closed operator in $L^2(\mathbb{R}_p^3)$. This shows that $\widehat{A}(k)$ generates a continuous contraction semigroup on $L^2(\mathbb{R}_p^3)$. We have that

$$\{\lambda : \text{Re}\lambda > -\nu_0\} \subset \text{Res}(\widehat{A}(k)), \quad \sigma_{ess}(\widehat{A}(k)) \subset \{\lambda : \text{Re}\lambda \leq -\nu_0\}.$$

In order to analyze the spectrum structure of $\widehat{B}(k)$, we need the following key estimates in the sequel.

Lemma 2.1. *There is a constant $C > 0$ such that the following estimates hold:*

(i) *For any $\delta > 0$, we have*

$$\sup_{\substack{\text{Re}\lambda \geq -\nu_0 + \delta, \\ \text{Im}\lambda \in \mathbb{R}}} \|(\lambda - \widehat{A}(k))^{-1}K\| \leq C\delta^{-1 + \frac{\eta}{3(\eta+3)}} |k|^{-\frac{\eta}{3(\eta+3)}} \quad \text{as } |k| \rightarrow \infty. \quad (2.9)$$

(ii) *If $|k| \leq k_0$ for any $k_0 > 0$, we have*

$$\sup_{\substack{\text{Re}\lambda \geq -\nu_0 + \delta, \\ |k| \leq k_0}} \|(\lambda - \widehat{A}(k))^{-1}K\| \leq C\delta^{-\frac{3}{\eta+3}} |\text{Im}\lambda|^{-\frac{\eta}{\eta+3}} \quad \text{as } |\text{Im}\lambda| \rightarrow \infty. \quad (2.10)$$

Here $\|\cdot\|$ is the operator norm in L^2 .

Proof. Let $\mathbf{1}_D(p)$ be the characteristic function of the domain $D \subset \mathbb{R}^3$. By virtue of (2.4) and (2.8), we see

$$\begin{aligned} \|\mathbf{1}_{|p| \leq R} (\lambda - \widehat{A}(k))^{-1} Kf\|^2 &= \int_{\mathbb{R}^3} \mathbf{1}_{|p| \leq R} |(\lambda - \widehat{A}(k))^{-1}|^2 |Kf(p)|^2 dp \\ &\leq C \|f\|^2 \int_{|p| \leq R} \frac{1}{|\operatorname{Re} \lambda + \nu(p)|^2 + |\operatorname{Im} \lambda + k \cdot \hat{p}|^2} dp. \end{aligned} \tag{2.11}$$

Denote the last integral by J . For any $R > 1$ and any small $\varrho \in (0, 1/2)$, we set

$$\Sigma_1 = \left\{ p \in \mathbb{R}^3 \mid |p| \leq R, |\operatorname{Im} \lambda + k \cdot \hat{p}| \leq \varrho |k| \right\}, \quad \Sigma_2 = \left\{ p \in \mathbb{R}^3 \mid |p| \leq R \right\} - \Sigma_1.$$

Denote $\operatorname{mes} D$ by the Lebesgue measure of the domain $D \subset \mathbb{R}^3$. One has

$$\begin{aligned} \operatorname{mes} \Sigma_1 &= \int_{|p| \leq R} \mathbf{1}_{|\operatorname{Im} \lambda + k \cdot \hat{p}| \leq \varrho |k|} (p) dp = \int_{|p| \leq R} \mathbf{1}_{[-\varrho |k| - \operatorname{Im} \lambda, \varrho |k| - \operatorname{Im} \lambda]} (k \cdot \hat{p}) dp \\ &= 2\pi \int_0^R \int_0^\pi \mathbf{1}_{[-\varrho |k| - \operatorname{Im} \lambda, \varrho |k| - \operatorname{Im} \lambda]} (|k| |\hat{p}| \cos \theta) \sin \theta |p|^2 d\theta dp \\ &= -2\pi |k|^{-1} \int_0^R |p|^2 |\hat{p}|^{-1} dp \\ &\quad \times \int_0^\pi \mathbf{1}_{[-\varrho |k| - \operatorname{Im} \lambda, \varrho |k| - \operatorname{Im} \lambda]} (|k| |\hat{p}| \cos \theta) d(|k| |\hat{p}| \cos \theta) \\ &= 2\pi |k|^{-1} \int_0^R |p|^2 |\hat{p}|^{-1} dp \int_{-|k| |\hat{p}|}^{|k| |\hat{p}|} \mathbf{1}_{[-\varrho |k| - \operatorname{Im} \lambda, \varrho |k| - \operatorname{Im} \lambda]} (y) dy \\ &\leq 2\pi |k|^{-1} \int_0^R |p|^2 |\hat{p}|^{-1} dp \int_{-|k|R/\sqrt{1+R^2}}^{|k|R/\sqrt{1+R^2}} \mathbf{1}_{[-\varrho |k| - \operatorname{Im} \lambda, \varrho |k| - \operatorname{Im} \lambda]} (y) dy. \end{aligned}$$

It follows from this that $\operatorname{mes} \Sigma_1 \leq C\varrho R^3$. Obviously $\operatorname{mes} \Sigma_2 \leq CR^3$. Thus one has

$$J \leq CR^3 (\delta^{-2} \varrho + (\varrho |k|)^{-2}).$$

If we choose $\varrho = (\delta/|k|)^{2/3}$, then $\varrho \in (0, 1/2)$ as $|k| \rightarrow \infty$. By these facts and (2.11), we arrive at

$$\|\mathbf{1}_{|p| \leq R} (\lambda - \widehat{A}(k))^{-1} K\| \leq CR^{\frac{3}{2}} \delta^{-\frac{2}{3}} |k|^{-\frac{1}{3}}. \tag{2.12}$$

By virtue of (2.3), we get

$$\begin{aligned} |Kf(p)|^2 &\leq \int_{\mathbb{R}^3} |K(p, q)| dq \int_{\mathbb{R}^3} |K(p, q)| |f(q)|^2 dq \\ &\leq C(1 + |p|^2)^{-\frac{\eta}{2}} \int_{\mathbb{R}^3} |K(p, q)| |f(q)|^2 dq. \end{aligned}$$

If $\operatorname{Re}\lambda \geq -\nu_0 + \delta$, we have from (2.1), (2.2) and this that

$$\begin{aligned} \|\mathbf{1}_{|p| \geq R}(\lambda - \widehat{A}(k))^{-1}Kf\|^2 &= \int_{\mathbb{R}^3} \mathbf{1}_{|p| \geq R} |(\lambda - \widehat{A}(k))^{-1}|^2 |Kf(p)|^2 dp \\ &= \int_{|p| \geq R} (|\operatorname{Re}\lambda + \nu(p)|^2 + |\operatorname{Im}\lambda + k \cdot \widehat{p}|^2)^{-1} |Kf(p)|^2 dp \\ &\leq C\delta^{-2}R^{-\eta} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |K(p,q)| |f(q)|^2 dq dp \leq C\delta^{-2}R^{-\eta} \|f\|^2. \end{aligned}$$

Thus we can obtain

$$\|\mathbf{1}_{|p| \geq R}(\lambda - \widehat{A}(k))^{-1}K\| \leq C\delta^{-1}R^{-\frac{\eta}{2}}. \quad (2.13)$$

It follows from (2.12) and (2.13) that

$$\|(\lambda - \widehat{A}(k))^{-1}K\| \leq C(\delta^{-1}R^{-\frac{\eta}{2}} + R^{\frac{3}{2}}\delta^{-\frac{2}{3}}|k|^{-\frac{1}{3}}).$$

If we choose $R = (|k|/\delta)^{2/(3(\eta+3))}$, then $R > 1$ as $|k| \rightarrow \infty$. As $|k| \rightarrow \infty$, one has

$$\|(\lambda - \widehat{A}(k))^{-1}K\| \leq C\delta^{-1+\frac{\eta}{3(\eta+3)}}|k|^{-\frac{\eta}{3(\eta+3)}}.$$

This completes the proof of (2.9). If $|k| \leq k_0$, $|p| \leq R$ and $|\operatorname{Im}\lambda| \geq 2k_0R$, one has

$$|\operatorname{Im}\lambda + k \cdot \widehat{p}| \geq |\operatorname{Im}\lambda| - |k||\widehat{p}| \geq |\operatorname{Im}\lambda| - \frac{k_0R}{\sqrt{1+R^2}} \geq \frac{|\operatorname{Im}\lambda|}{2}.$$

We have from this and (2.11) that

$$\|\mathbf{1}_{|p| \leq R}(\lambda - \widehat{A}(k))^{-1}K\|^2 \leq C(\delta^2 + |\operatorname{Im}\lambda|^2)^{-1}R^3.$$

It follows from this and (2.13) that

$$\|(\lambda - \widehat{A}(k))^{-1}K\| \leq C(\delta^{-1}R^{-\frac{\eta}{2}} + R^{\frac{3}{2}}|\operatorname{Im}\lambda|^{-1}).$$

If we choose $R = (|\operatorname{Im}\lambda|/\delta)^{2/(\eta+3)}$, then $R > 1$ as $|\operatorname{Im}\lambda| \rightarrow \infty$. As $|\operatorname{Im}\lambda| \rightarrow \infty$, one has

$$\|(\lambda - \widehat{A}(k))^{-1}K\| \leq C\delta^{-\frac{3}{\eta+3}}|\operatorname{Im}\lambda|^{-\frac{\eta}{\eta+3}}.$$

This concludes the proof of the lemma. \square

Lemma 2.2. For any $\delta > 0$, there exists a constant $\tau_1 > 0$ such that the following results hold:

- (i) For any $k \in \mathbb{R}^3$, $\{\lambda : \operatorname{Re}\lambda > 0\} \cup \{\lambda : -\nu_0 + \delta < \operatorname{Re}\lambda \leq 0, |\operatorname{Im}\lambda| \geq \tau_1\} \subset \operatorname{Res}(\widehat{B}(k))$.
- (ii) The set $\sigma(\widehat{B}(k)) \cap \{\lambda : -\nu_0 + \delta < \operatorname{Re}\lambda \leq 0\}$ consists of a finite number of eigenvalues of $\sigma(\widehat{B}(k))$.
- (iii) When $k \neq 0$, $\widehat{B}(k)$ has no eigenvalues on the imaginary axis. If λ is an eigenvalue of $\widehat{B}(k)$, then $\operatorname{Re}\lambda \leq 0$. If $\operatorname{Re}\lambda = 0$, $\lambda = 0$ if and only if $k = 0$.

Proof. If $\operatorname{Re}\lambda \geq -\nu_0 + \delta$, for any $k \in \mathbb{R}^3$, we have from (2.10) that $\|(\lambda - \widehat{A}(k))^{-1}K\| \rightarrow 0$, as $|\operatorname{Im}\lambda| \rightarrow \infty$. Then there exists $\tau_1 > 0$ such that for any k , $\operatorname{Re}\lambda \geq -\nu_0 + \delta$, and $|\operatorname{Im}\lambda| \geq \tau_1$, $\|(\lambda - \widehat{A}(k))^{-1}K\| \leq 1/2$. Then the Neumann series gives the bounded inverse $(I - (\lambda - \widehat{A}(k))^{-1}K)^{-1}$. If $\operatorname{Re}\lambda \geq -\nu_0 + \delta$ and $|\operatorname{Im}\lambda| \geq \tau_1$, we write the second resolvent equation

$$(\lambda - \widehat{B}(k))^{-1} = (\lambda - \widehat{A}(k))^{-1} + (\lambda - \widehat{A}(k))^{-1}K(\lambda - \widehat{B}(k))^{-1}. \tag{2.14}$$

This gives

$$(\lambda - \widehat{B}(k))^{-1} = (I - (\lambda - \widehat{A}(k))^{-1}K)^{-1}(\lambda - \widehat{A}(k))^{-1}. \tag{2.15}$$

By these we deduce that $\lambda \in \operatorname{Res}(\widehat{B}(k))$. Thus we have

$$\{\lambda : -\nu_0 + \delta < \operatorname{Re}\lambda \leq 0, |\operatorname{Im}\lambda| \geq \tau_1\} \subset \operatorname{Res}(\widehat{B}(k)). \tag{2.16}$$

Since K is a bounded operator on $L^2(\mathbb{R}_p^3)$, $\widehat{B}(k)$ is a bounded perturbation of a continuous semigroup generator $\widehat{A}(k)$. Hence, $\widehat{B}(k)$ generates a continuous semigroup on $L^2(\mathbb{R}_p^3)$, cf. [40]. Since the operator L is non-positive, we have

$$\operatorname{Re}\langle \widehat{B}(k)f, f \rangle = \langle Lf, f \rangle \leq 0.$$

Hence, $\widehat{B}(k)$ generates a continuous contraction semigroup on $L^2(\mathbb{R}_p^3)$. By this we have that $\{\lambda : \operatorname{Re}\lambda > 0\} \subset \operatorname{Res}(\widehat{B}(k))$. This and (2.16) imply that (i) holds.

On the other hand, since K is a compact operator on $L^2(\mathbb{R}_p^3)$, $\widehat{B}(k)$ is a compact perturbation of $\widehat{A}(k)$, thanks to [34, Theorem IV. 1.9], we have that

$$\sigma_{ess}(\widehat{B}(k)) = \sigma_{ess}(\widehat{A}(k)) \subset \{\lambda : \operatorname{Re}\lambda \leq -\nu_0\}.$$

Thus we know that $\sigma(\widehat{B}(k)) \cap \{\lambda : -\nu_0 < \operatorname{Re}\lambda\}$ consists of discrete eigenvalues with possible accumulation points on the boundary of $\{\lambda : -\nu_0 < \operatorname{Re}\lambda\}$. Since

$$\sigma(\widehat{B}(k)) \cap \overline{\{\lambda : -\nu_0 + \delta < \operatorname{Re}\lambda \leq 0\}} \subset \{\lambda : -\nu_0 + \delta < \operatorname{Re}\lambda \leq 0, |\operatorname{Im}\lambda| \leq \tau_1\}$$

is a compact set in $\{\lambda : -\nu_0 < \operatorname{Re}\lambda\}$ and it does not touch the boundary of $\{\lambda : -\nu_0 < \operatorname{Re}\lambda\}$, the number of eigenvalues in it is finite. Thus (ii) holds.

Let $\phi \neq 0$ such that $\widehat{B}(k)\phi = \lambda\phi$. Then we have

$$\lambda\|\phi\|^2 = \langle \widehat{B}(k)\phi, \phi \rangle = \langle L\phi, \phi \rangle - i\langle \hat{p} \cdot k\phi, \phi \rangle.$$

Noticing that L is non-positive, we have $\operatorname{Re}\lambda \leq 0$. If $k \neq 0$ and $\operatorname{Re}\lambda = 0$, we have $L\phi = 0$, which implies that $\phi \in \mathcal{N}$. The eigenvalue equation is reduced to $(\operatorname{Im}\lambda + \hat{p} \cdot k)\phi = 0$. We deduce from this that $k = 0$ and $\operatorname{Im}\lambda = 0$. This is a contradiction. This implies that $\widehat{B}(k)$ has no eigenvalues on the imaginary axis for $k \neq 0$. Thus (iii) holds. \square

Lemma 2.3. *There exists a constant $\mu_0 > 0$ such that for $g \in D(L) \cap \mathcal{N}^\perp$*

$$-\langle Lg, g \rangle \geq \mu_0 \|v^{\frac{1}{2}}g\|^2.$$

Proof. By Lemma 2.2, we have that $\sigma_{\text{ess}}(L) \subset (-\infty, -\nu_0]$. Thus $\sigma_d(L) \subset (-\nu_0, 0)$. If there is an isolated discrete eigenvalue of L in $(-\nu_0, 0)$, which is corresponding to the eigenfunction g , we take this eigenvalue to be $-\mu_1$. Otherwise we take $\mu_1 = \nu_0$. Hence, we have

$$-\langle Lg, g \rangle \geq \mu_1 \|g\|^2.$$

By using (2.1) and the fact that K is bounded in $L^2(\mathbb{R}_p^3)$, for any $\epsilon > 0$ small enough, we have

$$\begin{aligned} -\langle Lg, g \rangle &\geq (1-\epsilon)\mu_1 \|g\|^2 + \epsilon\langle \Lambda g, g \rangle - \epsilon\langle Kg, g \rangle \\ &\geq (1-\epsilon)\mu_1 \|g\|^2 + \nu_0\epsilon \|v^{\frac{1}{2}}g\|^2 - C\epsilon \|g\|^2 \geq \mu_0 \|v^{\frac{1}{2}}g\|^2. \end{aligned}$$

Here we take $\mu_0 = \nu_0\epsilon$. \square

In the following lemma we shall give the spectrum structure of $\widehat{B}(k)$ when k is away from zero.

Lemma 2.4. *For any $\kappa_0 > 0$, there exists $\tau_0 = \tau_0(\kappa_0) > 0$ such that for all $|k| > \kappa_0$,*

$$\sigma(\widehat{B}(k)) \subset \{\lambda : \operatorname{Re}\lambda < -\tau_0\}.$$

Proof. If $\operatorname{Re}\lambda \geq -\nu_0 + \delta$, we have from (2.9) that $\|(\lambda - \widehat{A}(k))^{-1}K\| \rightarrow 0$, as $|k| \rightarrow \infty$. Then there exists $\kappa_1 > 0$ such that for $|k| \geq \kappa_1$, $\|[I - (\lambda - \widehat{A}(k))^{-1}K]^{-1}\| \leq 2$. If $\operatorname{Re}\lambda \geq -\nu_0 + \delta$ and $|k| \geq \kappa_1$, we have from (2.15) that $\lambda \in \operatorname{Res}(\widehat{B}(k))$. Thus for any $|k| \geq \kappa_1$, we have that

$$\sigma_d(\widehat{B}(k)) \subset \{\lambda : \operatorname{Re}\lambda < -\nu_0 + \delta\}. \quad (2.17)$$

We claim that for any $\kappa_0 > 0$, there exists $\sigma_0 > 0$ such that for all $|k| \in (\kappa_0, \kappa_1]$,

$$\sigma_d(\widehat{B}(k)) \subset \{\lambda : \text{Re}\lambda \leq -\sigma_0\}. \tag{2.18}$$

By Lemma 2.2, $\sigma_{ess}(\widehat{B}(k)) \subset \{\lambda : \text{Re}\lambda \leq -\nu_0\}$. If we take $\tau_0 = \min\{\nu_0 - \delta, \sigma_0\}$, this completes the proof of the lemma.

In what follows we shall show (2.18). Suppose that (2.18) is violated. Then there exists the eigenvalues $\lambda_n \in \mathbf{C}$, $k_n \in \mathbb{R}^3$ and the corresponding eigenfunctions $\phi_n \in D(\widehat{B}(k_n)) \subset L^2(\mathbb{R}^3)$ such that

$$\widehat{B}(k_n)\phi_n = (L - i\hat{p} \cdot k_n)\phi_n = \lambda_n\phi_n, \quad \|\phi_n\| = 1, \quad n = 1, 2, 3, \dots, \tag{2.19}$$

and

$$-\frac{1}{n} \leq \text{Re}\lambda_n \leq 0, \quad \kappa_0 < |k_n| \leq \kappa_1. \tag{2.20}$$

Clearly $\{\lambda_n\}$ is bounded sequence. If not, we assume $\text{Im}\lambda_n \rightarrow \infty$. By Lemma 2.1, $\lim_{n \rightarrow \infty} \|(\lambda_n - \widehat{A}(k_n))^{-1}K\| = 0$ and thus $\lambda_n \in \text{Res}(\widehat{B}(k_n))$ by (2.15). This is a contradiction. Thus, up to a subsequence, we have that $k_n \rightarrow \bar{k}_0$ and $\lambda_n \rightarrow i\bar{\lambda}_0$ as $n \rightarrow \infty$, where $\bar{\lambda}_0$ is a real number and $\kappa_0 < |\bar{k}_0| \leq \kappa_1$. Since the operator K is compact, up to a subsequence, $K\phi_n \rightarrow h \in L^2$. It follow from (2.19) that

$$\phi_n = \frac{K\phi_n}{\lambda_n + i\hat{p} \cdot k_n + \nu(p)}.$$

Letting $n \rightarrow \infty$, one has

$$\phi_n = \frac{K\phi_n}{\lambda_n + i\hat{p} \cdot k_n + \nu(p)} \rightarrow \frac{h}{i\bar{\lambda}_0 + i\hat{p} \cdot \bar{k}_0 + \nu(p)} := \phi \neq 0.$$

It follows from this that $h = K\phi$ and $i\bar{\lambda}_0\phi = \widehat{B}(\bar{k}_0)\phi$. This shows that $\widehat{B}(\bar{k}_0)$ has a pure imaginary eigenvalue. By Lemma 2.2(iii) one has $k_0 = \lambda_0 = 0$. This is contradictive to the fact that $\kappa_0 < |k_0| \leq \kappa_1$. The proof of the lemma is complete. \square

In what follows we shall give the spectrum structure of $\widehat{B}(k)$ when k is near zero.

Lemma 2.5. *For any $\kappa_0 > 0$ small enough, there exists $\tau_0 = \tau_0(\kappa_0) > 0$ such that for any $|k| \leq \kappa_0$,*

(i) $\sigma(\widehat{B}(k)) \cap \{\lambda : \text{Re}\lambda \geq -\tau_0\} = \{\lambda_j(|k|)\}_{j=0}^4.$

- (ii) Let $\lambda_j(k)$ and $e_j(k)$ be the eigenvalue and the corresponding eigenfunction of the operator $\widehat{B}(k)$ with $0 \leq j \leq 4$. For $|k| \leq \kappa_0$, $\lambda_j(k)$ and $e_j(k, p)$ are analytic in $|k|$ and have the asymptotic expansions

$$\lambda_j(k) = \lambda_j(|k|) = \lambda_j^{(1)} |k| + \lambda_j^{(2)} |k|^2 + \mathcal{O}(|k|^3), \quad |k| \rightarrow 0, \quad (2.21)$$

$$e_j(k, p) = e_j(|k|, \hat{p} \cdot \omega, p) = e_j^{(0)} + |k| e_j^{(1)} + \mathcal{O}(|k|^2), \quad |k| \rightarrow 0 \quad (2.22)$$

with the coefficients $\lambda_j^{(1)} = -ic_j$ where c_j is in (2.28), $\lambda_j^{(2)} < 0$ is in (2.32). And $e_j^{(l)}$ with $l=0,1,2$ are (2.28), (2.30) and (2.31). Here $k = |k|\omega$ and $\{e_j(k, p)\}_{j=0}^4$ can be normalized as

$$\langle e_i(k, p), e_j(-k, p) \rangle = \delta_{ij}, \quad 0 \leq i, j \leq 4.$$

- (iii) Denote the eigen-projection and eigen-nilpotent corresponding to the eigenvalue $\lambda_j(k)$ by $P_j(k)$ and $Q_j(k)$, which are analytic in $|k|$ for $|k| \leq \kappa_0$. It holds that, for $0 \leq j \leq 4$,

$$P_j(k) = P_j^{(0)}(\omega) + |k| P_j^{(1)}(\omega) + |k|^2 P_j^{(2)}(\omega), \quad Q_j(k) = 0,$$

where $P_j^{(l)}(\omega)$ with $l=0,1,2$ are defined in (2.33) and (2.34) with $P_0 = \sum_{j=0}^4 P_j^{(0)}(\omega)$. Moreover, for any $\beta', \beta \in \mathbb{R}$, one has

$$\left\| P_j^{(l)} f \right\|_{\beta'} \leq C \|f\|_{\beta}. \quad (2.23)$$

Proof. Since L is invariant with respect to the rotation \mathcal{R} of $p \in \mathbb{R}^3$ by the direct calculations and $\mathcal{R}k \cdot \mathcal{R}\hat{p} = k \cdot \hat{p}$, we see that the eigenvalues only depend on $|k|$. Hence, $(L - i|k|(\hat{p} \cdot \omega))e(k, p) = \lambda(|k|)e(k, p)$ where $k = |k|\omega$. By this we know that $e(k, p) = e(|k|, \hat{p} \cdot \omega, p)$. Since $\widehat{B}(k) = L - i\hat{p} \cdot k$, we rewrite $\widehat{B}(k)$ as $\widehat{B}(z) = L + z\hat{p} \cdot \omega$ where $z = -i|k|$. It is easily seen that

$$\|\hat{p} \cdot \omega g\|^2 \leq \|g\|^2 \leq C(\|Lg\|^2 + \|g\|^2).$$

This shows that the operator $\omega \cdot \hat{p}I$ is L -bounded. Thus we regards $\widehat{B}(z)$ as the analytic families of Type (A) [40, p.16] for $|k| \leq \kappa_0$, where κ_0 is small enough. We see that $\widehat{B}(z)$ is self-adjoint in $L^2(\mathbb{R}_p^3)$ for any $z \in \mathbb{R}$. Since 0 is a discrete eigenvalue of multiplicity 5, by [40, Theorem XII.13] and its proof, there are 5 discrete eigenvalues $\lambda_j(|k|)$ of $\widehat{B}(k)$ with $\lambda_j(0) = 0$, which are analytic in $|k|$ for $|k| \leq \kappa_0$.

And the corresponding eigenfunctions $e_j(|k|, \hat{p} \cdot \omega, p)$ and eigen-projections $P_j(k)$ are also analytic in $|k|$ for $|k| \leq \kappa_0$. The definition of the eigen-nilpotents $Q_j(k)$ corresponding to the eigenvalue $\lambda_j(k)$ can be found in [34, p.181]. By a general theorem on the perturbation of symmetric operator [34, p.120], the eigen-nilpotents $Q_j(k)$ vanishes.

If $e_j(k, p)$ is an eigenfunction corresponding to the eigenvalue $\lambda_j(k)$ of $\widehat{B}(k)$, then $e_j(-k, p)$ is an eigenfunction corresponding to the eigenvalue $\lambda_j(-k)$ of $\widehat{B}(-k)$. Then we have for any $0 \leq i, j \leq 2$,

$$\begin{aligned} & \lambda_j(k) \langle e_j(k, p), e_i(-k, p) \rangle \\ &= \langle \widehat{B}(k) e_j(k, p), e_i(-k, p) \rangle = \langle e_j(k, p), \widehat{B}(-k) e_i(-k, p) \rangle \\ &= \langle e_j(k, p), \lambda_i(-k) e_i(-k, p) \rangle = \lambda_i(k) \langle e_j(k, p), e_i(-k, p) \rangle. \end{aligned}$$

Then we obtain $\langle e_j(k, p), e_i(-k, p) \rangle = 0$ when $0 \leq i \neq j \leq 2$. The normalization can be determined by $\langle e_j(k, p), e_j(-k, p) \rangle = 1$. For $j = 3, 4$, we use the Gram-Schmidt orthogonalization to determine $\langle e_j(k, p), e_i(-k, p) \rangle = \delta_{ij}$. Thus we have $\{e_j(k, p)\}_{j=0}^4$ can be normalized as

$$\langle e_i(k, p), e_j(-k, p) \rangle = \delta_{ij}, \quad 0 \leq i, j \leq 4.$$

After showing the existence and analyticity of the function $\{\lambda_j(|k|), e_j(k, p)\}_{j=0}^4$, we are in a position to compute the Taylor expansions by the idea of [7, 16]

$$\lambda_j(|k|) \cong \sum_{n=0}^{\infty} \lambda_j^{(n)} |k|^n, \quad e_j(k, p) \cong \sum_{n=0}^{\infty} e_j^{(n)} |k|^n. \tag{2.24}$$

Here $k = |k|\omega$. Comparing terms of the same order in $|k|$, we get

$$Le_j^{(0)} = \lambda_j^{(0)} e_j^{(0)}, \quad 0 \leq j \leq 4, \tag{2.25}$$

$$Le_j^{(1)} = \left(\lambda_j^{(1)} + i(\hat{p} \cdot \omega) \right) e_j^{(0)}, \quad 0 \leq j \leq 4, \tag{2.26}$$

$$Le_j^{(n)} = \left(\lambda_j^{(1)} + i(\hat{p} \cdot \omega) \right) e_j^{(n-1)} + \sum_{l=2}^n \lambda_j^{(l)} e_j^{(n-l)}, \quad n \geq 2, \quad 0 \leq j \leq 4. \tag{2.27}$$

By (2.25) and the fact that $\lambda_j(0) = 0$, one has that $\lambda_j^{(0)} = 0$ and $e_j^{(0)}$ is some linear combination of (2.5). We define the operator $\mathbf{A}(\omega) = P_0(\hat{p} \cdot \omega)P_0$. By using (2.5), we set

$$m = \langle \hat{p}_1 \psi_0, \psi_1 \rangle, \quad \bar{m} = \langle \hat{p}_1 \psi_2, \psi_4 \rangle.$$

We denote ω^t as the vector ω 's transpose. We have a matrix representation of $\mathbf{A}(\omega)$ as

$$\begin{pmatrix} 0 & m\omega & 0 \\ m\omega^t & 0 & \bar{m}\omega^t \\ 0 & \bar{m}\omega & 0 \end{pmatrix}.$$

It is easy to compute its eigenvalues c_i and normalized eigenvectors E_i which are given by

$$\left\{ \begin{array}{ll} c_0 = \sqrt{\bar{m}^2 + m^2}, & E_0 = \sqrt{\frac{m^2}{2\bar{m}^2 + 2m^2}}\psi_0 + \sqrt{\frac{1}{2}}\sum_{j=1}^3 \omega_j \psi_j + \sqrt{\frac{\bar{m}^2}{2\bar{m}^2 + 2m^2}}\psi_4, \\ c_1 = -\sqrt{\bar{m}^2 + m^2}, & E_1 = \sqrt{\frac{m^2}{2\bar{m}^2 + 2m^2}}\psi_0 - \sqrt{\frac{1}{2}}\sum_{j=1}^3 \omega_j \psi_j + \sqrt{\frac{\bar{m}^2}{2\bar{m}^2 + 2m^2}}\psi_4, \\ c_2 = 0, & E_2 = -\sqrt{\frac{m^2}{\bar{m}^2 + m^2}}\psi_0 + \sqrt{\frac{\bar{m}^2}{\bar{m}^2 + m^2}}\psi_4, \\ c_3 = 0, & E_3 = \sum_{j=1}^3 C_4^j(\omega)\psi_j, \\ c_4 = 0, & E_4 = \sum_{j=1}^3 C_5^j(\omega)\psi_j, \end{array} \right. \quad (2.28)$$

where $C_4^j(\omega)$ and $C_5^j(\omega)$ are the components of the unit vectors $C_4(\omega)$ and $C_5(\omega)$, which satisfies

$$C_4(\omega) \cdot C_5(\omega) = 0, \quad C_4(\omega) \cdot \omega = 0, \quad C_5(\omega) \cdot \omega = 0.$$

If we take $e_j^{(0)} = E_j$ in (2.28), we have from (2.26) and (2.28) that

$$\lambda_j^{(1)} = -i \langle (\hat{p} \cdot \omega) e_j^{(0)}, e_j^{(0)} \rangle = -i \langle P_0(\hat{p} \cdot \omega) P_0 E_j, E_j \rangle = -i c_j, \quad 0 \leq j \leq 4. \quad (2.29)$$

Noting that $e_j^{(0)} = E_j$, we have from (2.26) that

$$e_j^{(1)} = iL^{-1}Q_0(\hat{p} \cdot \omega)E_j + e'_j, \quad (2.30)$$

where $e'_j \in \mathcal{N}$ as (2.28) and L^{-1} denotes the inverse of L restricted to \mathcal{N}^\perp by Lemma 2.3. By (2.27), we have

$$\begin{aligned} Le_j^{(2)} &= \lambda_j^{(1)} e_j^{(1)} + i(\hat{p} \cdot \omega) e_j^{(1)} + \lambda_j^{(2)} e_j^{(0)}, \\ e_j^{(2)} &= L^{-1} \left(\lambda_j^{(1)} e_j^{(1)} + i(\hat{p} \cdot \omega) e_j^{(1)} \right) + e''_j. \end{aligned} \quad (2.31)$$

Here $e_j'' \in \mathcal{N}$ as (2.28). If we choose e_j' suitably, we have from (2.31) and (2.30) that

$$\lambda_j^{(2)} = - \left\langle i(\hat{p} \cdot \omega) e_j^{(1)}, e_j^{(0)} \right\rangle = \left\langle L^{-1} Q_0(\hat{p} \cdot \omega) E_j, Q_0(\hat{p} \cdot \omega) E_j \right\rangle < 0. \tag{2.32}$$

Continuing to use (2.24) and (2.27), we can obtain any order expansions of eigenvalue and eigenfunction. We also know that $e_j^{(n)}$ with $n \geq 3$ is some linear combinations of E_j plus some terms as $(L^{-1} Q_0(\hat{p} \cdot \omega))^k E_j$ where k is an integer. Next we will prove that the term $e_j^{(n)}$ is almost exponential decay about p , that is, $e_j^{(n)} = \mathcal{O}(|p|^{-\infty})$ for any j and n . In fact we only prove $L^{-1} Q_0(\hat{p} \cdot \omega) E_j$ and the other term can be proved by the induction.

We set

$$L^{-1} Q_0(\hat{p} \cdot \omega) E_j = g \in \mathcal{N}^\perp \quad \text{and} \quad \Lambda^{-1} Q_0(\hat{p} \cdot \omega) E_j = g - \Lambda^{-1} K g$$

For any $\beta > 0$, we have from (2.1) and (2.4) that

$$\|g\|_\beta \leq \|\Lambda^{-1} Q_0(\hat{p} \cdot \omega) E_j\|_\beta + \|\Lambda^{-1} K g\|_\beta \leq \|\Lambda^{-1} Q_0(\hat{p} \cdot \omega) E_j\|_\beta + C \|g\|_{\beta-\eta}.$$

We iterate this estimate successively to get

$$\|g\|_\beta \leq C \|\Lambda^{-1} Q_0(\hat{p} \cdot \omega) E_j\|_\beta + \|K g\|_0 \leq C \|\Lambda^{-1} Q_0(\hat{p} \cdot \omega) E_j\|_\beta + C \|g\|.$$

We have from Lemma 2.3 that

$$c_0 \|\Lambda^{\frac{1}{2}} g\|^2 \leq (Lg, g) = (Q_0(\hat{p} \cdot \omega) E_j, g) \leq C \|\Lambda^{-\frac{1}{2}} Q_0(\hat{p} \cdot \omega) E_j\|^2 + \frac{c_0}{2} \|\Lambda^{\frac{1}{2}} g\|^2.$$

It follows from the above two estimates that

$$\|L^{-1} Q_0(\hat{p} \cdot \omega) E_j\|_\beta = \|g\|_\beta \leq C \|Q_0(\hat{p} \cdot \omega) E_j\|_\beta \leq C_\beta.$$

Thus we prove that $e_j^{(n)} = \mathcal{O}(|p|^{-\infty})$. If we take the constant κ_0 small enough, we know that $e_j(k, p)$ is almost exponential decay about p .

Let C_j be a small circle which encloses $\lambda_j(k)$ with $0 \leq j \leq 4$ but encloses no other eigenvalue. Then for any $f \in L^2(\mathbb{R}_p^3)$, we have from the expression of $e_j(k, p)$ that,

$$\begin{aligned} P_j(k) f &= \frac{1}{2\pi i} \oint_{C_j} (\lambda - \widehat{B}(k))^{-1} f d\lambda = \langle f, e_j(-k, p) \rangle e_j(k, p) \\ &= \langle f, E_j \rangle E_j + |k| \left(\langle f, E_j \rangle e_j^{(1)} + \langle f, e_j^{(1)} \rangle E_j \right) + \mathcal{O}(|k|^2) f. \end{aligned} \tag{2.33}$$

Denote that

$$P_j^{(0)}(\omega)f = \langle f, E_j \rangle E_j, \quad P_j^{(1)}(\omega)f = \langle f, E_j \rangle e_j^{(1)} + \langle f, e_j^{(1)} \rangle E_j. \tag{2.34}$$

We use $P_j^{(2)}(\omega)$ to denote the remaining part of (2.33). Clearly

$$P_0f = \sum_{j=0}^4 P_j^{(0)}(\omega)f = \sum_{j=0}^4 \langle f, E_j \rangle E_j.$$

By virtue of (2.33), (2.34) and the decay of E_j and $e_j^{(n)}$ about p , we can prove (2.23). This completes the proof. □

Lemma 2.6. *For any given $\delta > 0$, the following results hold:*

(i) *If $\text{Re}\lambda \geq -\nu_0 + \delta$, for any $u \in L^2$, we have*

$$\int_{-\infty}^{+\infty} \|(\lambda - \widehat{A}(k))^{-1}u\|^2 d\text{Im}\lambda \leq C\|u\|^2.$$

(ii) *If $\text{Re}\lambda \geq -\nu_0 + \delta$, we have*

$$\sup_{\substack{\text{Re}\lambda \geq -\nu_0 + \delta, \\ \text{Im}\lambda \in \mathbb{R}, k \in \mathbb{R}^3}} \|(\lambda - \widehat{A}(k))^{-1}\| \leq C,$$

and $(\lambda - \widehat{A}(k))^{-1}$ is continuous in (λ, k) .

(iii) *If $\bar{\lambda} = -\sigma_0 + i\tau \in \text{Res}(\widehat{B}(k)) \cap \text{Res}(\widehat{A}(k))$ with any $\tau \in \mathbb{R}$ and a given $\sigma_0 \in (0, \nu_0 - \delta)$, we have that $1 \in \text{Res}((\bar{\lambda} - \widehat{A}(k))^{-1}K)$ and*

$$\sup_{\tau \in \mathbb{R}, k \in \mathbb{R}^3} \|(I - (\bar{\lambda} - \widehat{A}(k))^{-1}K)^{-1}\| \leq C.$$

Here $\|\cdot\|$ is the operator norm in L^2 .

Proof. If $\text{Re}\lambda \geq -\nu_0 + \delta$ for any $\delta > 0$, $(\lambda - \widehat{A}(k))^{-1}$ exists on L^2 . For any $u \in L^2$, one has

$$\begin{aligned} & \int_{-\infty}^{+\infty} \|(\lambda - \widehat{A}(k))^{-1}u\|^2 d\text{Im}\lambda \\ &= \int_{-\infty}^{+\infty} \int_{\mathbb{R}^3} \frac{|u(p)|^2}{|\text{Re}\lambda + \nu(p)|^2 + |\text{Im}\lambda + k \cdot \widehat{p}|^2} dp d\text{Im}\lambda \\ &\leq \int_{\mathbb{R}^3} |u(p)|^2 dp \int_{-\infty}^{+\infty} \frac{1}{\delta^2 + |\text{Im}\lambda + k \cdot \widehat{p}|^2} d\text{Im}\lambda \leq \frac{C}{\delta} \|u\|^2. \end{aligned}$$

For any $u \in L^2$, one has

$$\|(\lambda - \widehat{A}(k))^{-1}u\|^2 = \int_{\mathbb{R}^3} \frac{|u(p)|^2}{|\operatorname{Re}\lambda + \nu(p)|^2 + |\operatorname{Im}\lambda + k \cdot \widehat{p}|^2} dp \leq \frac{C}{\delta^2} \|u\|^2.$$

For any (λ_1, k_1) and (λ_2, k_2) with $\operatorname{Re}\lambda_i \geq -\nu_0 + \delta$ for $i = 1, 2$, we have

$$\|(\lambda_1 - \widehat{A}(k_1))^{-1}u - (\lambda_2 - \widehat{A}(k_2))^{-1}u\| \leq \frac{C}{\delta^2} \|u\| (|\lambda_1 - \lambda_2| + |k_1 - k_2|).$$

Thus $(\lambda - \widehat{A}(k))^{-1}$ is continuous in (λ, k) .

If $\bar{\lambda} = -\sigma_0 + i\tau \in \operatorname{Res}(\widehat{B}(k)) \cap \operatorname{Res}(\widehat{A}(k))$, we claim that $1 \in \operatorname{Res}((\bar{\lambda} - \widehat{A}(k))^{-1}K)$. If not, since $(\bar{\lambda} - \widehat{A}(k))^{-1}K$ is compact in L^2 , $1 \in \sigma_d((\bar{\lambda} - \widehat{A}(k))^{-1}K)$. For some $u \in L^2$, $u = (\bar{\lambda} - \widehat{A}(k))^{-1}Ku$ and $\widehat{B}(k)u = \bar{\lambda}u$. Then $\bar{\lambda}$ is a eigenvalue of $\widehat{B}(k)$ and this is a contradiction by the fact that $\bar{\lambda} \in \operatorname{Res}(B(k))$. Thus $1 \in \operatorname{Res}((\bar{\lambda} - \widehat{A}(k))^{-1}K)$. By (2.9), there exists $k_1 > 0$ such that if $|k| > k_1$ and any $\tau \in \mathbb{R}^3$, we have that $\|(\bar{\lambda} - \widehat{A}(k))^{-1}K\| \leq 1/2$ and

$$\sup_{|k| > k_1, \tau \in \mathbb{R}} \|(I - (\bar{\lambda} - \widehat{A}(k))^{-1}K)^{-1}\| \leq 2.$$

By (2.10), there exists $\tau_1 > 0$ such that if $|\tau| > \tau_1$ and any $|k| \leq k_1$, we have that $\|(\bar{\lambda} - \widehat{A}(k))^{-1}K\| \leq 1/2$ and

$$\sup_{|k| \leq k_1, |\tau| > \tau_1} \|(I - (\bar{\lambda} - \widehat{A}(k))^{-1}K)^{-1}\| \leq 2.$$

By (ii), we have that $(\bar{\lambda} - \widehat{A}(k))^{-1}K$ and $(I - (\bar{\lambda} - \widehat{A}(k))^{-1}K)^{-1}$ are continuous in (τ, k) by the Neumann series. On the compact set $\{|k| \leq k_1, |\tau| \leq \tau_1\}$, we have that

$$\sup_{|k| \leq k_1, |\tau| \leq \tau_1} \|(I - (\bar{\lambda} - \widehat{A}(k))^{-1}K)^{-1}\| \leq C.$$

Thus (iii) holds. This completes the proof of this lemma. □

Lemma 2.7. *There exists a constant $\kappa_0 > 0$ such that the following (i) and (ii) hold for any $g \in D(\widehat{B}(k))$:*

(i) *For any k with $|k| > \kappa_0$, there exists $\sigma_1 > 0$ such that*

$$\|e^{t\widehat{B}(k)}g\| \leq Ce^{-\sigma_1 t} \|g\|. \tag{2.35}$$

(ii) Put $Q(k) = I - \sum_{j=0}^4 P_j(k)$. For any $k, |k| \leq \kappa_0$, there exists $\sigma_2 > 0$ such that

$$\|e^{t\widehat{B}(k)} Q(k)g\| \leq Ce^{-\sigma_2 t} \|g\|. \tag{2.36}$$

Proof. For any $\lambda \in \text{Res}(\widehat{B}(k)) \cap \text{Res}(\widehat{A}(k))$, we have from (2.14) and (2.15) that

$$\begin{aligned} (\lambda - \widehat{B}(k))^{-1} &= (\lambda - \widehat{A}(k))^{-1} + (\lambda - \widehat{A}(k))^{-1} K (I - (\lambda - \widehat{A}(k))^{-1} K)^{-1} (\lambda - \widehat{A}(k))^{-1} \\ &=: (\lambda - \widehat{A}(k))^{-1} + Z(\lambda, k). \end{aligned} \tag{2.37}$$

By the Laplace inverse transform, for any $y_1 > 0, t > 0$ and any $g \in L^2(\mathbb{R}_p^3)$, one has

$$e^{t\widehat{B}(k)} g = \frac{1}{2\pi i} \int_{y_1 - i\infty}^{y_1 + i\infty} e^{\lambda t} (\lambda - \widehat{B}(k))^{-1} g(p) d\lambda. \tag{2.38}$$

By this and (2.37), we have

$$e^{t\widehat{B}(k)} g = e^{t\widehat{A}(k)} g + \lim_{y_2 \rightarrow \infty} \frac{1}{2\pi i} \int_{y_1 - iy_2}^{y_1 + iy_2} e^{\lambda t} Z(\lambda, k) g(p) d\lambda. \tag{2.39}$$

It follows from (2.1) that

$$\|e^{t\widehat{A}(k)} g\| \leq e^{-\nu_0 t} \|g\|. \tag{2.40}$$

We shift the integral path of (2.39) from $\text{Re}\lambda = y_1$ to $\text{Re}\lambda = -\tau_0$ where $\tau_0 > 0$ is the one given in Lemma 2.4. We can choose τ_0 such that the contour integral of $e^{\lambda t} Z(\lambda, k)$ on the rectangular path does not contain the eigenvalues when $|k| > \kappa_0$. Thus we arrive at

$$\begin{aligned} &\int_{y_1 - iy_2}^{y_1 + iy_2} e^{\lambda t} Z(\lambda, k) g(p) d\lambda \\ &= \int_{-\tau_0 - iy_2}^{-\tau_0 + iy_2} e^{\lambda t} Z(\lambda, k) g(p) d\lambda + \int_{-\tau_0 + iy_2}^{y_1 + iy_2} e^{\lambda t} Z(\lambda, k) g(p) d\lambda \\ &\quad + \int_{y_1 - iy_2}^{-\tau_0 - iy_2} e^{\lambda t} Z(\lambda, k) g(p) d\lambda. \end{aligned} \tag{2.41}$$

By Lemma 2.1, we have that

$$\lim_{y_2 \rightarrow \infty} \sup_{\substack{\text{Re}\lambda \geq -\tau_0, \\ k \in \mathbb{R}^3}} \|(\text{Re}\lambda \pm iy_2 - \widehat{A}(k))^{-1} K\| = 0.$$

It follows from this that

$$\sup_{\substack{\operatorname{Re}\lambda \geq -\tau_0, \\ k \in \mathbb{R}^3}} \|(I - (\operatorname{Re}\lambda \pm iy_2 - \widehat{A}(k))^{-1}K)^{-1}\| \leq C,$$

if y_2 is large enough. By this, (2.37) and Lemma 2.6(ii), we have

$$\lim_{y_2 \rightarrow \infty} \sup_{\substack{\operatorname{Re}\lambda \geq -\tau_0, \\ k \in \mathbb{R}^3}} \|Z(\operatorname{Re}\lambda \pm iy_2, k)\| \leq C \lim_{y_2 \rightarrow \infty} \sup_{\substack{\operatorname{Re}\lambda \geq -\tau_0, \\ k \in \mathbb{R}^3}} \|(\operatorname{Re}\lambda \pm iy_2 - \widehat{A}(k))^{-1}K\| = 0.$$

It follows from this that

$$\left\| \int_{-\tau_0+iy_2}^{y_1+iy_2} e^{\lambda t} Z(\lambda, k) g(p) d\lambda \right\| + \left\| \int_{y_1-iy_2}^{-\tau_0-iy_2} e^{\lambda t} Z(\lambda, k) g(p) d\lambda \right\| \rightarrow 0 \text{ as } y_2 \rightarrow \infty.$$

Next we shall consider the first term of (2.41). Since $\{\lambda; \operatorname{Re}\lambda = -\tau_0\} \subset \operatorname{Res}(\widehat{A}(k))$ and $\widehat{B}(k)$ has no eigenvalue on the line $\operatorname{Re}\lambda = -\tau_0$ for $|\operatorname{Im}\lambda|$ being bounded, $-\tau_0 \pm iy_2 \in \operatorname{Res}(\widehat{A}(k)) \cap \operatorname{Res}(\widehat{B}(k))$. By Lemma 2.6(iii) and (2.37), for any $g, h \in L^2(\mathbb{R}_p^3)$, one has

$$\begin{aligned} & \left| \left\langle \int_{-\tau_0-iy_2}^{-\tau_0+iy_2} e^{\lambda t} Z(\lambda, k) g(p) d\lambda, h \right\rangle \right| = \left| \left\langle \int_{-y_2}^{y_2} e^{(-\tau_0+is)t} Z(-\tau_0+is, k) g(p) ds, h \right\rangle \right| \\ & \leq C e^{-\tau_0 t} \|K\| \int_{-y_2}^{y_2} \|(-\tau_0+is - \widehat{A}(k))^{-1} g\| \|(-\tau_0+is - \widehat{A}(-k))^{-1} h\| ds, \end{aligned}$$

which is less than $C e^{-\tau_0 t} \|g\| \|h\|$ by Lemma 2.6(i). By (2.39) and the above estimates, (2.35) holds. By using Lemmas 2.4 and 2.5 and the similar arguments as the above, we can prove that (2.36) holds and we omit the proof. This completes the proof of this lemma. \square

Remark 2.1. The time decay rates of the solutions to the linearized relativistic Boltzmann equation have been proved in [20]. Since we know the precise spectrum structure of the linearized equation in Lemmas 2.4 and 2.5, we can make use of the arguments as the above and the direct calculations to get the time decay rates of the solutions to the linearized equation as that in [20].

Lemma 2.8. *There exist constants $\kappa_0 > 0, \sigma_0 > 0$ and $C > 0$ such that the following results hold:*

(i) For $|k| \leq \kappa_0$, one has

$$e^{t\widehat{B}(k)} = \sum_{j=0}^4 e^{\lambda_j(|k|)t} P_j(k) + U(t, k),$$

where $\lambda_j(|k|)$ and $P_j(k)$ are in Lemma 2.5. The operator $U = U(t, k)$ is given by

$$U = e^{t\widehat{B}(k)}Q(k) = Q(k)e^{t\widehat{B}(k)},$$

and for any $\beta \in \mathbb{R}$, the operator U has the decomposition

$$U = e^{t\widehat{A}(k)}Q(k) + U_1(t, k), \quad \|U_1u\|_\beta \leq Ce^{-\sigma_0 t}(\|u\|_{\beta-\eta} + \|u\|).$$

(ii) For any $|k| > \kappa_0$, we have

$$e^{t\widehat{B}(k)} = e^{t\widehat{A}(k)} + U_2(t, k), \quad \|U_2u\|_\beta \leq Ce^{-\sigma_0 t}(\|u\|_{\beta-\eta} + \|u\|).$$

Proof. For any $\lambda \in \text{Res}(\widehat{A}(k)) \cap \text{Res}(\widehat{B}(k))$, we have

$$(\lambda - \widehat{B}(k))^{-1} = (\lambda - \widehat{A}(k))^{-1} + (\lambda - \widehat{A}(k))^{-1}K(\lambda - \widehat{B}(k))^{-1}.$$

Taking the inverse Laplace transform on both sides of this yields

$$e^{t\widehat{B}(k)} = e^{t\widehat{A}(k)} + e^{t\widehat{A}(k)}K * e^{t\widehat{B}(k)} \quad (2.42)$$

for all $t \geq 0$, where $*$ means the convolution in t ,

$$g * h = \int_0^t g(t-s)h(s)ds. \quad (2.43)$$

Then we iterate it $l-1$ times with $l \in \mathbb{N}_+$ to obtain that

$$e^{t\widehat{B}(k)} = e^{t\widehat{A}(k)} + \sum_{\ell=1}^{l-1} \left\{ [e^{t\widehat{A}(k)}K *]^{\ell} e^{t\widehat{A}(k)} \right\} + [e^{t\widehat{A}(k)}K *]^l e^{t\widehat{B}(k)}. \quad (2.44)$$

It follows from (2.1) that $\|e^{t\widehat{A}(k)}u\| \leq Ce^{-\nu_0 t}\|u\|$. For some fixed $l \in \mathbb{N}_+$ with $l \geq |\beta|/\eta + 2$ and any $\ell = 1, \dots, l-1$, we have from (2.1) and (2.4) that

$$\left\| [e^{t\widehat{A}(k)}K *]^{\ell} e^{t\widehat{A}(k)}u \right\|_{\beta} \leq Ct^{\ell}e^{-\nu_0 t}\|u\|_{\beta-\ell\eta} \leq Ce^{-\sigma_3 t}\|u\|_{\beta-\eta}.$$

For any $|k| > \kappa_0$, one has from Lemma 2.7 that $\|e^{t\widehat{B}(k)}u\| \leq Ce^{-\sigma_1 t}\|u\|$. By using this, (2.1), (2.4) and the fact that K is bounded in $L^2(\mathbb{R}_p^3)$, we have from the direct calculations that

$$\left\| [e^{t\widehat{A}(k)}K *]^l e^{t\widehat{B}(k)}u \right\|_{\beta} \leq Ce^{-\sigma_4 t}\|u\|.$$

For any $|k| > \kappa_0$, if we denote the last two term in (2.44) by $U_2(t, k)$ and $\sigma_0 = \min\{\sigma_3, \sigma_4\}$, we have from (2.44) and the above estimates that (ii) holds.

For any $|k| \leq \kappa_0$, since $P_j(k)$ with $j=0, 1, \dots, 4$ is the eigen-projection corresponding to the eigenvalue $\lambda_j(|k|)$ of the operator $\widehat{B}(k)$ by Lemma 2.5, we have

$$\widehat{B}(k)P_j(k)u = \lambda_j(|k|)P_j(k)u.$$

Thus one has

$$(\lambda - \widehat{B}(k))P_j(k)u = (\lambda - \lambda_j(|k|))P_j(k)u,$$

which implies, for any $\lambda \in \text{Res}(\widehat{B}(k))$,

$$(\lambda - \widehat{B}(k))^{-1}P_j(k)u = (\lambda - \lambda_j(|k|))^{-1}P_j(k)u.$$

It follows that:

$$(\lambda - \widehat{B}(k))^{-1}u = \sum_{j=0}^4 (\lambda - \lambda_j(|k|))^{-1}P_j(k)u + (\lambda - \widehat{B}(k))^{-1}Q(k)u.$$

By using the inverse Laplace transform on both sides and (2.44), we arrive at

$$\begin{aligned} e^{t\widehat{B}(k)}u &= \sum_{j=0}^4 e^{\lambda_j(|k|)t}P_j(k)u + e^{t\widehat{B}(k)}Q(k)u \\ &= \sum_{j=0}^4 e^{\lambda_j(|k|)t}P_j(k)u + e^{t\widehat{A}(k)}Q(k)u + \sum_{\ell=1}^{l-1} \left\{ [e^{t\widehat{A}(k)}K*]^\ell e^{t\widehat{A}(k)} \right\} Q(k)u \\ &\quad + [e^{t\widehat{A}(k)}K*]^l e^{t\widehat{B}(k)}Q(k)u. \end{aligned} \tag{2.45}$$

We denote $U(t, k)$ by the sum of the three terms in (2.45). For any $|k| \leq \kappa_0$, one has from Lemma 2.7 that $\|e^{t\widehat{B}(k)}Q(k)u\| \leq Ce^{-\sigma_1 t}\|u\|$. It follows from (2.45) and the similar arguments as (ii) that (i) holds. The proof of the lemma is complete. \square

Notice that $\widehat{B}(k) = \widehat{B}^1(k)$ and $\widehat{B}^\epsilon(k) = \widehat{B}(\epsilon k)/\epsilon$. By Lemmas 2.4, 2.5 and 2.8, we have the spectral analysis results of the operator $\widehat{B}^\epsilon(k)$.

Theorem 2.1. *There exist positive constants κ_0 , σ_0 and $C > 0$ such that the followings hold for each $\epsilon > 0$:*

(i) *For any $|k| \leq \kappa_0/\epsilon$, we have that*

$$e^{t\widehat{B}^\epsilon(k)} = \sum_{j=0}^4 e^{\lambda_j(|\epsilon k|)\frac{t}{\epsilon}}P_j(\epsilon k) + U(\epsilon, t, k),$$

where

- (a) Let $\lambda_j(|\epsilon k|)$ and $e_j(|\epsilon k|)$ with $0 \leq j \leq 4$ be the eigenvalue and the corresponding eigenfunction of the operator $\widehat{B}^\epsilon(k)$. And $\lambda_j(|\epsilon k|)$ and $e_j(|\epsilon k|, p)$ are analytic in $|k|$ as in Lemma 2.5 and has the following asymptotic expansions:

$$\begin{aligned}\lambda_j(|\epsilon k|) &= \lambda_j^{(1)} |\epsilon k| + \lambda_j^{(2)} |\epsilon k|^2 + \mathcal{O}(|\epsilon k|^3), & |\epsilon k| \rightarrow 0, \\ e_j(|\epsilon k|, p) &= e_j(|\epsilon k|, \hat{p} \cdot \omega, p) = e_j^{(0)} + |\epsilon k| e_j^{(1)} + \mathcal{O}(|\epsilon k|^2), & |\epsilon k| \rightarrow 0\end{aligned}$$

with the coefficients $\lambda_j^{(1)} = -ic_j$ with $c_j \in \mathbb{R}$ and $\lambda_j^{(2)} \in \mathbb{R}$ with $\lambda_j^{(2)} < 0$.

- (b) The eigen-projection corresponding to the eigenvalue $\lambda_j(\epsilon|k|)$ on $L^2(\mathbb{R}_p^3)$ is denoted by $P_j(\epsilon k)$. For each j , one has

$$P_j(\epsilon k) = P_j^{(0)}(\omega) + |\epsilon k| P_j^{(1)}(\omega) + |\epsilon k|^2 P_j^{(2)}(\omega).$$

Moreover, for any $\beta', \beta \in \mathbb{R}$, one has

$$\left\| P_j^{(l)}(\omega) f \right\|_{\beta'} \leq C \|f\|_{\beta}, \quad l = 0, 1, 2.$$

- (c) Putting $Q(\epsilon k) = I - \sum_{j=0}^4 P_j(\epsilon k)$, the operator $U = U(\epsilon, t, k)$ is given by

$$U = e^{t\widehat{B}^\epsilon(k)} Q(\epsilon k) = Q(\epsilon k) e^{t\widehat{B}^\epsilon(k)},$$

and the operator U has the decomposition

$$U = e^{t\widehat{A}^\epsilon(k)} Q(\epsilon k) + U_1(\epsilon, t, k), \quad \|U_1 u\|_{\beta} \leq C e^{-\frac{\sigma_0 t}{\epsilon}} (\|u\|_{\beta-\eta} + \|u\|).$$

- (ii) For any $|k| > \kappa_0/\epsilon$, one has

$$e^{t\widehat{B}^\epsilon(k)} = e^{t\widehat{A}^\epsilon(k)} + U_2(\epsilon, t, k), \quad \|U_2 u\|_{\beta} \leq C e^{-\frac{\sigma_0 t}{\epsilon}} (\|u\|_{\beta-\eta} + \|u\|).$$

This theorem is main results in this section and it will be used crucially in the later analysis.

3 Existence of local solutions

In this section, we shall prove the existence of the solution $f^\epsilon(t)$ to the relativistic Boltzmann equation for any $\epsilon > 0$ in a finite time interval independent of ϵ . We make use of the spectral analysis results in the previous section and the contraction mapping principle to (2.7). For this, we first consider the first term in (2.7).

Lemma 3.1. *Let $f_0 \in X_\beta^{\alpha,l}$ with $\alpha > 0, l > 0$ and $\beta > 3/2$. For any $t > 0, \gamma > 0$ and $\alpha - \gamma t > 0$, one has*

$$\|e^{tB^\epsilon} f_0\|_{\alpha-\gamma t, l, \beta} \leq C \|f_0\|_{\alpha, l, \beta}. \tag{3.1}$$

Proof. By virtue of Theorem 2.1, we define

$$\begin{cases} e^{tB^\epsilon} f_0 = \sum_{j=0}^6 E_j^\epsilon(t) f_0, \\ \mathcal{F}_x E_j^\epsilon(t) f_0 = \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) e^{\lambda_j(\epsilon|k|)\frac{t}{\epsilon}} P_j(\epsilon k) \widehat{f}_0, \quad j=0,1,\dots,4, \\ \mathcal{F}_x E_5^\epsilon(t) f_0 = \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) U(\epsilon, t, k) \widehat{f}_0, \\ \mathcal{F}_x E_6^\epsilon(t) f_0 = \chi\left(|k| > \frac{\kappa_0}{\epsilon}\right) e^{t\widehat{B}^\epsilon} \widehat{f}_0. \end{cases} \tag{3.2}$$

By using Theorem 2.1(i), we have

$$\left\| \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) e^{\lambda_j(\epsilon|k|)\frac{t}{\epsilon}} P_j(\epsilon k) \widehat{f}_0 \right\|_\beta \leq C \|\widehat{f}_0\|_\beta.$$

For any $\beta > 3/2$, we easily see

$$\left\| \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) U(\epsilon, t, k) \widehat{f}_0 \right\|_\beta + \left\| \chi\left(|k| > \frac{\kappa_0}{\epsilon}\right) e^{t\widehat{B}^\epsilon} \widehat{f}_0 \right\|_\beta \leq C \|\widehat{f}_0\|_\beta + C \|\widehat{f}_0\| \leq C \|\widehat{f}_0\|_\beta.$$

These estimates, (3.2) and (1.31) imply that this lemma holds. This completes the proof. \square

Recalling (2.1), (3.2), (2.43) and $Q_0 = I - P_0$, we define the operator H^ϵ by

$$H^\epsilon f^\epsilon(t) := e^{tB^\epsilon} * Q_0 \frac{1}{\epsilon} \Lambda f^\epsilon(t). \tag{3.3}$$

In what follows we assume that

$$\alpha > 0, \quad \gamma > 0, \quad \tau = \frac{\alpha}{\gamma}, \quad t \in [0, \tau]. \tag{3.4}$$

Lemma 3.2. *Letting $f^\epsilon(t) \in Y_\beta^{\alpha, \gamma, l}([0, \tau])$ with any $l > 0$ and $\beta > 5/2$, for any $t \in [0, \tau]$, one has*

$$\|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l, \beta, t} \leq C \left(1 + \frac{1}{\gamma}\right) \|f^\epsilon\|_{\alpha, \gamma, l, \beta, t} \tag{3.5}$$

where the constant $C > 0$ is independent of γ and t .

Proof. Thanks to (3.2), we can rewrite (3.3) as

$$H_j^\epsilon f^\epsilon(t) := E_j^\epsilon(t) * Q_0 \frac{1}{\epsilon} \Lambda f^\epsilon(t), \quad 0 \leq j \leq 6. \tag{3.6}$$

For $0 \leq j \leq 4$, by Theorem 2.1(i), one has

$$P_j(\epsilon k) Q_0 = |\epsilon k| \left(P_j^{(1)}(\omega) + |\epsilon k| P_j^{(2)}(\omega) \right) Q_0.$$

And for any $\beta' > 0$, we also have

$$\|P_j(\epsilon k) Q_0 \Lambda \widehat{f}^\epsilon(s)\|_{\beta'} \leq C |\epsilon k| \|\widehat{f}^\epsilon(s)\|_{\beta'}.$$

For $0 \leq j \leq 4$, we have from these facts that

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^l (1+|p|)^{\beta-1} |\mathcal{F}_x H_j^\epsilon f^\epsilon(t)| \\ & \leq e^{(\alpha-\gamma t)|k|} (1+|k|)^l \int_0^t \left\| \left(|k| \leq \frac{\kappa_0}{\epsilon} \right) e^{\lambda_j(\epsilon|k|)\frac{(t-s)}{\epsilon}} P_j(\epsilon k) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(s) \right\|_{\beta-1} ds \\ & \leq C \int_0^t e^{-\gamma|k|(t-s)} |k| \|f^\epsilon(s)\|_{\alpha-\gamma s, l, \beta-1} ds \leq \frac{C}{\gamma} \sup_{s \in [0, t]} \|f^\epsilon\|_{\alpha-\gamma s, l, \beta-1}. \end{aligned}$$

For $0 \leq j \leq 4$, one has from this that

$$\|H_j^\epsilon f^\epsilon(t)\|_{\alpha-\gamma t, l, \beta-1} \leq \frac{C}{\gamma} \sup_{s \in [0, t]} \|f^\epsilon(s)\|_{\alpha-\gamma s, l, \beta-1}. \tag{3.7}$$

It follows from (3.6), (3.2) and Theorem 2.1 that

$$\mathcal{F}_x H_5^\epsilon f^\epsilon(t) = \int_0^t \chi \left(|k| \leq \frac{\kappa_0}{\epsilon} \right) \left(e^{(t-s)\widehat{A}^\epsilon(k)} Q(\epsilon k) + U_1(\epsilon, t-s, k) \right) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(s) ds. \tag{3.8}$$

By virtue of Theorem 2.1(i), one has

$$\|\Lambda^{-1} Q(\epsilon k) Q_0 \Lambda \widehat{f}^\epsilon(s)\|_{\beta-1} \leq C \|\widehat{f}^\epsilon(s)\|_{\beta-1}.$$

For the first term in (3.8), we have from Theorem 2.1 and this that

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^l (1+|p|)^{\beta-1} \left| \int_0^t \chi \left(|k| \leq \frac{\kappa_0}{\epsilon} \right) e^{(t-s)\widehat{A}^\epsilon(k)} Q(\epsilon k) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(s) ds \right| \\ & \leq C e^{(\alpha-\gamma t)|k|} (1+|k|)^l (1+|p|)^{\beta-1} \\ & \quad \times \int_0^t \chi \left(|k| \leq \frac{\kappa_0}{\epsilon} \right) e^{-\frac{\nu(p)}{\epsilon}(t-s)} \frac{\nu(p)}{\epsilon} |\Lambda^{-1} Q(\epsilon k) Q_0 \Lambda \widehat{f}^\epsilon(s)| ds \\ & \leq C \sup_{s \in [0, t]} \|f^\epsilon\|_{\alpha-\gamma s, l, \beta-1} \int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)} \frac{\nu(p)}{\epsilon} ds \leq C \sup_{s \in [0, t]} \|f^\epsilon\|_{\alpha-\gamma s, l, \beta-1}. \end{aligned} \tag{3.9}$$

For the second term in (3.8), for any $\beta > 5/2$, we have from (2.1) and Theorem 2.1 that

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|}(1+|k|)^l(1+|p|)^{\beta-1} \left| \int_0^t \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) U_1(\epsilon, t-s, k) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(s) ds \right| \\ & \leq e^{(\alpha-\gamma t)|k|}(1+|k|)^l \int_0^t \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) \left\| U_1(\epsilon, t-s, k) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(s) \right\|_{\beta-1} ds \\ & \leq e^{(\alpha-\gamma t)|k|}(1+|k|)^l \int_0^t e^{-\frac{\sigma_0}{\epsilon}(t-s)} \frac{1}{\epsilon} \left(\left\| Q_0 \Lambda \widehat{f}^\epsilon(s) \right\|_{\beta-1-\eta} + \left\| Q_0 \Lambda \widehat{f}^\epsilon(s) \right\| \right) ds \\ & \leq C \sup_{s \in [0, t]} \|f^\epsilon\|_{\alpha-\gamma s, l, \beta} \int_0^t e^{-\frac{\sigma_0}{\epsilon}(t-s)} \frac{1}{\epsilon} ds \leq C \sup_{s \in [0, t]} \|f^\epsilon\|_{\alpha-\gamma s, l, \beta}. \end{aligned} \tag{3.10}$$

By using (3.8)-(3.10), we can obtain

$$\|H_5^\epsilon f^\epsilon(t)\|_{\alpha-\gamma t, l, \beta-1} \leq C \sup_{s \in [0, t]} \|f^\epsilon(s)\|_{\alpha-\gamma s, l, \beta}. \tag{3.11}$$

By the similar arguments as (3.11), the estimate of (3.11) is true for $H_6^\epsilon f^\epsilon$. By using this, (3.11), (3.7), (3.6), and (3.3), we arrive at

$$\|H^\epsilon f^\epsilon(t)\|_{\alpha-\gamma t, l, \beta-1} \leq C \left(1 + \frac{1}{\gamma}\right) \sup_{s \in [0, t]} \|f^\epsilon(s)\|_{\alpha-\gamma s, l, \beta}.$$

If we use any $t_1 \in [0, t]$ to replace t in the above estimate, we can obtain the similar estimate. Thus we have from this fact that

$$\|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l, \beta-1, t} \leq C \left(1 + \frac{1}{\gamma}\right) \|f^\epsilon\|_{\alpha, \gamma, l, \beta, t}. \tag{3.12}$$

We shall recover the loss of the p -weight by the smoothing property of K in (2.4). It follows from (2.42) and (3.3) that

$$\mathcal{F}_x H^\epsilon f^\epsilon(t) = \int_0^t e^{(t-s)\widehat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(s) ds + \int_0^t e^{(t-s)\widehat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x H^\epsilon f^\epsilon(s) ds. \tag{3.13}$$

For the first term in (3.13), one has

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|}(1+|k|)^l(1+|p|)^\beta \left| \int_0^t e^{(t-s)\widehat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(s) ds \right| \\ & \leq C e^{(\alpha-\gamma t)|k|}(1+|k|)^l(1+|p|)^\beta \int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)} \frac{\nu(p)}{\epsilon} |\Lambda^{-1} Q_0 \Lambda \widehat{f}^\epsilon(s)| ds \\ & \leq C \sup_{s \in [0, t]} \|f^\epsilon\|_{\alpha-\gamma s, l, \beta} \int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)} \frac{\nu(p)}{\epsilon} ds \leq C \sup_{s \in [0, t]} \|f^\epsilon\|_{\alpha-\gamma s, l, \beta}. \end{aligned} \tag{3.14}$$

For the second term, one has from (2.4) that

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|}(1+|k|)^l(1+|p|)^\beta \left| \int_0^t e^{(t-s)\widehat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x H^\epsilon f^\epsilon(s) ds \right| \\ & \leq C e^{(\alpha-\gamma t)|k|}(1+|k|)^l \int_0^t e^{-\frac{\nu_0}{\epsilon}(t-s)} \frac{1}{\epsilon} \|\mathcal{F}_x H^\epsilon f^\epsilon(s)\|_{\beta-\eta} ds \\ & \leq C \sup_{s \in [0,t]} \|H^\epsilon f^\epsilon(s)\|_{\alpha-\gamma s, l, \beta-\eta}. \end{aligned} \tag{3.15}$$

Thus, for any $t \in [0, \tau]$, we have from (1.31) and (3.13)-(3.15) that

$$\|H^\epsilon f^\epsilon(t)\|_{\alpha-\gamma t, l, \beta} \leq C \|f^\epsilon\|_{\alpha, \gamma, l, \beta, t} + C \|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l, \beta-\eta, t}.$$

It follows from this that

$$\|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l, \beta, t} \leq C \|f^\epsilon\|_{\alpha, \gamma, l, \beta, t} + C \|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l, \beta-\eta, t}. \tag{3.16}$$

Noticing that $\eta \in (0, 1]$, we iterate the above estimate in finite times to get

$$\|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l, \beta, t} \leq C \|f^\epsilon\|_{\alpha, \gamma, l, \beta, t} + C \|H^\epsilon f^\epsilon(s)\|_{\alpha, \gamma, l, \beta-1, t}. \tag{3.17}$$

Combining this and (3.12), we conclude the proof of (3.5). □

Lemma 3.3. *Suppose that any $l > 3, \beta \geq 0$, and $h_1, h_2 \in Y_\beta^{\alpha, \gamma, l}([0, \tau])$. Then one has*

$$\|\Lambda^{-1}\Gamma(h_1, h_2)\|_{\alpha-\gamma t, l, \beta} \leq C \|h_1\|_{\alpha-\gamma t, l, \beta} \|h_2\|_{\alpha-\gamma t, l, \beta}. \tag{3.18}$$

Proof. It is easily seen that

$$\mathcal{F}_x \Lambda^{-1}\Gamma(h_1, h_2) = (2\pi)^{\frac{3}{2}} \int_{\mathbb{R}^3} \Lambda^{-1}\Gamma(\widehat{h}_1(t, k-k', \cdot), \widehat{h}_2(t, k', \cdot)) dk'.$$

By [19, Theorem 2.1], one has

$$\|\Lambda^{-1}\Gamma(h_1, h_2)\|_\beta \leq C \|h_1\|_\beta \|h_2\|_\beta. \tag{3.19}$$

By this, we arrive at

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|}(1+|k|)^l(1+|p|)^\beta |\mathcal{F}_x \Lambda^{-1}\Gamma(h_1, h_2)| \\ & \leq C e^{(\alpha-\gamma t)|k|}(1+|k|)^l \int_{\mathbb{R}^3} \|\widehat{h}_1(t, k-k', \cdot)\|_\beta \|\widehat{h}_2(t, k', \cdot)\|_\beta dk' \\ & \leq C \|h_1\|_{\alpha-\gamma t, l, \beta} \|h_2\|_{\alpha-\gamma t, l, \beta} e^{(\alpha-\gamma t)|k|}(1+|k|)^l \\ & \quad \times \int_{\mathbb{R}^3} (1+|k-k'|)^{-l} (1+|k'|)^{-l} e^{-(\alpha-\gamma t)(|k-k'|+|k'|)} dk' \\ & \leq C \|h_1\|_{\alpha-\gamma t, l, \beta} \|h_2\|_{\alpha-\gamma t, l, \beta} (1+|k|)^l \int_{\mathbb{R}^3} (1+|k-k'|)^{-l} (1+|k'|)^{-l} dk' \\ & \leq C \|h_1\|_{\alpha-\gamma t, l, \beta} \|h_2\|_{\alpha-\gamma t, l, \beta} \end{aligned}$$

where we have used the inequality that for any $l > 3$,

$$(1 + |k|)^l \int_{\mathbb{R}^3} (1 + |k - k'|)^{-l} (1 + |k'|)^{-l} dk' \leq C.$$

This completes the proof of the lemma. □

Theorem 3.1. *Suppose that $\beta > 5/2, l > 3$ and (3.4) holds. Then there are positive constants a_0 and a_1 such that if $f_0 \in X_\beta^{\alpha, l}$ and $\|f_0\|_{\alpha, l, \beta} < a_0$ and for any $\epsilon > 0$, (2.7) has a unique solution $f^\epsilon \in Y_\beta^{\alpha, \gamma, l}([0, \tau])$ with $\|f^\epsilon\|_{\alpha, \gamma, l, \beta, \tau} \leq a_1 \|f_0\|_{\alpha, l, \beta}$.*

Proof. By (2.7), we define the nonlinear map N^ϵ by

$$N^\epsilon[f^\epsilon](t) = e^{tB^\epsilon} f_0 + H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon, f^\epsilon). \tag{3.20}$$

Note that $Q_0 \Gamma(f^\epsilon, f^\epsilon) = \Gamma(f^\epsilon, f^\epsilon)$. By Lemmas 3.1-3.3, one has

$$\|N^\epsilon[f^\epsilon]\|_{\alpha, \gamma, l, \beta, \tau} \leq C_1 \|f_0\|_{\alpha, l, \beta} + C_2 \left(1 + \frac{1}{\gamma}\right) \|f^\epsilon\|_{\alpha, \gamma, l, \beta, \tau}^2 \tag{3.21}$$

and

$$\begin{aligned} \|N^\epsilon[f^\epsilon] - N^\epsilon[h^\epsilon]\|_{\alpha, \gamma, l, \beta, \tau} &= \|H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon + h^\epsilon, f^\epsilon - h^\epsilon)\|_{\alpha, \gamma, l, \beta, \tau} \\ &\leq C_2 \left(1 + \frac{1}{\gamma}\right) \|f^\epsilon + h^\epsilon\|_{\alpha, \gamma, l, \beta, \tau} \|f^\epsilon - h^\epsilon\|_{\alpha, \gamma, l, \beta, \tau}. \end{aligned} \tag{3.22}$$

Let $a_0 = (4C_1 C_2)^{-1}$ and $\|f_0\|_{\alpha, l, \beta} < a_0$. We choose $\gamma > 0$ such that

$$\gamma > \frac{\|f_0\|_{\alpha, l, \beta}}{a_0 - \|f_0\|_{\alpha, l, \beta}}.$$

If we put

$$a_1 = \left(2C_2 \left(1 + \frac{1}{\gamma}\right) \|f_0\|_{\alpha, l, \beta}\right)^{-1} \mu, \quad \mu = 1 - \left(1 - \left(1 + \frac{1}{\gamma}\right) \frac{1}{a_0} \|f_0\|_{\alpha, l, \beta}\right)^{\frac{1}{2}},$$

then $\mu \in (0, 1)$. Denote by Y_0 the closed ball

$$Y_0 = \left\{ f^\epsilon \in Y_\beta^{\alpha, \gamma, l}([0, \tau]) \mid \|f^\epsilon\|_{\alpha, \gamma, l, \beta, \tau} \leq a_1 \|f_0\|_{\alpha, l, \beta} \right\}$$

in the space $Y_\beta^{\alpha,\gamma,l}([0,\tau])$. Clearly, Y_0 is a complete metric space with the metric induced by the norm $\|\cdot\|_{\alpha,\gamma,l,\beta,\tau}$. From the above estimates, it follows that for any $f^\epsilon, h^\epsilon \in Y_0$,

$$\|N^\epsilon[f^\epsilon]\|_{\alpha,\gamma,l,\beta,\tau} \leq a_1 \|f_0\|_{\alpha,l,\beta}, \tag{3.23}$$

and

$$\begin{aligned} \|N^\epsilon[f^\epsilon] - N^\epsilon[h^\epsilon]\|_{\alpha,\gamma,l,\beta,\tau} &\leq 2C_2 \left(1 + \frac{1}{\gamma}\right) a_1 \|f_0\|_{\alpha,l,\beta} \|f^\epsilon - h^\epsilon\|_{\alpha,\gamma,l,\beta,\tau} \\ &\leq \mu \|f^\epsilon - h^\epsilon\|_{\alpha,\gamma,l,\beta,\tau}. \end{aligned} \tag{3.24}$$

Therefore N^ϵ is a contraction map on Y_0 if $\|f_0\|_{\alpha,\gamma,l,\beta} < a_0$, and has a unique fixed point $f^\epsilon = f^\epsilon(t) \in Y_0$. This complete the proof of this theorem. \square

In what follows we shall prove that $f = f^\epsilon(t)$ constructed in Theorem 3.1 is a classical solution to (1.27) if $f_0 \in \dot{X}_\beta^{\alpha,l}$. Once we prove this, we complete the proof of Theorem 1.1(i). For this we will prove the following theorem.

Theorem 3.2. *Let f^ϵ and f_0 be those of Theorem 3.1. If, in addition, $f_0 \in \dot{X}_\beta^{\alpha,l}$, then $f = f^\epsilon \in Z_{\beta,\tau}^{\alpha,\gamma,l}$ and is a classical solution to (1.27).*

Proof. Let $\widehat{X}_\beta^{\alpha,l}$ denote the space of $\widehat{f} \in S'(\mathbb{R}_x^3 \times \mathbb{R}_p^3)$ satisfying (1.31) and the infinity condition in (1.32). It is a Banach space with the norm (1.31) and \mathcal{F}_x is isometric from $\dot{X}_\beta^{\alpha,l}$ to $\widehat{X}_\beta^{\alpha,l}$. Thus if $\widehat{A}^\epsilon(k)$ is a generator in $\widehat{X}_\beta^{\alpha,l}$, so is A^ϵ .

Notice that $|e^{-(ik \cdot p + \nu(p)/\epsilon)t}| \leq e^{-\nu_0 t/\epsilon}$. It holds that

$$\left\| e^{-(ik \cdot p + \frac{1}{\epsilon}\nu(p))t} \widehat{f} \right\|_{\beta'} \leq e^{-\frac{\nu_0 t}{\epsilon}} \|\widehat{f}\|_{\beta},$$

and for any compact set $\Omega \subset \mathbb{R}_k^3 \times \mathbb{R}_p^3$,

$$e^{-(ik \cdot p + \frac{1}{\epsilon}\nu(p))t} \in B^0((0,\infty)_\epsilon \times [0,\infty)_t; L^\infty(\Omega)).$$

By this and the infinity condition in (1.32) we prove

$$e^{t\widehat{A}^\epsilon(k)} \widehat{f} = e^{-(ik \cdot p + \frac{1}{\epsilon}\nu(p))t} \widehat{f} \in B^0\left((0,\infty)_\epsilon \times [0,\infty)_t; \widehat{X}_\beta^{\alpha,l}\right). \tag{3.25}$$

This continuity in t shows that $e^{t\widehat{A}^\epsilon(k)}$ defines a semigroup on $\widehat{X}_\beta^{\alpha,l}$. It is generator is $\widehat{A}^\epsilon(k)$ with the domain $D(\widehat{A}^\epsilon(k))$. Hence, for any $\epsilon > 0$, A^ϵ is a generator

in $\dot{X}_\beta^{\alpha,l}$ with the domain $D(A^\epsilon)$ induced from that of $\widehat{A}^\epsilon(k)$ satisfies (3.25) with an obvious modification. In addition, $\dot{X}_{\beta+1}^{\alpha,l+1} \subset D(A^\epsilon)$.

We redefine B^ϵ as

$$B^\epsilon = A^\epsilon + \frac{1}{\epsilon}K, \quad D(B^\epsilon) = D(A^\epsilon).$$

Since the operator K is bounded and A^ϵ is a semigroup generator in $\dot{X}_\beta^{\alpha,l}$, for any $\epsilon > 0$, B^ϵ is also a semigroup generator in $\dot{X}_\beta^{\alpha,l}$. For any $f \in \dot{X}_\beta^{\alpha,l}$, it holds that

$$e^{tB^\epsilon} f = \sum_{l=0}^{\infty} \left(e^{tA^\epsilon} \frac{1}{\epsilon} K * \right)^l e^{tA^\epsilon} f.$$

Thanks to (2.1), (2.4) and (3.25), for the operator A^ϵ , the convergence is uniform strongly in $\dot{X}_\beta^{\alpha,l}$ for $(\epsilon, t) \in [\delta, \infty) \times [0, t_0]$ for any $\delta, t_0 > 0$ and

$$\left(e^{tA^\epsilon} \frac{1}{\epsilon} K * \right)^l e^{tA^\epsilon} f \in B^0([\delta, \infty) \times [0, t_0]; \dot{X}_\beta^{\alpha,l}).$$

Thus, if $f \in \dot{X}_\beta^{\alpha,l}$,

$$e^{tB^\epsilon} f \in B^0((0, \infty) \times [0, \infty); \dot{X}_\beta^{\alpha,l}) \tag{3.26}$$

as a function of $(\epsilon, t) \in (0, \infty) \times [0, \infty)$.

If $f = f^\epsilon \in Z_{\beta,\tau}^{\alpha,\gamma,l}$, for any fixed $R > 0$, $\Lambda \mathcal{F}_x^{-1}(\chi(|p| \leq R) \widehat{f}^\epsilon) \in Z_{\beta,\tau}^{\alpha,\gamma,l}$ by (2.1). By this and (3.26), one has

$$H^\epsilon \mathcal{F}_x^{-1}(\chi(|p| \leq R) \widehat{f}^\epsilon) \in C^0((0, \infty)_\epsilon; \dot{Y}_\beta^{\alpha,\gamma,l}([0, \tau])).$$

By this and Lemma 3.2, one has

$$\begin{aligned} & \|H^\epsilon \mathcal{F}_x^{-1}(\chi(|p| > R) \widehat{f}^\epsilon)\|_{\alpha,\gamma,l,\beta,\tau} \\ & \leq C \left(1 + \frac{1}{\gamma}\right) \|\mathcal{F}_x^{-1}(\chi(|k| + |p| > R) \widehat{f}^\epsilon)\|_{\alpha,\gamma,l,\beta,\tau} \rightarrow 0 \quad \text{as } R \rightarrow \infty. \end{aligned}$$

By these facts and $f = f^\epsilon \in Z_{\beta,\tau}^{\alpha,\gamma,l}$, we arrive at $H^\epsilon f^\epsilon \in Z_{\beta,\tau}^{\alpha,\gamma,l}$.

If $h_i = h_i^\epsilon(t, x, p) \in Z_{\beta,\tau}^{\alpha,\gamma,l}$ with $i = 1, 2$, we will show that $\Lambda^{-1}\Gamma(h_1, h_2) \in Z_{\beta,\tau}^{\alpha,\gamma,l}$. Let

$$V^{l,\beta} = B^0((0, \infty) \times [0, \tau]; X_\beta^{\alpha,l}) \quad \text{and} \quad V = \bigcap_{l' \geq l, \beta' \geq \beta} V^{l',\beta'}.$$

Then $Z_{\beta,\tau}^{\alpha,\gamma,l}$ is a strong closure of V in $V^{l,\beta}$ by the infinity condition in (1.32). For $h_1, h_2 \in V$, then $\Lambda^{-1}\Gamma(h_1, h_2) \in V$ by the similar arguments as (3.18). By using these and the proof of (3.18), we can obtain the desired results.

By the above facts, if $f_0 \in \dot{X}_\beta^{\alpha,l}$, we have from (3.20) that $N^\epsilon[f^\epsilon](t) \in Z_{\beta,\tau}^{\alpha,\gamma,l}$. Define by Z_0 the closed ball

$$Z_0 = \left\{ f = f^\epsilon \in Z_{\beta,\tau}^{\alpha,\gamma,l} \mid \|f\|_{\alpha,\gamma,l,\beta,\tau} \leq a_1 \|f_0\|_{\alpha,l,\beta} \right\}. \quad (3.27)$$

By the similar arguments as Theorem 3.1, we can obtain a solution $f = f^\epsilon(t)$ to (2.7) in Z_0 . Then we show that $f = f^\epsilon(t)$ is a classical solution to (1.27). For any $h > 0$ and $t \in [0, \tau - \delta]$ with any $\delta > 0$ small enough, we have from (3.20) that

$$\begin{aligned} & \frac{1}{h} (N^\epsilon[f^\epsilon](t+h) - N^\epsilon[f^\epsilon](t)) \\ &= \frac{1}{h} (e^{hB^\epsilon} - I) N^\epsilon[f^\epsilon](t) + \frac{1}{h} \int_t^{t+h} e^{(t+h-s)B^\epsilon} Q_0 \frac{1}{\epsilon} \Gamma(f^\epsilon(s), f^\epsilon(s)) ds \\ &=: v_1 + v_2. \end{aligned} \quad (3.28)$$

Since $N^\epsilon[f^\epsilon] \in Z_{\beta,\tau}^{\alpha,\gamma,l}$, $N^\epsilon[f^\epsilon] \in D(B^\epsilon)$ if B^ϵ is considered in the space $\dot{X}_{\beta-1}^{\alpha-\gamma t, l-1}$. Then, for fixed ϵ and $t, v_1 \rightarrow B^\epsilon N^\epsilon[f^\epsilon]$ ($h \rightarrow 0$) strongly there by the semigroup property. Put $\Gamma(s) = \Gamma(f^\epsilon(s), f^\epsilon(s))$ and write

$$\begin{aligned} v_2 - \frac{1}{\epsilon} \Gamma(t) &= \frac{1}{h} \int_0^h (e^{sB^\epsilon} - I) ds Q_0 \frac{1}{\epsilon} \Gamma(t) \\ &\quad + \frac{1}{h} \int_0^h e^{sB^\epsilon} \frac{1}{\epsilon} Q_0 (\Gamma(t+h-s) - \Gamma(t)) ds \\ &=: v_3 + v_4. \end{aligned} \quad (3.29)$$

Since $\Gamma(t) \in Z_{\beta-1,\tau}^{\alpha,\gamma,l}$, $\Gamma(t) \in \dot{Y}_{\beta-1}^{\alpha,\gamma,l}([0, \tau])$ for any fixed $\epsilon > 0$, the first term v_3 in (3.29) tends to 0 as $h \rightarrow 0$ by the semigroup property. Next we estimate the term v_4 . We easily see

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-1} (\widehat{\Gamma}(t+h-s) - \widehat{\Gamma}(t)) \\ &= e^{\alpha|k|} (1+|k|)^{l-1} (e^{-\gamma(t+h-s)|k|} \widehat{\Gamma}(t+h-s) - e^{-\gamma t|k|} \widehat{\Gamma}(t)) \\ &\quad + e^{(\alpha-\gamma(t+h-s))|k|} (1+|k|)^{l-1} (e^{\gamma(h-s)|k|} - 1) \widehat{\Gamma}(t+h-s) \\ &=: \Gamma_1(t, h-s, k, p) + \Gamma_2(t, h-s, k, p). \end{aligned} \quad (3.30)$$

For $0 < s \leq h$ with $h \rightarrow 0$, we see

$$\begin{aligned} |\Gamma_2(t, h-s, k, p)| &\leq e^{(\alpha-\gamma(t+h-s))|k|} (1+|k|)^{l-1} |(e^{\gamma(h-s)|k|} - 1)\widehat{\Gamma}(t+h-s)| \\ &\leq C\gamma(h-s)|k|e^{(\alpha-\gamma(t+h-s))|k|} (1+|k|)^{l-1} |\widehat{\Gamma}(t+h-s)| \\ &\leq C(h-s)e^{(\alpha-\gamma(t+h-s))|k|} (1+|k|)^l |\widehat{\Gamma}(t+h-s)|. \end{aligned} \tag{3.31}$$

We have from this, (3.18), (3.30) and Theorem 2.1 that

$$\begin{aligned} &\left\| \int_0^h \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) e^{\lambda_j(\epsilon|k|)\frac{s}{\epsilon}} P_j(\epsilon k) Q_0 \frac{1}{\epsilon} \Gamma_2(t, h-s, k, p) ds \right\|_{\beta-1} \\ &\leq C \int_0^h \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) |k|(h-s) e^{(\alpha-\gamma(t+h-s))|k|} (1+|k|)^l \|\widehat{\Gamma}(t+h-s)\|_{\beta-1} ds \\ &\leq \frac{C}{\epsilon} h^2 \sup_{0 \leq s \leq h} \|\Gamma(t+h-s)\|_{\alpha-\gamma(t+h-s), l, \beta-1} \leq \frac{C}{\epsilon} h^2 \|f\|_{\alpha, \gamma, l, \beta, \tau}^2 \end{aligned}$$

and

$$\begin{aligned} &\left\| \int_0^h \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) e^{\lambda_j(\epsilon|k|)\frac{s}{\epsilon}} P_j(\epsilon k) Q_0 \frac{1}{\epsilon} \Gamma_1(t, h-s, k, p) ds \right\|_{\beta-1} \\ &\leq C \int_0^h \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) |k| e^{\alpha|k|} (1+|k|)^{l-1} \\ &\quad \times \left\| (e^{-\gamma(t+h-s)|k|} \widehat{\Gamma}(t+h-s) - e^{-\gamma t|k|} \widehat{\Gamma}(t)) \right\|_{\beta-1} ds \\ &\leq \frac{C}{\epsilon} h \sup_{0 \leq s \leq h} \left\| \mathcal{F}_x^{-1}(e^{-\gamma(t+h-s)|k|} \widehat{\Gamma}(t+h-s) - e^{-\gamma t|k|} \widehat{\Gamma}(t)) \right\|_{\alpha, l, \beta-1}. \end{aligned}$$

We also have

$$\begin{aligned} &\left\| \int_0^h \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) e^{s\widehat{A}^\epsilon(k)} Q(\epsilon k) Q_0 \frac{1}{\epsilon} \Gamma_2(t, h-s, k, p) ds \right\|_{\beta-1} \\ &\leq \int_0^h \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) \frac{1}{\epsilon} (h-s) e^{(\alpha-\gamma(t+h-s))|k|} (1+|k|)^l \|\widehat{\Gamma}(t+h-s)\|_{\beta-1} ds \\ &\leq \frac{C}{\epsilon} h^2 \sup_{0 \leq s \leq h} \|\Gamma(t+h-s)\|_{\alpha-\gamma(t+h-s), l, \beta-1} \leq \frac{C}{\epsilon} h^2 \|f\|_{\alpha, \gamma, l, \beta, \tau}^2 \end{aligned}$$

and

$$\left\| \int_0^h \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) e^{s\widehat{A}^\epsilon(k)} Q(\epsilon k) Q_0 \frac{1}{\epsilon} \Gamma_1(t, h-s, k, p) ds \right\|_{\beta-1}$$

$$\begin{aligned} &\leq C \int_0^h \chi\left(|k| \leq \frac{\kappa_0}{\epsilon}\right) \frac{1}{\epsilon} e^{\alpha|k|} (1+|k|)^{l-1} \\ &\quad \times \|e^{-\gamma(t+h-s)|k|} \widehat{\Gamma}(t+h-s) - e^{-\gamma t|k|} \widehat{\Gamma}(t)\|_{\beta-1} ds \\ &\leq \frac{C}{\epsilon} h \sup_{0 \leq s \leq h} \|\mathcal{F}_x^{-1}(e^{-\gamma(t+h-s)|k|} \widehat{\Gamma}(t+h-s) - e^{-\gamma t|k|} \widehat{\Gamma}(t))\|_{\alpha,l,\beta-1}. \end{aligned}$$

For the term with U_1 with $|k| \leq \kappa_0/\epsilon$ and the case that $|k| > \kappa_0/\epsilon$, we can treat it similarly. Thus we can obtain

$$\begin{aligned} &\|v_4\|_{\alpha-\gamma t, l-1, \beta-1} \\ &\leq \frac{C}{\epsilon} \left(\sup_{0 \leq s \leq h} \|\mathcal{F}_x^{-1}(e^{-\gamma(t+h-s)|k|} \widehat{\Gamma}(t+h-s) - e^{-\gamma t|k|} \widehat{\Gamma}(t))\|_{\alpha,l,\beta-1} + h \|f\|_{\alpha,\gamma,l,\beta,\tau}^2 \right). \end{aligned}$$

For fixed ϵ and t , clearly the second term in the above inequality tends to 0 as $h \rightarrow 0$. Since $\Gamma(t) \in \dot{Y}_{\beta-1}^{\alpha,\gamma,l}([0,\tau])$, we also have that the first term in the above inequality tends to 0 as $h \rightarrow 0$. Thus we have that $v_4 \rightarrow 0$ as $h \rightarrow 0$ in $\dot{X}_{\beta-1}^{\alpha-\gamma t, l-1}$. For any $h < 0$ and $t \in [\delta, \tau]$ with any small $\delta > 0$, we can obtain the similar results. By using (3.28), (3.29) and these facts we have from $f^\epsilon = N[f^\epsilon]$ that df^ϵ/dt exists and satisfies

$$\frac{df^\epsilon}{dt} = B^\epsilon f^\epsilon(t) + \frac{1}{\epsilon} \Gamma(f^\epsilon(t), f^\epsilon(t)) \in \dot{Y}_{\beta-1}^{\alpha,\gamma,l}([0,\tau]).$$

Since $f^\epsilon(t)$ is analytic in $x \in \mathbb{R}^3 + iB_{\alpha-\gamma t}$, this shows that $f^\epsilon(t)$ is a classical solution to (1.27). This completes the proof the theorem. \square

4 Hydrodynamic limit of the local solutions

In this section, we will prove the Euler limit of the relativistic Boltzmann equation. We shall use the contraction mapping principle to obtain the relevant limit. For $\beta > 5/2$, we define the space

$$\begin{aligned} W_{\beta,\tau}^{\alpha,\gamma,l} = \left\{ f = f^\epsilon \in Z_{\beta,\tau}^{\alpha,\gamma,l} \mid \exists f^0 \in \dot{Y}_\beta^{\alpha,\gamma,l}((0,\tau]), \forall \zeta > 0, \right. \\ \left. \|f^\epsilon - f^0\|_{\dot{Y}_{\beta-1}^{\alpha,\gamma,l}([\zeta,\tau])} \rightarrow 0 \text{ as } \epsilon \rightarrow 0 \right\}. \end{aligned} \tag{4.1}$$

In the below we assume that $\beta > 5/2, \epsilon \in (0,1], \sigma \in [0,1], l > \sigma$ and (3.4) holds. And there exists an absolute constant $C > 0$ such that $\tau \leq C$ and $\gamma \leq C$. We define the

norm as

$$[f^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} = \sup_{\substack{0 < \epsilon' < \epsilon, \\ 0 < s \leq \tau}} \left(\frac{s^2}{\epsilon'}\right)^{\frac{\sigma}{2}} \|f^{\epsilon'}(s)\|_{\alpha-\gamma s, l-\sigma, \beta-1}, \quad \sigma \in (0,1). \tag{4.2}$$

For $0 \leq j \leq 4$, we further define the operators $E^0(t)$ and $E_j^0(t)$ by $E^0(t) = \sum_{j=0}^4 E_j^0(t)$ with

$$\mathcal{F}_x E_j^0(t) f_0 = e^{i\lambda_j^{(1)}|k|t} P_j^{(0)} \widehat{f}_0(k, \cdot), \quad t \in (0, \tau]. \tag{4.3}$$

Lemma 4.1. *Suppose $f_0 \in \dot{X}_\beta^{\alpha,l}$. Then one has*

- (i) $e^{tB^\epsilon} f_0 \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit $E^0(t)f_0$.
- (ii) $[e^{tB^\epsilon} f_0 - E^0(t)f_0]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \leq C \|f_0\|_{\alpha,l,\beta}$.

Proof. For any $\beta_1, \beta_2 \in \mathbb{R}$, we have from Theorem 2.1 that

$$\left\| e^{i\lambda_j^{(1)}|k|t} P_j^{(0)} \widehat{f}_0 \right\|_{\beta_1} \leq C \left\| P_j^{(0)} \widehat{f}_0 \right\|_{\beta_1} \leq C \|\widehat{f}_0\|_{\beta_2}.$$

For any $t \in (0, \tau]$, we have from this and (4.3) that

$$\|E^0(t)f_0\|_{\alpha-\gamma t, l, \beta} \leq C \|f_0\|_{\alpha, l, \beta}. \tag{4.4}$$

Noting that $f_0 \in \dot{X}_\beta^{\alpha,l}$, we have from the above similar arguments as (4.4) that

$$\begin{aligned} & \|E^0(t)\mathcal{F}_x^{-1}(\chi(|k|+|p|>R)\widehat{f}_0)\|_{\alpha-\gamma t, l, \beta} \\ & \leq C \|\mathcal{F}_x^{-1}(\chi(|k|+|p|>R)\widehat{f}_0)\|_{\alpha, l, \beta} \rightarrow 0 \quad \text{as } R \rightarrow +\infty. \end{aligned}$$

Thus we have from these facts that $E^0(t)f_0 \in \dot{Y}_\beta^{\alpha,\gamma,l}((0, \tau])$. By Lemma 3.1, for any $t \in [0, \tau]$, one has

$$\|e^{tB^\epsilon} f_0\|_{\alpha-\gamma t, l, \beta} \leq C \|f_0\|_{\alpha, l, \beta}.$$

We also have

$$\begin{aligned} & \|e^{tB^\epsilon} \mathcal{F}_x^{-1}(\chi(|k|+|p|>R)\widehat{f}_0)\|_{\alpha-\gamma t, l, \beta} \\ & \leq C \|\mathcal{F}_x^{-1}(\chi(|k|+|p|>R)\widehat{f}_0)\|_{\alpha, l, \beta} \rightarrow 0 \quad \text{as } R \rightarrow +\infty. \end{aligned}$$

Thus we prove that $e^{tB^\epsilon} f_0 \in Z_{\beta,\tau}^{\alpha,\gamma,l}$.

Then we shall show the convergence property in $W_{\beta,\tau}^{\alpha,\gamma,l}$. If $|k| < R$ with $R < \kappa_0/\epsilon$, for any $\sigma \in [0,1]$, one has from Theorem 2.1 that

$$\begin{aligned} \left| e^{\lambda_j(\epsilon|k|)\frac{t}{\epsilon}} - e^{i\lambda_j^{(1)}|k|t} \right| &= \left| e^{i\lambda_j^{(1)}|k|t} \left(e^{\lambda_j(\epsilon|k|)\frac{t}{\epsilon} - i\lambda_j^{(1)}|k|t} - 1 \right) \right| \\ &\leq C\epsilon |\lambda_j^{(2)}| |k|^2 t \leq C\epsilon R^{2-\sigma} |k|^\sigma, \end{aligned} \quad (4.5)$$

where we have used that for any $a \in \mathbb{C}$ with $\operatorname{Re} a < 0$ and $\sigma \in [0,1]$,

$$|e^a - 1| \leq C|a|^\sigma. \quad (4.6)$$

For any $\beta_1, \beta_2 \in \mathbb{R}$, we have from Theorem 2.1 that

$$\left\| \left(P_j(\epsilon k) - P_j^{(0)} \right) \widehat{f}_0 \right\|_{\beta_1} \leq C\epsilon |k| \|\widehat{f}_0\|_{\beta_2} \leq C\epsilon R^{1-\sigma} |k|^\sigma \|\widehat{f}_0\|_{\beta_2}. \quad (4.7)$$

We decompose f_0 as

$$f_0 = \mathcal{F}_x^{-1} \chi(|k| \leq R) \widehat{f}_0 + \mathcal{F}_x^{-1} \chi(|k| > R) \widehat{f}_0 := f_0' + f_0''. \quad (4.8)$$

By using (3.2), (4.3) and (4.8), for any $0 \leq j \leq 4$, we have from Theorem 2.1 that

$$\begin{aligned} &\mathcal{F}_x(E_j^\epsilon(t) - E_j^0(t)) f_0' \\ &= \chi(|k| \leq R) \left(e^{\lambda_j(\epsilon|k|)\frac{t}{\epsilon}} P_j(\epsilon k) - e^{i\lambda_j^{(1)}|k|t} P_j^{(0)} \right) \widehat{f}_0 \\ &= \chi(|k| \leq R) \left\{ e^{\lambda_j(\epsilon|k|)\frac{t}{\epsilon}} (P_j(\epsilon k) - P_j^{(0)}) \widehat{f}_0 + (e^{\lambda_j(\epsilon|k|)\frac{t}{\epsilon}} - e^{i\lambda_j^{(1)}|k|t}) P_j^{(0)} \widehat{f}_0 \right\}. \end{aligned}$$

By using this, (4.5) and (4.7), for any $0 \leq j \leq 4$, we have

$$\left\| \mathcal{F}_x(E_j^\epsilon(t) - E_j^0(t)) f_0' \right\|_{\beta-1} \leq C\epsilon R^{2-\sigma} |k|^\sigma \|\widehat{f}_0\|_\beta.$$

This implies that

$$\left\| (E_j^\epsilon(t) - E_j^0(t)) f_0' \right\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C\epsilon R^{2-\sigma} \|f_0\|_{\alpha, l, \beta}. \quad (4.9)$$

For any $t > 0$ and $\beta > 5/2$, we have from (3.2) and Theorem 2.1 that

$$\left\| \mathcal{F}_x E_5^\epsilon(t) f_0' \right\|_{\beta-1} \leq C e^{-\mu t/\epsilon} \|\widehat{f}_0\|_\beta, \quad \mu = \min(\nu_0, \sigma_0).$$

This implies that

$$\left\| E_5^\epsilon(t) f_0' \right\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C e^{-\mu t/\epsilon} \|f_0\|_{\alpha, l-\sigma, \beta}. \quad (4.10)$$

By the choice of R , one has that $\mathcal{F}_x E_6^\epsilon f'_0 = 0$. It follows from (4.9), (4.10) and this that

$$\|(e^{tB^\epsilon} - E^0(t))f'_0\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C \left(\epsilon R^{2-\sigma} + e^{-\frac{\mu t}{\epsilon}} \right) \|f_0\|_{\alpha, l, \beta}. \tag{4.11}$$

For any $\varsigma > 0$, we choose $t \in [\varsigma, \tau]$, $\sigma = 0$ and $R = \kappa_0 \epsilon^{-1/4}$ in (4.11) to get

$$\begin{aligned} & \|(e^{tB^\epsilon} - E^0(t))f'_0\|_{\alpha-\gamma t, l, \beta-1} \\ & \leq C \left(e^{-\frac{\mu \varsigma}{\epsilon}} + \kappa_0^2 \epsilon^{\frac{1}{2}} \right) \|f_0\|_{\alpha, l, \beta} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned} \tag{4.12}$$

For $\beta > 5/2$, we have from Lemma 3.1 that $\|\mathcal{F}_x e^{tB^\epsilon} f\|_{\beta-1} \leq C \|\widehat{f}\|_{\beta-1}$. This, (4.8) and (1.32) imply

$$\begin{aligned} \|e^{tB^\epsilon} f''_0\|_{\alpha-\gamma t, l, \beta-1} & \leq C \|\mathcal{F}_x^{-1}(\chi(|k| > R)\widehat{f}_0)\|_{\alpha, l, \beta-1} \\ & \leq C \|\mathcal{F}_x^{-1}(\chi(|k| + |p| > R)\widehat{f}_0)\|_{\alpha, l, \beta} \rightarrow 0 \end{aligned} \tag{4.13}$$

as $R \rightarrow +\infty$. It follows from (4.3) and (4.8) that, as $R \rightarrow +\infty$,

$$\begin{aligned} \|E^0(t)f''_0\|_{\alpha-\gamma t, l, \beta-1} & \leq C \|\mathcal{F}_x^{-1}(\chi(|k| > R)\widehat{f}_0)\|_{\alpha, l, \beta-1} \\ & \leq C \|\mathcal{F}_x^{-1}(\chi(|k| + |p| > R)\widehat{f}_0)\|_{\alpha, l, \beta} \rightarrow 0. \end{aligned} \tag{4.14}$$

Thus we can obtain

$$\begin{aligned} & \|(e^{tB^\epsilon} - E^0(t))f''_0\|_{\alpha-\gamma t, l, \beta-1} \\ & \leq C \|\mathcal{F}_x^{-1}(\chi(|k| + |p| > R)\widehat{f}_0)\|_{\alpha, l, \beta} \rightarrow 0 \text{ as } \epsilon \rightarrow 0. \end{aligned} \tag{4.15}$$

By using (4.8), (4.12) and (4.15), we have proved (i). Then we prove (ii).

It follows from (4.13) and (4.14) that

$$\begin{aligned} \|(e^{tB^\epsilon} - E^0(t))f''_0\|_{\alpha-\gamma t, l-\sigma, \beta-1} & \leq C \|\mathcal{F}_x^{-1}(\chi(|k| > R)\widehat{f}_0)\|_{\alpha, l-\sigma, \beta} \\ & \leq C(1+R)^{-\sigma} \|f_0\|_{\alpha, l, \beta}. \end{aligned} \tag{4.16}$$

If we choose $t \in (0, \tau]$, $\sigma \in (0, 1)$ and $R = \kappa_0 \epsilon^{-1/2}$ in (4.11), we get

$$\begin{aligned} \|(e^{tB^\epsilon} - E^0(t))f'_0\|_{\alpha-\gamma t, l-\sigma, \beta-1} & \leq C \left(\left(\frac{\epsilon}{t^2} \right)^{\frac{\sigma}{2}} t^{\frac{\sigma}{2}} + \kappa_0^{2-\sigma} \epsilon^{\frac{\sigma}{2}} \right) \|f_0\|_{\alpha, l, \beta} \\ & \leq C \left(\frac{\epsilon}{t^2} \right)^{\frac{\sigma}{2}} \|f_0\|_{\alpha, l, \beta}. \end{aligned} \tag{4.17}$$

Here we have used the fact that

$$\sup_{x \geq 0} x^\alpha e^{-xt} \leq Ct^{-\alpha} \quad \text{for any } \alpha, t > 0. \tag{4.18}$$

By using (4.16), (4.17) and (4.2), we have proved (ii). This completes the proof of the lemma. \square

For $0 \leq j \leq 4$, we define the operators H_j^0 by

$$\mathcal{F}_x H_j^0 f(t) = e^{i\lambda_j^{(1)}|k|t} |k| P_j^{(1)}(\omega) * Q_0 \Lambda \widehat{f}(t), \quad t \in (0, \tau]. \tag{4.19}$$

Lemma 4.2. *Suppose that $f = f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$ with the limit f^0 . For $0 \leq j \leq 4$, one has*

- (i) $H_j^0 f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$ with the limit $H_j^0 f^0$.
- (ii) $[H_j^0 (f^\epsilon - f^0)]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \leq (C/\gamma) [f^\epsilon - f^0]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}$.

Proof. For any $\beta_1, \beta_2 \in \mathbb{R}$ and $1 \leq j \leq 4$, we have from Theorem 2.1 that

$$\left\| P_j^{(1)}(\omega) Q_0 \Lambda \widehat{h}(t) \right\|_{\beta_1} \leq C \|\widehat{h}(t)\|_{\beta_2}.$$

For any $t_1, t_2 \in [0, t]$ with $t_1 < t_2$, we arrive at

$$\begin{aligned} & \left\| \int_{t_1}^{t_2} e^{i\lambda_j^{(1)}|k|(t-s)} |k| P_j^{(1)}(\omega) Q_0 \Lambda \widehat{h}(s) ds \right\|_{\beta_1} \\ & \leq C \int_{t_1}^{t_2} |k| \|\widehat{h}(s)\|_{\beta_2} ds \leq C \int_{t_1}^{t_2} |k| e^{-(\alpha-\gamma s)|k|} (1+|k|)^{-l} \|h(s)\|_{\alpha-\gamma s, l, \beta_2} ds. \end{aligned} \tag{4.20}$$

By this, for any $\delta > 0$, one has

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^l \left\| \int_{\delta}^t e^{i\lambda_j^{(1)}|k|(t-s)} |k| P_j^{(1)}(\omega) Q_0 \Lambda \widehat{h}(s) ds \right\|_{\beta-\sigma} \\ & \leq C \sup_{s \in [\delta, t]} \|h(s)\|_{\alpha-\gamma s, l, \beta-\sigma} \int_{\delta}^t e^{-\gamma|k|(t-s)} |k| ds \leq \frac{C}{\gamma} \sup_{s \in [\delta, t]} \|h(s)\|_{\alpha-\gamma s, l, \beta-\sigma}. \end{aligned} \tag{4.21}$$

It follows from (4.20) that

$$e^{(\alpha-\gamma t)|k|} (1+|k|)^l \left\| \int_0^{\delta} e^{i\lambda_j^{(1)}|k|(t-s)} |k| P_j^{(1)}(\omega) Q_0 \Lambda \widehat{h}(s) ds \right\|_{\beta-\sigma}$$

$$\begin{aligned}
 &\leq C \sup_{s \in (0, \delta]} \|h(s)\|_{\alpha-\gamma s, l, \beta-\sigma} \int_0^\delta e^{-\gamma|k|(t-s)} |k| ds \\
 &\leq \frac{C}{\gamma} \sup_{s \in (0, \delta]} \|h(s)\|_{\alpha-\gamma s, l, \beta-\sigma} \int_0^\delta (t-s)^{-1} ds \\
 &\leq \frac{C\delta}{\gamma} (t-\delta)^{-1} \sup_{s \in (0, \delta]} \|h(s)\|_{\alpha-\gamma s, l, \beta-\sigma}. \tag{4.22}
 \end{aligned}$$

Here we used the fact that $e^{-\gamma(t-s)|k|} \gamma |k| (t-s) \leq C$ for $t > s$.

For any $t \in (0, \tau]$ and $\sigma = 0$, we have from (4.19), (4.21) and (4.22) with $\delta = t/2$ that

$$\sup_{t \in (0, \tau]} \|H_j^0 f^0\|_{\alpha-\gamma t, l, \beta} \leq \frac{C}{\gamma},$$

that is, $H_j^0 f^0 \in \dot{Y}_\beta^{\alpha, \gamma, l}((0, \tau])$. Similarly $H_j^0 f^\epsilon \in Z_{\beta, \tau}^{\alpha, \gamma, l}$. Since $f = f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$ with the limit f^0 , there exists $C > 0$ independent of ϵ such that

$$\|f\|_{\alpha, \gamma, l, \beta, \tau} + \sup_{s \in (0, \tau]} \|f^0(s)\|_{\alpha-\gamma s, l, \beta} \leq C.$$

For any $\varsigma > 0, \delta \in (0, \varsigma), \sigma = 1$ and $t \in [\varsigma, \tau]$, we have from this, (4.21) and (4.22) that

$$\begin{aligned}
 &\sup_{t \in [\varsigma, \tau]} \|H_j^0 (f^\epsilon - f^0)(t)\|_{\alpha-\gamma t, l, \beta-1} \\
 &\leq \frac{C}{\gamma} \sup_{s \in [\delta, \tau]} \|(f^\epsilon - f^0)(s)\|_{\alpha-\gamma s, l, \beta-1} + \frac{C\delta}{\gamma} (\varsigma - \delta)^{-1}. \tag{4.23}
 \end{aligned}$$

Since $f = f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$ with the limit f^0 , we have from this, (4.1) and (4.23) that $H_j^0 f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$ with the limit $H_j^0 f^0$. This concludes the proof of the first result in the lemma.

For any $t \in (0, \tau]$ and $\sigma \in (0, 1)$, we have from (4.20) and (4.2) that

$$\begin{aligned}
 &e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \left\| \int_0^t e^{i\lambda_j^{(1)}|k|(t-s)} |k| P_j^{(1)}(\omega) Q_0 \Lambda(\widehat{f}^{\epsilon'} - \widehat{f}^0)(s) ds \right\|_{\beta-1} \\
 &\leq C [f^\epsilon - f^0]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \int_0^t e^{-\gamma|k|(t-s)} |k| \left(\frac{\epsilon'}{s^2}\right)^{\frac{\sigma}{2}} ds \leq \frac{C}{\gamma} \left(\frac{\epsilon'}{t^2}\right)^{\frac{\sigma}{2}} [f^\epsilon - f^0]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}. \tag{4.24}
 \end{aligned}$$

Here we used the fact that, for any $\sigma \in (0, 1)$,

$$\int_0^t e^{-b(t-s)} b s^{-\sigma} ds \leq C t^{-\sigma}. \tag{4.25}$$

This, (4.19) and (4.2) will give the second result in the lemma. This completes the proof of the lemma. \square

Assume that $R \in (1, \kappa_0/\epsilon)$. For any $f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$, we define as

$$f^\epsilon = \mathcal{F}_x^{-1}(\chi(|k| \leq R)\widehat{f}^\epsilon) + \mathcal{F}_x^{-1}(\chi(|k| > R)\widehat{f}^\epsilon) := f_R^\epsilon + \widetilde{f}_R^\epsilon. \tag{4.26}$$

Lemma 4.3. Recall H_j^ϵ of (3.6) and H_j^0 of (4.19) with $0 \leq j \leq 4$. For $f = f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$, one has

$$(i) \quad \|(H_j^\epsilon - H_j^0)f_R^\epsilon\|_{\alpha - \gamma t, l, \beta - 1} \leq (C/\gamma)\epsilon R^2 \|f\|_{\alpha, \gamma, l, \beta, \tau}.$$

$$(ii) \quad [(H_j^\epsilon - H_j^0)f_R^\epsilon]^{\epsilon, \tau, \sigma}_{\alpha, \gamma, l, \beta - 1} \leq (C/\gamma)\epsilon^{1 - \frac{\sigma}{2}} R^{2 - \sigma} \|f\|_{\alpha, \gamma, l, \beta, \tau}.$$

Proof. For any $\beta_1, \beta_2 \in \mathbb{R}, 0 \leq j \leq 4$ and $\ell = 0, 1, 2$, we have from Theorem 2.1 that

$$\left\| P_j^{(\ell)}(\omega) Q_0 \Lambda \widehat{h}(t) \right\|_{\beta_1} \leq C \|\widehat{h}(t)\|_{\beta_2}.$$

Noting that $\chi(|k| \leq (\kappa_0/\epsilon))\chi(|k| \leq R) = \chi(|k| \leq R)$, we have from (4.5) and (4.6) that

$$\begin{aligned} & e^{(\alpha - \gamma t)|k|} (1 + |k|)^{l - \sigma} \\ & \times \left\| \int_0^t \left(e^{\lambda_j(\epsilon|k|)\frac{(t-s)}{\epsilon}} - e^{i\lambda_j^{(1)}|k|(t-s)} \right) P_j(\epsilon|k|) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(s) ds \right\|_{\beta - 1} \\ & \leq C e^{(\alpha - \gamma t)|k|} (1 + |k|)^{l - \sigma} \int_0^t \epsilon |k|^3 \chi(|k| \leq R) \|\widehat{f}^\epsilon(s)\|_{\beta - 1} ds \\ & \leq C \epsilon R^{2 - \sigma} \sup_{s \in [0, t]} \|f^\epsilon\|_{\alpha - \gamma s, l, \beta - 1} \int_0^t e^{-\gamma(t-s)|k|} |k| ds \\ & \leq \frac{C\epsilon}{\gamma} R^{2 - \sigma} \sup_{s \in [0, t]} \|f^\epsilon\|_{\alpha - \gamma s, l, \beta}. \end{aligned} \tag{4.27}$$

Here we have used the fact that

$$P_j(\epsilon|k|) Q_0 = |\epsilon k| \left(P_j^{(1)}(\omega) + |\epsilon k| P_j^{(2)}(\omega) \right) Q_0.$$

It follows that

$$e^{(\alpha - \gamma t)|k|} (1 + |k|)^{l - \sigma} (1 + |p|)^{\beta - 1} \left| \int_0^t e^{i\lambda_j^{(1)}|k|(t-s)} \left(P_j(\epsilon|k|) - \epsilon|k| P_j^{(1)} \right) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(s) ds \right|$$

$$\begin{aligned}
 &\leq C e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \int_0^t \epsilon |k|^2 \chi(|k| \leq R) \|\widehat{f^\epsilon}(s)\|_{\beta-1} ds \\
 &\leq C \epsilon R^{1-\sigma} \sup_{s \in [0,t]} \|f^\epsilon\|_{\alpha-\gamma s, l, \beta-1} \int_0^t e^{-\gamma(t-s)|k|} |k| ds \\
 &\leq \frac{C \epsilon}{\gamma} R^{1-\sigma} \sup_{s \in [0,t]} \|f^\epsilon\|_{\alpha-\gamma s, l, \beta}. \tag{4.28}
 \end{aligned}$$

Recalling H_j^ϵ of (3.6) and H_j^0 of (4.19), for $\sigma = 0$ and $t \in [0, \tau]$, we have from (4.27) and (4.28) that

$$\|(H_j^\epsilon - H_j^0) f_R^\epsilon\|_{\alpha-\gamma t, l, \beta-1} \leq \frac{C}{\gamma} \epsilon R^2 \|f\|_{\alpha, \gamma, l, \beta, \tau}.$$

Thus we have proved (i) and then we prove (ii). For any $\epsilon' \in (0, \epsilon)$, $\sigma \in (0, 1)$ and $t \in [0, \tau]$, we have from (4.27) and (4.28) that

$$\|(H_j^{\epsilon'} - H_j^0) f_R^{\epsilon'}\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq \frac{C_1}{\gamma} \epsilon' R^{2-\sigma} \|f\|_{\alpha, \gamma, l, \beta, \tau}.$$

By using this and (4.2), we arrive at

$$\left[(H_j^\epsilon - H_j^0) f_R^\epsilon \right]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \leq \frac{C_1}{\gamma} \tau^\sigma \epsilon^{1-\frac{\sigma}{2}} R^{2-\sigma} \|f\|_{\alpha, \gamma, l, \beta, \tau} \leq \frac{C}{\gamma} \epsilon^{1-\frac{\sigma}{2}} R^{2-\sigma} \|f\|_{\alpha, \gamma, l, \beta, \tau}.$$

This completes the proof of the lemma. □

For the study of H_5^ϵ in (3.6), we need the following operator:

$$\widetilde{H}_5^\epsilon f^\epsilon(t) = e^{\frac{i}{\epsilon} L} * Q_0 \frac{1}{\epsilon} \Lambda f^\epsilon(t). \tag{4.29}$$

Lemma 4.4. Recall H_5^ϵ of (3.6) and \widetilde{H}_5^ϵ of (4.29). For $f = f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$, one has

- (i) $\|(H_5^\epsilon - \widetilde{H}_5^\epsilon) f_R^\epsilon\|_{\alpha-\gamma t, l, \beta-1} \leq C \epsilon R \|f\|_{\alpha, \gamma, l, \beta, \tau}.$
- (ii) $[(H_5^\epsilon - \widetilde{H}_5^\epsilon) f_R^\epsilon]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \leq C \epsilon^{1-\sigma/2} R^{1-\sigma} \|f\|_{\alpha, \gamma, l, \beta, \tau}.$

Proof. It follows from (2.8) that

$$\frac{1}{\epsilon} L = \widehat{B}^\epsilon(k) + ik \cdot \widehat{p}.$$

Thus we can obtain

$$e^{\frac{i}{\epsilon} L} \widehat{u} = e^{t \widehat{B}^\epsilon(k)} \widehat{u} + e^{t \widehat{B}^\epsilon(k)} (ik \cdot \widehat{p}) * e^{\frac{i}{\epsilon} L} \widehat{u}. \tag{4.30}$$

Letting $u \in \dot{X}_\beta^{\alpha,l}$ with $\beta > 3/2$, we have from Theorem 3.2 that $e^{t\widehat{B}^\epsilon(k)}\widehat{u} \in C^0((0,\infty)_\epsilon \times [0,\infty)_t, \dot{X}_\beta^{\alpha,l-1})$. Thus (4.30) holds in $\dot{X}_\beta^{\alpha,l-1}$ if $u \in \dot{X}_\beta^{\alpha,l}$.

Noting that $e^{tL/\epsilon}Q_0 = Q_0e^{tL/\epsilon} = Q_0e^{tL/\epsilon}Q_0$, we have from (3.6), (4.29), (4.30) and Theorem 2.1 that

$$\begin{aligned} & \mathcal{F}_x[(H_5^\epsilon - \widetilde{H}_5^\epsilon)f_R^\epsilon] \\ &= (Q(\epsilon k) - Q_0)e^{\frac{t}{\epsilon}L} * Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t) - Q(\epsilon k)e^{t\widehat{B}^\epsilon(k)}(ik \cdot \hat{p}) * e^{\frac{t}{\epsilon}L} * Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t) \\ &= \left\{ (Q(\epsilon k) - Q_0) - U(\epsilon, t, k)(ik \cdot \hat{p}) * \right\} \mathcal{F}_x \widetilde{H}_5^\epsilon f_R^\epsilon. \end{aligned} \quad (4.31)$$

For any $\beta_1, \beta_2 \in \mathbb{R}, 1 \leq j \leq 4$ and $\ell = 0, 1, 2$, we have from Theorem 2.1 that

$$\begin{aligned} & \|(Q(\epsilon k) - Q_0)f\|_{\beta_1} \\ &= \left\| \left(|\epsilon k| \sum_j P_j^{(1)}(\omega) + |\epsilon k|^2 \sum_j P_j^{(2)}(\omega) \right) f \right\|_{\beta_1} \leq C\epsilon |k| \|f\|_{\beta_2}. \end{aligned} \quad (4.32)$$

It follows from (4.29) and Theorem 2.1 that

$$\mathcal{F}_x \widetilde{H}_5^\epsilon f_R^\epsilon(t) = \int_0^t \left(e^{-\frac{t-s}{\epsilon}v(p)} + U_1(\epsilon, t-s, 0) \right) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(s) ds. \quad (4.33)$$

For the first term in (4.33), we have that

$$\begin{aligned} & \left| \epsilon |k| e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} \int_0^t e^{-\frac{t-s}{\epsilon}v(p)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(s) ds \right| \\ & \leq C\epsilon |k|^{1-\sigma} e^{(\alpha-\gamma t)|k|} (1+|k|)^l (1+|p|)^{\beta-1} \int_0^t e^{-\frac{t-s}{\epsilon}v(p)} \frac{v(p)}{\epsilon} |\Lambda^{-1} Q_0 \Lambda \widehat{f}_R^\epsilon(s)| ds \\ & \leq C\epsilon R^{1-\sigma} \|f^\epsilon\|_{\alpha, \gamma, l, \beta-1, \tau} \int_0^t e^{-\frac{t-s}{\epsilon}v(p)} \frac{v(p)}{\epsilon} ds \leq C\epsilon R^{1-\sigma} \|f\|_{\alpha, \gamma, l, \beta-1, \tau}. \end{aligned} \quad (4.34)$$

For the second term in (4.33), for $\beta > 5/2$, we have from Theorem 2.1 that

$$\begin{aligned} & \left| \epsilon |k| e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} \int_0^t U_1(\epsilon, t-s, 0) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(s) ds \right| \\ & \leq C\epsilon |k|^{1-\sigma} e^{(\alpha-\gamma t)|k|} (1+|k|)^l \int_0^t \left\| U_1(\epsilon, t-s, 0) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(s) \right\|_{\beta-1} ds \\ & \leq C\epsilon R^{1-\sigma} e^{(\alpha-\gamma t)|k|} (1+|k|)^l \int_0^t e^{-\frac{\sigma_0(t-s)}{\epsilon}} \frac{1}{\epsilon} \left(\|Q_0 \Lambda \widehat{f}_R^\epsilon\|_{\beta-1-\eta} + \|Q_0 \Lambda \widehat{f}_R^\epsilon(s)\| \right) ds \\ & \leq C\epsilon R^{1-\sigma} \|f^\epsilon\|_{\alpha, \gamma, l, \beta, \tau} \int_0^t e^{-\frac{\sigma_0(t-s)}{\epsilon}} \frac{1}{\epsilon} ds \leq C\epsilon R^{1-\sigma} \|f\|_{\alpha, \gamma, l, \beta, \tau}. \end{aligned} \quad (4.35)$$

Thus we have from (4.33)-(4.35) that

$$\epsilon|k|e^{(\alpha-\gamma t)|k|}(1+|k|)^{l-\sigma}\|\mathcal{F}_x(\tilde{H}_5^\epsilon f_R^\epsilon)\|_{\beta-1} \leq C\epsilon R^{1-\sigma}\|f\|_{\alpha,\gamma,l,\beta,\tau}.$$

By using this, (4.32) and (1.31), we arrive at

$$\|\mathcal{F}_x^{-1}[(Q(\epsilon k) - Q_0)\mathcal{F}_x(\tilde{H}_5^\epsilon f_R^\epsilon)]\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C\epsilon R^{1-\sigma}\|f\|_{\alpha,\gamma,l,\beta,\tau}. \tag{4.36}$$

Now we estimate the second term in (4.31). We see

$$e^{\frac{t}{\epsilon}L}f_0 = e^{-\frac{\nu(p)}{\epsilon}t}f_0 + \int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)}\frac{1}{\epsilon}Ke^{\frac{s}{\epsilon}L}f_0 ds. \tag{4.37}$$

It follows from this and (4.29) that

$$\mathcal{F}_x\tilde{H}_5^\epsilon f_R^\epsilon(t) = \int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)}Q_0\frac{1}{\epsilon}\Lambda\hat{f}_R^\epsilon(s)ds + \int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)}\frac{1}{\epsilon}K\mathcal{F}_x\tilde{H}_5^\epsilon f_R^\epsilon(s)ds. \tag{4.38}$$

For the first term in (4.38), we have that

$$\begin{aligned} & |k \cdot \hat{p}|e^{(\alpha-\gamma t)|k|}(1+|k|)^{l-\sigma}(1+|p|)^{\beta-1}\left|\int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)}Q_0\frac{1}{\epsilon}\Lambda\hat{f}_R^\epsilon(s)ds\right| \\ & \leq C|k|^{1-\sigma}e^{(\alpha-\gamma t)|k|}(1+|k|)^l(1+|p|)^{\beta-1}\int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)}\frac{\nu(p)}{\epsilon}|\Lambda^{-1}Q_0\Lambda\hat{f}_R^\epsilon(s)|ds \\ & \leq CR^{1-\sigma}\|f^\epsilon\|_{\alpha,\gamma,l,\beta-1,\tau}\int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)}\frac{\nu(p)}{\epsilon}ds \leq CR^{1-\sigma}\|f\|_{\alpha,\gamma,l,\beta,\tau}. \end{aligned} \tag{4.39}$$

For the second term, one has from (2.4) that

$$\begin{aligned} & |k \cdot \hat{p}|e^{(\alpha-\gamma t)|k|}(1+|k|)^{l-\sigma}(1+|p|)^{\beta-1}\left|\int_0^t e^{-\frac{\nu(p)}{\epsilon}(t-s)}\frac{1}{\epsilon}K\mathcal{F}_x\tilde{H}_5^\epsilon f_R^\epsilon(s)ds\right| \\ & \leq C|k|^{1-\sigma}e^{(\alpha-\gamma t)|k|}(1+|k|)^l\int_0^t e^{-\frac{\nu_0}{\epsilon}(t-s)}\frac{1}{\epsilon}\|\mathcal{F}_x\tilde{H}_5^\epsilon f_R^\epsilon(s)\|_{\beta-1-\eta}ds \\ & \leq CR^{1-\sigma}\|\tilde{H}_5^\epsilon f_R^\epsilon\|_{\alpha,\gamma,l,\beta-1,\tau}. \end{aligned} \tag{4.40}$$

Thus we can obtain

$$\begin{aligned} & \|\mathcal{F}_x^{-1}[(ik \cdot \hat{p})\mathcal{F}_x(\tilde{H}_5^\epsilon f_R^\epsilon)]\|_{\alpha-\gamma t, l-\sigma, \beta-1} \\ & \leq CR^{1-\sigma}(\|f\|_{\alpha,\gamma,l,\beta,\tau} + \|\tilde{H}_5^\epsilon f_R^\epsilon\|_{\alpha,\gamma,l,\beta-1,\tau}). \end{aligned} \tag{4.41}$$

By the similar arguments as (4.33)-(4.35), for $\beta > 5/2$, we can obtain

$$\|\tilde{H}_5^\epsilon f_R^\epsilon\|_{\alpha,\gamma,l,\beta-1,\tau} \leq C\|f\|_{\alpha,\gamma,l,\beta,\tau}.$$

In view of this and (4.41), we arrive at

$$\|\mathcal{F}_x^{-1}[(ik \cdot \hat{p})\mathcal{F}_x(\tilde{H}_5^\epsilon f_R^\epsilon)]\|_{\alpha,\gamma,l-\sigma,\beta-1,\tau} \leq CR^{1-\sigma}\|f\|_{\alpha,\gamma,l,\beta,\tau}. \quad (4.42)$$

Set $g = g^\epsilon = \mathcal{F}_x^{-1}[(ik \cdot \hat{p})\mathcal{F}_x(\tilde{H}_5^\epsilon f_R^\epsilon)]$. It follows from Theorem 2.1 that

$$U(\epsilon, t, k) * \hat{g}(t) = \int_0^t \left(e^{(t-s)\hat{A}^\epsilon(k)} Q(\epsilon k) + U_1(\epsilon, t-s, k) \right) \hat{g}(s) ds.$$

We see that

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} \left| \int_0^t e^{(t-s)\hat{A}^\epsilon(k)} Q(\epsilon k) \hat{g}^\epsilon(s) ds \right| \\ & \leq C e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \int_0^t e^{-\frac{\nu_0}{\epsilon}(t-s)} \|\hat{g}^\epsilon(s)\|_{\beta-1} ds \\ & \leq C \|g^\epsilon\|_{\alpha,\gamma,l-\sigma,\beta-1,\tau} \int_0^t e^{-\frac{\nu_0}{\epsilon}(t-s)} ds \leq C\epsilon \|g^\epsilon\|_{\alpha,\gamma,l-\sigma,\beta-1,\tau}. \end{aligned}$$

And we also have

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} \left| \int_0^t U_1(\epsilon, t-s, k) \hat{g}^\epsilon(s) ds \right| \\ & \leq C e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \int_0^t e^{-\frac{\sigma_0}{\epsilon}(t-s)} (\|\hat{g}^\epsilon(s)\|_{\beta-1-\eta} + \|\hat{g}^\epsilon(s)\|) ds \\ & \leq C \|g^\epsilon\|_{\alpha,\gamma,l-\sigma,\beta-1,\tau} \int_0^t e^{-\frac{\sigma_0}{\epsilon}(t-s)} ds \leq C\epsilon \|g^\epsilon\|_{\alpha,\gamma,l-\sigma,\beta-1,\tau}. \end{aligned}$$

Thus we can obtain

$$\|\mathcal{F}_x^{-1}[U(\epsilon, t, k) * \hat{g}^\epsilon(t)]\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C\epsilon \|g^\epsilon\|_{\alpha,\gamma,l-\sigma,\beta-1,\tau}.$$

By view of this and (4.42), we arrive at

$$\|\mathcal{F}_x^{-1}[U(\epsilon, t, k) * (ik \cdot \hat{p})\mathcal{F}_x(\tilde{H}_5^\epsilon f_R^\epsilon)]\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C\epsilon R^{1-\sigma} \|f\|_{\alpha,\gamma,l,\beta,\tau}.$$

We have from (4.31), (4.36) and this that

$$\|(H_5^\epsilon - \tilde{H}_5^\epsilon) f_R^\epsilon\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C\epsilon R^{1-\sigma} \|f\|_{\alpha,\gamma,l,\beta,\tau}.$$

By view of this and (4.2), we arrive at

$$[(H_5^\epsilon - \tilde{H}_5^\epsilon) f_R^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \leq C\tau^\sigma \epsilon^{1-\frac{\sigma}{2}} R^{1-\sigma} \|f\|_{\alpha,\gamma,l,\beta,\tau} \leq C\epsilon^{1-\frac{\sigma}{2}} R^{1-\sigma} \|f\|_{\alpha,\gamma,l,\beta,\tau}.$$

This completes the proof of the lemma. \square

Since the operator L has 0 as an isolated eigenvalue, L^{-1} does not exist but $L^{-1}Q_0$ does. Put

$$H_5^0 = -L^{-1}Q_0\Lambda. \tag{4.43}$$

Next we will prove that $L^{-1}Q_0\Lambda$ is bounded in $L_{\bar{\beta}}^\infty$ with $\bar{\beta} > 3/2$. Note that

$$L^{-1}Q_0\Lambda h_1 = -Q_0h_1 + L^{-1}Q_0Kh_1.$$

Assume that $h_2 \in \mathcal{N}^\perp$ and h_1 such that $L^{-1}Q_0Kh_1 = h_2$. Then

$$\Lambda^{-1}Q_0Kh_1 = -h_2 + \Lambda^{-1}Kh_2.$$

One has

$$\begin{aligned} \|h_2\|_{\bar{\beta}} &\leq \|\Lambda^{-1}Q_0Kh_1\|_{\bar{\beta}} + \|\Lambda^{-1}Kh_2\|_{\bar{\beta}} \\ &\leq C\|h_1\|_{\bar{\beta}-\eta-\frac{a}{2}} + C\|h_2\|_{\bar{\beta}-\eta-\frac{a}{2}}. \end{aligned}$$

We iterate this estimate successively to get

$$\begin{aligned} \|h_2\|_{\bar{\beta}} &\leq C\|h_1\|_{\bar{\beta}} + C\|Kh_1\|_0 + C\|Kh_2\|_0 \\ &\leq C\|h_1\|_{\bar{\beta}} + C\|h_1\| + C\|h_2\| \leq C\|h_1\|_{\bar{\beta}} + C\|h_2\|. \end{aligned}$$

We have from Lemma 2.3 that

$$\begin{aligned} \mu_0\|v^{\frac{1}{2}}h_2\|^2 &\leq -\langle Lh_2, h_2 \rangle = -\langle Q_0Kh_1, h_2 \rangle \\ &\leq \frac{\mu_0}{2}\|h_2\|^2 + C\|Kh_1\|^2 \leq \frac{\mu_0}{2}\|v^{\frac{1}{2}}h_2\|^2 + C\|h_1\|^2. \end{aligned}$$

For any $\bar{\beta} > 3/2$, we have from the above two estimates that

$$\|L^{-1}Q_0Kh_1\|_{\bar{\beta}} = \|h_2\|_{\bar{\beta}} \leq C\|h_1\|_{\bar{\beta}}.$$

Thus we arrive at

$$\|H_5^0h_1\|_{\bar{\beta}} = \|L^{-1}Q_0\Lambda h_1\|_{\bar{\beta}} = \|Q_0h_1 - L^{-1}Q_0Kh_1\|_{\bar{\beta}} \leq C\|h_1\|_{\bar{\beta}}. \tag{4.44}$$

This implies that H_5^0 is bounded on $\dot{Y}_{\bar{\beta}}^{\alpha, \gamma, l}((0, \tau])$ with any $\bar{\beta} > 3/2$.

Lemma 4.5. Recall H_5^0 of (4.43) and \tilde{H}_5^ϵ of (4.29). For $f = f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$ and any $\delta \in (0, t)$, one has

$$\|(\tilde{H}_5^\epsilon - H_5^0)f_R^\epsilon(t)\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C(e^{-\mu \frac{\delta}{\epsilon}} + \delta R^{1-\sigma})\|f\|_{\alpha, \gamma, l, \beta, \tau} + C\left(\frac{\epsilon}{t}\right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha, l-\sigma, \beta}^{t, \delta}.$$

Here $\mu = \min(\nu_0, \sigma_0)$ and the norm $\llbracket \cdot \rrbracket_{\alpha, l-\sigma, \beta}^{t, \delta}$ is defined as

$$\llbracket f^\epsilon \rrbracket_{\alpha, l-\sigma, \beta}^{t, \delta} = \sup_{t-\delta \leq s < t} \left(\frac{s}{t-s}\right)^\sigma \|\mathcal{F}_x^{-1}(e^{-\gamma|k|t}\widehat{f}^\epsilon(t) - e^{-\gamma|k|s}\widehat{f}^\epsilon(s))\|_{\alpha, l-\sigma, \beta}. \tag{4.45}$$

Proof. By using (4.29), for any $\delta \in (0, t)$, we make the decomposition

$$\begin{aligned}
 \mathcal{F}_x \tilde{H}_5^\epsilon f_R^\epsilon(t) &= \int_0^\delta e^{\frac{s}{\epsilon}L} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t) ds + \int_0^\delta e^{\frac{s}{\epsilon}L} Q_0 \frac{1}{\epsilon} \Lambda (\widehat{f}_R^\epsilon(t-s) - \widehat{f}_R^\epsilon(t)) ds \\
 &\quad + \int_\delta^t e^{\frac{s}{\epsilon}L} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t-s) ds \\
 &= \int_0^\infty e^{\frac{s}{\epsilon}L} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t) ds - \int_\delta^\infty e^{\frac{s}{\epsilon}L} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t) ds \\
 &\quad + \int_0^\delta e^{\frac{s}{\epsilon}L} Q_0 \frac{1}{\epsilon} \Lambda (\widehat{f}_R^\epsilon(t-s) - \widehat{f}_R^\epsilon(t)) ds + \int_\delta^t e^{\frac{s}{\epsilon}L} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t-s) ds \\
 &= \mathcal{F}_x H_5^0 f_R^\epsilon + \sum_{j=2}^4 w_j. \tag{4.46}
 \end{aligned}$$

Here we have used the fact that the Laplace transform of a semigroup is the resolvent of its generator. Then we first estimate the term w_4 . It follows from Theorem 2.1 that

$$w_4 = \int_\delta^t e^{\frac{s}{\epsilon}L} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t-s) ds = \int_\delta^t (e^{-\frac{s}{\epsilon}v(p)} Q_0 + U_1(\epsilon, s, 0)) \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t-s) ds.$$

We have from this and (4.26) that

$$\begin{aligned}
 &e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} \left| \int_\delta^t e^{-\frac{s}{\epsilon}v(p)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t-s) ds \right| \\
 &\leq e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \int_\delta^t e^{-\frac{s}{\epsilon}v(p)} \frac{v(p)}{\epsilon} \|\Lambda^{-1} Q_0 \Lambda \widehat{f}_R^\epsilon(t-s)\|_{\beta-1} ds \\
 &\leq C e^{-\nu_0 \frac{\delta}{\epsilon}} \|f\|_{\alpha, \gamma, l, \beta, \tau}.
 \end{aligned}$$

For the second term, for $\beta > 5/2$, one has

$$\begin{aligned}
 &e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} \left| \int_\delta^t U_1(\epsilon, s, 0) \frac{1}{\epsilon} \Lambda \widehat{f}_R^\epsilon(t-s) ds \right| \\
 &\leq e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \int_\delta^t e^{-\sigma_0 \frac{s}{\epsilon}} \frac{1}{\epsilon} \left(\|\Lambda \widehat{f}_R^\epsilon(t-s)\|_{\beta-1-\eta} + \|\Lambda \widehat{f}_R^\epsilon(t-s)\| \right) ds \\
 &\leq C e^{-\sigma_0 \frac{\delta}{\epsilon}} \|f\|_{\alpha, \gamma, l, \beta, \tau}.
 \end{aligned}$$

Thus we have from the above two estimates that

$$\|\mathcal{F}_x^{-1} w_4\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C e^{-\mu \frac{\delta}{\epsilon}} \|f\|_{\alpha, \gamma, l, \beta, \tau}. \tag{4.47}$$

By using Theorem 2.1 and the similar arguments as (4.47), we can obtain

$$\| \mathcal{F}_x^{-1} w_2 \|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C e^{-\mu \frac{\delta}{\epsilon}} \| f \|_{\alpha, \gamma, l, \beta, \tau}. \tag{4.48}$$

Next we estimate the term w_3 . We easily see

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (\widehat{f}_R^\epsilon(t-s) - \widehat{f}_R^\epsilon(t)) \\ &= e^{\alpha|k|} (1+|k|)^{l-\sigma} (e^{-\gamma(t-s)|k|} \widehat{f}_R^\epsilon(t-s) - e^{-\gamma t|k|} \widehat{f}_R^\epsilon(t)) \\ & \quad + e^{(\alpha-\gamma(t-s))|k|} (1+|k|)^{l-\sigma} (e^{-\gamma s|k|} - 1) \widehat{f}_R^\epsilon(t-s) \\ & := h_1(t, s, k, p) + h_2(t, s, k, p). \end{aligned} \tag{4.49}$$

For the term $h_2(t, s, k, p)$, we have from Theorem 2.1 that

$$\int_0^\delta e^{\frac{s}{\epsilon} L} Q_0 \frac{1}{\epsilon} \Lambda h_2(t, s, k, p) ds = \int_0^\delta (e^{-\frac{s}{\epsilon} \nu(p)} Q_0 + U_1(\epsilon, s, 0)) \frac{1}{\epsilon} \Lambda h_2(t, s, k, p) ds. \tag{4.50}$$

For $0 < s \leq t$, by using (4.6), (4.26) and (4.49), we see

$$\begin{aligned} |h_2(t, s, k, p)| &\leq e^{(\alpha-\gamma(t-s))|k|} (1+|k|)^{l-\sigma} |e^{-\gamma s|k|} - 1| \widehat{f}_R^\epsilon(t-s)| \\ &\leq C \gamma s |k| e^{(\alpha-\gamma(t-s))|k|} (1+|k|)^{l-\sigma} |\widehat{f}_R^\epsilon(t-s)| \\ &\leq C s R^{1-\sigma} e^{(\alpha-\gamma(t-s))|k|} (1+|k|)^l |\widehat{f}^\epsilon(t-s)|. \end{aligned}$$

For the term $h_2(t, s, k, p)$, we have from this, (4.18) and (4.50) that

$$\begin{aligned} & \left\| \int_0^\delta e^{-\frac{s}{\epsilon} \nu(p)} Q_0 \frac{1}{\epsilon} \Lambda h_2(t, s, k, p) ds \right\|_{\beta-1} \\ &\leq \sup_p \int_0^\delta e^{-\frac{s}{\epsilon} \nu(p)} \frac{\nu(p)}{\epsilon} \|\Lambda^{-1} Q_0 \Lambda h_2(t, s, k, p)\|_{\beta-1} ds \\ &\leq C R^{1-\sigma} \|f^\epsilon\|_{\alpha, \gamma, l, \beta, \tau} \sup_p \int_0^\delta e^{-\frac{s}{\epsilon} \nu(p)} \frac{\nu(p)}{\epsilon} s ds \leq C \delta R^{1-\sigma} \|f\|_{\alpha, \gamma, l, \beta, \tau}. \end{aligned}$$

For the second term in (4.50), for $\beta > 5/2$, we have from Theorem 2.1 that

$$\begin{aligned} & \left\| \int_0^\delta U_1(\epsilon, s, 0) \frac{1}{\epsilon} \Lambda h_2(t, s, k, p) ds \right\|_{\beta-1} \\ &\leq \int_0^\delta e^{-\sigma_0 \frac{s}{\epsilon}} \frac{1}{\epsilon} (\|\Lambda h_2(t, s, k, p)\|_{\beta-1-\eta} + \|\Lambda h_2(t, s, k, p)\|) ds \\ &\leq C R^{1-\sigma} \|f^\epsilon\|_{\alpha, \gamma, l, \beta, \tau} \int_0^\delta e^{-\sigma_0 \frac{s}{\epsilon}} \frac{1}{\epsilon} s ds \leq C \delta R^{1-\sigma} \|f\|_{\alpha, \gamma, l, \beta, \tau}. \end{aligned}$$

Here we used the fact that

$$\|\Lambda h_2(t,s,k,p)\|_{\beta-1-\eta} + \|\Lambda h_2(t,s,k,p)\| \leq C \|h_2(t,s,k,p)\|_{\beta}.$$

For $0 < s < \delta$, we have from (4.49) and (4.45) that

$$\begin{aligned} \|h_1(t,s,k,p)\|_{\beta-1} &\leq \|\mathcal{F}_x^{-1}(e^{-\gamma t|k|}\widehat{f}_R^\epsilon(t) - e^{-\gamma(t-s)|k|}\widehat{f}_R^\epsilon(t-s))\|_{\alpha,l-\sigma,\beta-1} \\ &\leq C\left(\frac{s}{t-s}\right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha,l-\sigma,\beta-1}^{t,\delta}. \end{aligned}$$

For the term $h_1(t,s,k,p)$, we have from (4.50) that

$$\begin{aligned} &\left\| \int_0^\delta e^{-\frac{s}{\epsilon}v(p)} Q_0 \frac{1}{\epsilon} \Lambda h_1(t,s,k,p) ds \right\|_{\beta-1} \\ &\leq \sup_p \int_0^\delta e^{-\frac{s}{\epsilon}v(p)} \frac{v(p)}{\epsilon} \|\Lambda^{-1} Q_0 \Lambda h_1(t,s,k,p)\|_{\beta-1} ds \\ &\leq C \llbracket f^\epsilon \rrbracket_{\alpha,l-\sigma,\beta-1}^{t,\delta} \sup_p \int_0^\delta e^{-\frac{s}{\epsilon}v(p)} \frac{v(p)}{\epsilon} \left(\frac{s}{t-s}\right)^\sigma ds \leq C \left(\frac{\epsilon}{t}\right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha,l-\sigma,\beta-1}^{t,\delta}. \end{aligned}$$

Here we used the fact that

$$\int_0^t e^{-b(t-s)} b^{1+\sigma} \left(\frac{t-s}{s}\right)^\sigma ds \leq C t^{-\sigma}. \tag{4.51}$$

For the second term in (4.50), for $\beta > 5/2$, one has

$$\begin{aligned} &\left\| \int_0^\delta U_1(\epsilon,s,0) \frac{1}{\epsilon} \Lambda h_1(t,s,k,p) ds \right\|_{\beta-1} \\ &\leq \int_0^\delta e^{-\sigma_0 \frac{s}{\epsilon}} \frac{1}{\epsilon} (\|\Lambda h_1(t,s,k,p)\|_{\beta-1-\eta} + \|\Lambda h_1(t,s,k,p)\|) ds \\ &\leq C \llbracket f^\epsilon \rrbracket_{\alpha,l-\sigma,\beta}^{t,\delta} \int_0^\delta e^{-\sigma_0 \frac{s}{\epsilon}} \frac{1}{\epsilon} \left(\frac{s}{t-s}\right)^\sigma ds \leq C \left(\frac{\epsilon}{t}\right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha,l-\sigma,\beta}^{t,\delta}. \end{aligned}$$

By using (4.46), (4.49) and the estimates of the term $h_1(t,s,k,p)$ and $h_2(t,s,k,p)$ as the above, we can obtain

$$e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} |w_3| \leq C \delta R^{1-\sigma} \|f\|_{\alpha,\gamma,l,\beta,\tau} + C \left(\frac{\epsilon}{t}\right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha,l-\sigma,\beta}^{t,\delta}.$$

This implies

$$\|\mathcal{F}_x^{-1} w_3\|_{\alpha-\gamma t,l-\sigma,\beta-1} \leq C \delta R^{1-\sigma} \|f\|_{\alpha,\gamma,l,\beta,\tau} + C \left(\frac{\epsilon}{t}\right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha,l-\sigma,\beta}^{t,\delta}.$$

By using (4.46)-(4.48) and this, we can get the desired estimate. This completes the proof of the lemma. □

Lemma 4.6. *Suppose that $f = f^\epsilon \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit f^0 . Recall H^ϵ of (3.3) and put $H^0 = \sum_{j=0}^5 H_j^0$. Then one has*

(i) $H^\epsilon f^\epsilon \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit $H^0 f^0$.

(ii)

$$\begin{aligned}
 & [H^\epsilon f^\epsilon - H^0 f^0]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \\
 & \leq C \left(1 + \frac{1}{\gamma} \right) \left(\|f\|_{\alpha,\gamma,l,\beta,\tau} + [f^\epsilon - f^0]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} + \sup_{0 < \epsilon' \leq \epsilon} \llbracket f^{\epsilon'} \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma} \right).
 \end{aligned}$$

Here $\sigma \in (0,1)$ and the norm $\llbracket f^\epsilon \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma}$ is defined as

$$\llbracket f^\epsilon \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma} = \sup_{0 \leq s < t \leq \tau} \left(\frac{s}{t-s} \right)^\sigma \|f^\epsilon(t) - f^\epsilon(s)\|_{\alpha-\gamma t, l-\sigma, \beta}. \tag{4.52}$$

Proof. It follows from Theorem 3.2 that $H^\epsilon f^\epsilon \in Z_{\beta,\tau}^{\alpha,\gamma,l}$ with any $\beta > 5/2$. By using (4.44) and Lemma 4.2(i), we know that $H^0 f^0 \in \dot{Y}_\beta^{\alpha,\gamma,l}((0,\tau])$ with any $\beta > 5/2$. We have from (4.26) and the assumptions that

$$H^\epsilon f^\epsilon - H^0 f^0 = (H^\epsilon - H^0) f_R^\epsilon + (H^\epsilon - H^0) \tilde{f}_R^\epsilon + H^0 (f^\epsilon - f^0). \tag{4.53}$$

It follows from (4.44) that $H_5^0 f^\epsilon \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit $H_5^0 f^0$. And for any $0 \leq j \leq 4$, we have from Lemma 4.2(i) that $H_j^0 f^\epsilon \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit $H_j^0 f^0$. Thus, for any $\varsigma > 0$, $\|H^0 (f^\epsilon - f^0)\|_{\dot{Y}_{\beta-1}^{\alpha,\gamma,l}([\varsigma,\tau])} \rightarrow 0$ as $\epsilon \rightarrow 0$.

Put $R = \kappa_0 \epsilon^{-1/4}$. By (4.26), (4.44) and the similar arguments as (3.12), we arrive at

$$\begin{aligned}
 & \|(H^\epsilon - H^0) \tilde{f}_R^\epsilon\|_{\alpha-\gamma t, l, \beta-1} \\
 & \leq C \left(1 + \frac{1}{\gamma} \right) \|\mathcal{F}_x^{-1}(\chi(|k| + |p| > R) \hat{f}^\epsilon)\|_{\alpha,\gamma,l,\beta,\tau} \rightarrow 0 \text{ as } \epsilon \rightarrow 0.
 \end{aligned}$$

Recalling (4.26) and H_6^ϵ of (3.6), then $H_6^\epsilon f_R^\epsilon = 0$. Thus, we have from this, Lemmas 4.3(i), 4.4(i) and 4.5 with $\sigma = 0$ that

$$\|(H^\epsilon - H^0) f_R^\epsilon\|_{\alpha-\gamma t, l, \beta-1} \leq C \left(1 + \frac{1}{\gamma} \right) (\epsilon R^2 + \delta R + e^{-\mu \frac{\delta}{\epsilon}}) \|f\|_{\alpha,\gamma,l,\beta,\tau} + C \llbracket f^\epsilon \rrbracket_{\alpha,l,\beta}^{t,\delta}.$$

Note that

$$\mathcal{F}_x^{-1}(e^{-\gamma t|k|}\widehat{f^\epsilon}(t)) \in B^0([0, \infty)_\epsilon \times [0, \tau]_t; \dot{X}_\beta^{\alpha, l}).$$

We further put $\delta = \epsilon^{1/2}$ and we have from (4.45) that $\|f^\epsilon\|_{\alpha, l, \beta}^{t, \delta} \rightarrow 0$ as $\epsilon \rightarrow 0$. Thus we arrive at

$$\|(H^\epsilon - H^0)f_R^\epsilon\|_{\alpha - \gamma t, l, \beta - 1} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

By virtue of (4.53) and the above estimates, for any $\zeta > 0$, we have

$$\|H^\epsilon f^\epsilon - H^0 f^0\|_{\dot{Y}_{\beta-1}^{\alpha, \gamma, l}([\zeta, \tau])} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Thus we have proved (i). Then we prove (ii). For any $\beta > 5/2$ and $\sigma \in (0, 1)$, we have from (4.44) that

$$\|H_5^0(f^\epsilon - f^0)\|_{\alpha - \gamma t, l - \sigma, \beta - 1} \leq C \|f^\epsilon - f^0\|_{\alpha - \gamma t, l - \sigma, \beta - 1}.$$

It follows from Lemma 4.2(ii) that, for any $0 \leq j \leq 4$,

$$[H_j^0(f^\epsilon - f^0)]_{\alpha, \gamma, l, \beta - 1}^{\epsilon, \tau, \sigma} \leq \frac{C}{\gamma} [f^\epsilon - f^0]_{\alpha, \gamma, l, \beta - 1}^{\epsilon, \tau, \sigma}.$$

By (4.2) and the above two estimates we arrive at

$$[H^0(f^\epsilon - f^0)]_{\alpha, \gamma, l, \beta - 1}^{\epsilon, \tau, \sigma} \leq C \left(1 + \frac{1}{\gamma}\right) [f^\epsilon - f^0]_{\alpha, \gamma, l, \beta - 1}^{\epsilon, \tau, \sigma}. \tag{4.54}$$

Put $R = \kappa_0 \epsilon^{-1/2}$. For $\beta > 5/2$ and $l > \sigma$, by (4.26), (4.44) and the similar arguments as (3.12), we arrive at

$$\begin{aligned} \|(H^\epsilon - H^0)\widetilde{f}_R^\epsilon\|_{\alpha - \gamma t, l - \sigma, \beta - 1} &\leq C \left(1 + \frac{1}{\gamma}\right) \|\mathcal{F}_x^{-1}(\chi(|k| > R)\widehat{f^\epsilon})\|_{\alpha, \gamma, l - \sigma, \beta, \tau} \\ &\leq C \left(1 + \frac{1}{\gamma}\right) (1 + R)^{-\sigma} \|f\|_{\alpha, \gamma, l, \beta, \tau}. \end{aligned}$$

By (4.2) and this, one has

$$[(H^\epsilon - H^0)\widetilde{f}_R^\epsilon]_{\alpha, \gamma, l, \beta - 1}^{\epsilon, \tau, \sigma} \leq C \tau^\sigma \left(1 + \frac{1}{\gamma}\right) \|f\|_{\alpha, \gamma, l, \beta, \tau} \leq C \left(1 + \frac{1}{\gamma}\right) \|f\|_{\alpha, \gamma, l, \beta, \tau}. \tag{4.55}$$

For any $s \in (0, t)$, by (4.6), one has

$$e^{(\alpha - \gamma s)|k|} (1 + |k|)^{l - \sigma} \|(1 - e^{-\gamma(t-s)|k|})\widehat{f^\epsilon}(s)\|_\beta \leq C \gamma^\sigma (t - s)^\sigma \|f^\epsilon\|_{\alpha - \gamma s, l, \beta}.$$

Thus we can obtain

$$\begin{aligned} & \|\mathcal{F}_x^{-1}(e^{-\gamma|k|t}\widehat{f}^\epsilon(t) - e^{-\gamma|k|s}\widehat{f}^\epsilon(s))\|_{\alpha,l-\sigma,\beta} \\ & \leq C\|f^\epsilon(t) - f^\epsilon(s)\|_{\alpha-\gamma t,l-\sigma,\beta} + C\gamma^\sigma(t-s)^\sigma\|f^\epsilon\|_{\alpha-\gamma s,l,\beta}. \end{aligned}$$

By using this, (4.45) and (4.52), we arrive at

$$\llbracket f^\epsilon \rrbracket_{\alpha,l-\sigma,\beta}^{t,\delta} \leq C\llbracket f^\epsilon \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma} + C(\gamma\tau)^\sigma\|f^\epsilon\|_{\alpha,\gamma,l,\beta,\tau} \leq C(\llbracket f^\epsilon \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma} + \|f\|_{\alpha,\gamma,l,\beta,\tau}).$$

Put $R = \kappa_0\epsilon^{-1/2}$ and $\delta = \epsilon^{1/2}t$. We have from (4.2), (4.18), Lemma 4.5 and these facts that

$$\begin{aligned} & [(\widetilde{H}_5^\epsilon - H_5^0)f_R^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \\ & \leq C(1 + \tau^{1+\sigma})\|f\|_{\alpha,\gamma,l,\beta,\tau} + C\epsilon^{\frac{\sigma}{2}} \left(\sup_{0 < \epsilon' \leq \epsilon} \llbracket f^{\epsilon'} \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma} + \|f\|_{\alpha,\gamma,l,\beta,\tau} \right) \\ & \leq C\|f\|_{\alpha,\gamma,l,\beta,\tau} + C \sup_{0 < \epsilon' \leq \epsilon} \llbracket f^{\epsilon'} \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma}. \end{aligned} \tag{4.56}$$

It follows from Lemma 4.4(ii) that

$$[(H_5^\epsilon - \widetilde{H}_5^\epsilon)f_R^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \leq C\epsilon^{1-\frac{\sigma}{2}}R^{1-\sigma}\|f\|_{\alpha,\gamma,l,\beta,\tau} \leq C\epsilon^{\frac{1}{2}}\|f\|_{\alpha,\gamma,l,\beta,\tau}. \tag{4.57}$$

We have from (4.56) and (4.57) that

$$[(H_5^\epsilon - H_5^0)f_R^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \leq C\|f\|_{\alpha,\gamma,l,\beta,\tau} + C \sup_{0 < \epsilon' \leq \epsilon} \llbracket f^{\epsilon'} \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma}. \tag{4.58}$$

It follows from Lemma 4.3(ii) that, for any $0 \leq j \leq 4$,

$$[(H_j^\epsilon - H_j^0)f_R^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \leq \frac{C}{\gamma}\epsilon^{1-\frac{\sigma}{2}}R^{2-\sigma}\|f\|_{\alpha,\gamma,l,\beta,\tau} \leq \frac{C}{\gamma}\|f\|_{\alpha,\gamma,l,\beta,\tau}. \tag{4.59}$$

Noticing that $H_6^\epsilon f_R^\epsilon = 0$, we have from (4.58) and (4.59) that

$$[(H^\epsilon - H^0)f_R^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \leq C\left(1 + \frac{1}{\gamma}\right)\|f\|_{\alpha,\gamma,l,\beta,\tau} + C \sup_{0 < \epsilon' \leq \epsilon} \llbracket f^{\epsilon'} \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma}. \tag{4.60}$$

By using (4.53)-(4.55) and (4.60) we can deduce Lemma 4.6(ii). This completes the proof of the lemma. \square

In what follows we shall give the proof of Theorem 1.1(ii) and (iii).

Proof of assertions (ii) and (iii) of Theorem 1.1. Suppose that $f = f^\epsilon \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit f^0 . For any $\beta > 5/2$, we have from Theorem 3.2 that

$$\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) \in Z_{\beta,\tau}^{\alpha,\gamma,l}, \quad \Lambda^{-1}\Gamma(f^0, f^0) \in \dot{Y}_\beta^{\alpha,\gamma,l}((0, \tau]).$$

By Lemma 3.3 one has

$$\begin{aligned} & \|\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) - \Lambda^{-1}\Gamma(f^0, f^0)\|_{\alpha-\gamma t, l, \beta-1} \\ &= \|\Lambda^{-1}\Gamma(f^\epsilon - f^0, f^\epsilon + f^0)\|_{\alpha-\gamma t, l, \beta-1} \\ &\leq C\|f^\epsilon + f^0\|_{\alpha-\gamma t, l, \beta-1}\|f^\epsilon - f^0\|_{\alpha-\gamma t, l, \beta-1}. \end{aligned}$$

For any $\varsigma > 0$, we have from this that

$$\begin{aligned} & \|\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) - \Lambda^{-1}\Gamma(f^0, f^0)\|_{\dot{Y}_{\beta-1}^{\alpha,\gamma,l}([\varsigma, \tau])} \\ &\leq C\left(\|f\|_{\alpha,\gamma,l,\beta,\tau} + \|f^0\|_{\dot{Y}_\beta^{\alpha,\gamma,l}((0,\tau])}\right)\|f^\epsilon - f^0\|_{\dot{Y}_{\beta-1}^{\alpha,\gamma,l}([\varsigma, \tau])}. \end{aligned}$$

This implies

$$\|\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) - \Lambda^{-1}\Gamma(f^0, f^0)\|_{\dot{Y}_{\beta-1}^{\alpha,\gamma,l}([\varsigma, \tau])} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Hence, $\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit $\Lambda^{-1}\Gamma(f^0, f^0)$. By using this and Lemma 4.6(i), we deduce that $H^\epsilon \Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit $H^0 \Lambda^{-1}\Gamma(f^0, f^0)$. We have from Lemma 4.1(i) that $e^{tB^\epsilon} f_0 \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit $E^0(t) f_0$. Hence, N^ϵ in (3.20) is a contraction map in the space $W_{\beta,\tau}^{\alpha,\gamma,l} \cap Z_0$ by Theorem 3.2 and then the solution $f^\epsilon(t)$ in Theorem 1.1(i) is in $W_{\beta,\tau}^{\alpha,\gamma,l}$. This completes the proof of Theorem 1.1(ii). Then, for any $t \in (0, \tau]$, we arrive at

$$f^0(t) = E^0(t) f_0 + H^0 \Lambda^{-1}\Gamma(f^0(t), f^0(t)). \tag{4.61}$$

Noting that $Q_0 P_j^{(0)}(\omega) = 0$, we have from (4.3) that $Q_0 E^0(t) = 0$. It follows from (2.34) that $Q_0 P_j^{(1)}(\omega) Q_0 = 0$. Recalling (4.19), we deduce that $Q_0 H_j^0 = 0$ for $0 \leq j \leq 4$. It follows from these facts, (4.19), (4.43) and (4.61) that

$$\begin{aligned} Q_0 f^0(t) &= Q_0 H_5^0 \Lambda^{-1}\Gamma(f^0(t), f^0(t)) = -L^{-1}\Gamma(f^0(t), f^0(t)), \\ L f^0(t) &= -\Gamma(f^0(t), f^0(t)). \end{aligned} \tag{4.62}$$

If we put $F^0(t) = J_0 + \sqrt{J_0} f^0(t)$, we arrive at the relation (1.15). Thus $F^0(t)$ is a local relativistic Maxwellian as (1.12).

By using (2.34) and (4.61), we have from (4.3), (4.19) and (4.43) that $P_0 f^0(0) = P_0 f_0$, which gives the initial data (1.25) of the relativistic Euler equations (1.24). Since $f^\epsilon(t)$ is analytic in $x \in \mathbb{R}^3 + iB_{\alpha-\gamma t}$, then the hydrodynamical quantities of $F^0(t)$ is a classical solution to the system (1.24). This completes the proof of Theorem 1.1(iii). \square

5 Convergence rate of the solution

In this section we will prove the convergence rate about ϵ of $f^\epsilon(t) \rightarrow f^0(t)$ as $\epsilon \rightarrow 0$ in (4.1). For this, we first consider the Hölder continuity in t of the solutions to the linearized relativistic Boltzmann equation uniformly for $\epsilon \in (0,1)$.

Lemma 5.1. *Suppose that $f_0 \in \dot{X}_\beta^{\alpha,l}$. Recalling (4.52), for any $\beta > 3/2$ and $l > \sigma$ with $\sigma \in (0,1)$, one has*

$$\| [e^{tB^\epsilon} f_0]_{\alpha,\gamma,l,\beta}^{\tau,\sigma} \leq C \| f_0 \|_{\alpha,l,\beta}. \tag{5.1}$$

Proof. For any $\beta_1, \beta_2 > 0$, $0 \leq j \leq 4$, and $0 < s < t \leq \tau$, we have from (3.2), (4.6) and Theorem 2.1 that

$$\begin{aligned} \| \mathcal{F}_x(E_j^\epsilon(t) - E_j^\epsilon(s)) f_0 \|_{\beta_1} &= \left\| \chi \left(|k| < \frac{\kappa_0}{\epsilon} \right) e^{\lambda_j(|\epsilon k| \frac{s}{\epsilon})} (e^{\lambda_j(|\epsilon k| \frac{t-s}{\epsilon})} - 1) P_j(\epsilon k) \widehat{f}_0 \right\|_{\beta_1} \\ &\leq C s^\sigma |k|^\sigma \left(\frac{t-s}{s} \right)^\sigma \| \widehat{f}_0 \|_{\beta_2}. \end{aligned}$$

By this we can obtain

$$\| (E_j^\epsilon(t) - E_j^\epsilon(s)) f_0 \|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C \left(\frac{t-s}{s} \right)^\sigma \| f_0 \|_{\alpha,l,\beta-1}. \tag{5.2}$$

We notice that

$$\begin{aligned} \mathcal{F}_x E_6^\epsilon(t) f_0 &= \chi \left(|k| > \frac{\kappa_0}{\epsilon} \right) e^{t\widehat{B}^\epsilon(k)} f_0 \\ &= \chi \left(|k| > \frac{\kappa_0}{\epsilon} \right) e^{t\widehat{A}^\epsilon(k)} f_0 + \chi \left(|k| > \frac{\kappa_0}{\epsilon} \right) e^{t\widehat{A}^\epsilon(k)} \frac{1}{\epsilon} K * \mathcal{F}_x E_6^\epsilon(t) f_0. \end{aligned} \tag{5.3}$$

For any $\bar{\beta} \geq 0$, we first prove that

$$\| \mathcal{F}_x^{-1} (e^{t\widehat{A}^\epsilon(k)} \widehat{f}_0 - e^{s\widehat{A}^\epsilon(k)} \widehat{f}_0) \|_{\alpha-\gamma t, l-\sigma, \bar{\beta}} \leq C \left(\frac{t-s}{s} \right)^\sigma \| f_0 \|_{\alpha,l,\bar{\beta}}. \tag{5.4}$$

In fact, we have from (4.6), (4.18), (2.1) and the bounded-ness of \hat{p} that

$$\begin{aligned}
 |e^{t\hat{A}^\epsilon(k)}\hat{f}_0 - e^{s\hat{A}^\epsilon(k)}\hat{f}_0| &= |e^{\frac{s}{\epsilon}(-\nu(p)+i\epsilon k\cdot\hat{p})} (e^{\frac{t-s}{\epsilon}(-\nu(p)+i\epsilon k\cdot\hat{p})} - 1)\hat{f}_0| \\
 &\leq Ce^{-\nu(p)\frac{s}{\epsilon}} \left(\left(\nu(p)\frac{t-s}{\epsilon} \right)^\sigma + |(t-s)k\cdot\hat{p}|^\sigma \right) |\hat{f}_0| \\
 &\leq Ce^{-\nu(p)\frac{s}{\epsilon}} \left(\nu(p)\frac{s}{\epsilon} \right)^\sigma \left(\frac{t-s}{s} \right)^\sigma \left(1 + \left| \frac{\epsilon k\cdot\hat{p}}{\nu(p)} \right|^\sigma \right) |\hat{f}_0| \\
 &\leq C \left(\frac{t-s}{s} \right)^\sigma (1+|k|)^\sigma |\hat{f}_0|.
 \end{aligned}
 \tag{5.5}$$

Thus (5.4) follows from this. For the second term in (5.3), one has

$$\begin{aligned}
 &\int_0^t e^{(t-s_1)\hat{A}^\epsilon(k)} \frac{1}{\epsilon} K\mathcal{F}_x E_6^\epsilon(s_1) f_0 ds_1 - \int_0^s e^{(s-s_1)\hat{A}^\epsilon(k)} \frac{1}{\epsilon} K\mathcal{F}_x E_6^\epsilon(s_1) f_0 ds_1 \\
 &= (e^{(t-s)\hat{A}^\epsilon(k)} - 1) \int_0^s e^{(s-s_1)\hat{A}^\epsilon(k)} \frac{1}{\epsilon} K\mathcal{F}_x E_6^\epsilon(s_1) f_0 ds_1 \\
 &\quad + \int_s^t e^{(t-s_1)\hat{A}^\epsilon(k)} \frac{1}{\epsilon} K\mathcal{F}_x E_6^\epsilon(s_1) f_0 ds_1.
 \end{aligned}
 \tag{5.6}$$

We denote the two terms in (5.6) by I_1 and I_2 . By the similar arguments as (5.5), one has

$$|e^{(t-s)\hat{A}^\epsilon(k)} - 1| \leq C \left(\nu(p)\frac{t-s}{\epsilon} \right)^\sigma (1+|k|)^\sigma.$$

Note that $\eta \in (0,1]$, $\sigma \in (0,1)$. For the term I_1 , we have from this, (2.1), (4.18) and Theorem 2.1 that

$$\begin{aligned}
 |I_1| &\leq C \left(\nu(p)\frac{t-s}{\epsilon} \right)^\sigma (1+|k|)^\sigma \left| \int_0^s e^{(s-s_1)\hat{A}^\epsilon(k)} \frac{1}{\epsilon} K\mathcal{F}_x E_6^\epsilon(s_1) f_0 ds_1 \right| \\
 &\leq C \left(\frac{t-s}{\epsilon} \right)^\sigma (1+|k|)^\sigma (1+|p|)^{-(\beta-1)} \int_0^s e^{-\frac{\nu_0}{\epsilon}(s-s_1)} \frac{1}{\epsilon} \|\mathcal{F}_x E_6^\epsilon(s_1) f_0\|_{\beta-1-\eta+\sigma} ds_1 \\
 &\leq C \left(\frac{t-s}{\epsilon} \right)^\sigma (1+|k|)^\sigma (1+|p|)^{-(\beta-1)} \\
 &\quad \times \int_0^s e^{-\frac{\nu_0}{\epsilon}(s-s_1)} e^{-\frac{\mu}{\epsilon}s_1} \frac{1}{\epsilon} (\|\hat{f}_0\|_{\beta-1-\eta+\sigma} + \|\hat{f}_0\|) ds_1 \\
 &\leq C \left(\frac{t-s}{\epsilon} \right)^\sigma (1+|k|)^\sigma (1+|p|)^{-(\beta-1)} e^{-\frac{\mu}{\epsilon}s} \left(\frac{s}{\epsilon} \right) \|\hat{f}_0\|_\beta \\
 &\leq C \left(\frac{t-s}{s} \right)^\sigma (1+|k|)^\sigma (1+|p|)^{-(\beta-1)} \|\hat{f}_0\|_\beta.
 \end{aligned}$$

By (4.18), for any $\sigma \in (0,1)$ and $s \in (0,t)$, one has

$$e^{-\frac{\mu}{\epsilon}t} \left(\frac{t-s}{\epsilon}\right) = \left(\frac{t-s}{s}\right)^\sigma e^{-\frac{\mu}{\epsilon}(t-s)} \left(\frac{t-s}{\epsilon}\right)^{1-\sigma} e^{-\frac{\mu}{\epsilon}s} \left(\frac{s}{\epsilon}\right)^\sigma \leq C \left(\frac{t-s}{s}\right)^\sigma. \tag{5.7}$$

For the term I_2 , we have from this, (2.4) and Theorem 2.1 that

$$\begin{aligned} |I_2| &\leq C(1+|p|)^{-(\beta-1)} \int_s^t e^{-\frac{\nu_0}{\epsilon}(t-s_1)} \frac{1}{\epsilon} \|\mathcal{F}_x E_6^\epsilon(s_1) f_0\|_{\beta-1-\eta} ds_1 \\ &\leq C(1+|p|)^{-(\beta-1)} \int_s^t e^{-\frac{\nu_0}{\epsilon}(t-s_1)} e^{-\frac{\mu}{\epsilon}s_1} \frac{1}{\epsilon} (\|\widehat{f}_0\|_{\beta-1-2\eta} + \|\widehat{f}_0\|) ds_1 \\ &\leq C(1+|p|)^{-(\beta-1)} e^{-\frac{\mu}{\epsilon}t} \left(\frac{t-s}{\epsilon}\right) \|\widehat{f}_0\|_\beta \leq C \left(\frac{t-s}{s}\right)^\sigma (1+|p|)^{-(\beta-1)} \|\widehat{f}_0\|_\beta. \end{aligned}$$

Thus, by using (5.6) and the above estimates, we arrive at

$$\begin{aligned} &\left\| \mathcal{F}_x^{-1} \int_0^t e^{(t-s_1)\widehat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x E_6^\epsilon(s_1) f_0 ds_1 \right. \\ &\quad \left. - \mathcal{F}_x^{-1} \int_0^s e^{(s-s_1)\widehat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x E_6^\epsilon(s_1) f_0 ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta-1} \\ &\leq C \left(\frac{t-s}{s}\right)^\sigma \|f_0\|_{\alpha, l, \beta}. \end{aligned} \tag{5.8}$$

By using (5.3), (5.4) and this, we can obtain

$$\|(E_6^\epsilon(t) - E_6^\epsilon(s)) f_0\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C \left(\frac{t-s}{s}\right)^\sigma \|f_0\|_{\alpha, l, \beta}. \tag{5.9}$$

By the similar arguments as this, the term $E_5^\epsilon(t) f_0$ shares the same estimate. Thus we have from (5.2), (5.9) and (3.2) that

$$\|e^{tB^\epsilon} f_0 - e^{sB^\epsilon} f_0\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C \left(\frac{t-s}{s}\right)^\sigma \|f_0\|_{\alpha, l, \beta}.$$

Recalling (4.52), we have from this that

$$\| [e^{tB^\epsilon} f_0]_{\alpha, \gamma, l, \beta-1}^{\tau, \sigma} \| \leq C \|f_0\|_{\alpha, l, \beta}. \tag{5.10}$$

We shall recover the loss of the p -weight by the smoothing property of K in (2.4). We notice that

$$\widehat{g}^\epsilon(t) = e^{t\widehat{B}^\epsilon(k)} \widehat{f}_0 = e^{t\widehat{A}^\epsilon(k)} \widehat{f}_0 + e^{t\widehat{A}^\epsilon(k)} \frac{1}{\epsilon} K * \widehat{g}^\epsilon(t). \tag{5.11}$$

For any $b > 0$, by (4.6) and (4.18), one has

$$\int_s^t e^{-bs_1} b ds_1 = e^{-bs} (1 - e^{-b(t-s)}) \leq C e^{-bs} (bs)^\sigma (bs)^{-\sigma} (b(t-s))^\sigma \leq C \left(\frac{t-s}{s}\right)^\sigma. \tag{5.12}$$

For any $b > 0$, we easily see

$$\int_0^s e^{-bs_1} b (s-s_1)^{-\sigma} ds_1 \leq C s^{-\sigma}. \tag{5.13}$$

For the second term in (5.11), we have from (3.1), (5.10), (5.12) and (5.13) that

$$\begin{aligned} & \left\| \mathcal{F}_x^{-1} \int_0^t e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K \hat{g}^\epsilon(t-s_1) ds_1 - \mathcal{F}_x^{-1} \int_0^s e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K \hat{g}^\epsilon(s-s_1) ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta} \\ &= \left\| \mathcal{F}_x^{-1} \int_s^t e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K \hat{g}^\epsilon(t-s_1) ds_1 \right. \\ & \quad \left. + \mathcal{F}_x^{-1} \int_0^s e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K (\hat{g}^\epsilon(t-s_1) - \hat{g}^\epsilon(s-s_1)) ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta} \\ &\leq C \|g^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta-\eta, \tau} \int_s^t e^{-\nu_0 \frac{s_1}{\epsilon}} \frac{1}{\epsilon} ds_1 \\ & \quad + C \int_0^s e^{-\nu_0 \frac{s_1}{\epsilon}} \frac{1}{\epsilon} \|\hat{g}^\epsilon(t-s_1) - \hat{g}^\epsilon(s-s_1)\|_{\alpha-\gamma t, l-\sigma, \beta-\eta} ds_1 \\ &\leq C \left(\frac{t-s}{s}\right)^\sigma \|f_0\|_{\alpha, l-\sigma, \beta} + C \|g^\epsilon\|_{\alpha, \gamma, l, \beta-\eta}^{\tau, \sigma} \int_0^s e^{-\nu_0 \frac{s_1}{\epsilon}} \frac{1}{\epsilon} \left(\frac{t-s}{s-s_1}\right)^\sigma ds_1 \\ &\leq C \left(\frac{t-s}{s}\right)^\sigma \left(\|f_0\|_{\alpha, l, \beta} + \|g^\epsilon\|_{\alpha, \gamma, l, \beta-\eta}^{\tau, \sigma} \right). \end{aligned}$$

In view of (5.4), (4.52), (5.11) and this, we have

$$\|e^{tB^\epsilon} f_0\|_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \leq C \left(\|f_0\|_{\alpha, l, \beta} + \|e^{tB^\epsilon} f_0\|_{\alpha, \gamma, l, \beta-\eta}^{\tau, \sigma} \right).$$

By using (5.4), (5.11) and (5.10), for $\eta \in (0, 1]$, we iterate this successively in the finite times to get (5.1). This completes the proof of the lemma. \square

Recalling (4.52), for any $\beta > 5/2$ and $l > \sigma$ with $\sigma \in (0, 1)$, we assume

$$\|f^\epsilon\|_{\alpha, \gamma, l, \beta}^{\tau, \sigma} = \sup_{0 \leq s < t \leq \tau} \left(\frac{s}{t-s}\right)^\sigma \|f^\epsilon(t) - f^\epsilon(s)\|_{\alpha-\gamma t, l-\sigma, \beta} \leq C. \tag{5.14}$$

Lemma 5.2. *Assume that (5.14) holds. Recalling (3.3), one has*

$$\llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \leq C \left(1 + \frac{1}{\gamma} \right) \left(\|f^\epsilon\|_{\alpha, \gamma, l - \sigma, \beta, \tau} + \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \right).$$

Proof. For any $0 < s < t \leq \tau$, by using (3.3), one has

$$\begin{aligned} & \mathcal{F}_x(H^\epsilon f^\epsilon(t) - H^\epsilon f^\epsilon(s)) \\ &= \int_s^t e^{s_1 \widehat{B}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(t - s_1) ds_1 \\ & \quad + \int_0^s e^{s_1 \widehat{B}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \{ \widehat{f}^\epsilon(t - s_1) - \widehat{f}^\epsilon(s - s_1) \} ds_1 \\ & := I_3 + I_4. \end{aligned} \tag{5.15}$$

We first estimate the term I_3 . For any $\beta_1, \beta_2 > 0$, we have from (3.2), (5.15), (5.12) and Theorem 2.1 that

$$\begin{aligned} & \left| \int_s^t \chi \left(|k| < \frac{\kappa_0}{\epsilon} \right) e^{\lambda_j(\epsilon|k|) \frac{s_1}{\epsilon}} P_j(\epsilon k) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(t - s_1) ds_1 \right| \\ & \leq C(1 + |p|)^{-\beta_1} e^{-(\alpha - \gamma t)|k|} (1 + |k|)^{-(l - \sigma)} \|f^\epsilon\|_{\alpha, \gamma, l - \sigma, \beta_2, \tau} \int_s^t e^{-\gamma s_1 |k|} |k| ds_1 \\ & \leq \frac{C}{\gamma} \left(\frac{t - s}{s} \right)^\sigma (1 + |p|)^{-\beta_1} e^{-(\alpha - \gamma t)|k|} (1 + |k|)^{-(l - \sigma)} \|f^\epsilon\|_{\alpha, \gamma, l - \sigma, \beta_2, \tau}. \end{aligned}$$

By this, for any $0 \leq j \leq 4$, we have

$$\left\| \int_s^t E_j^\epsilon(s_1) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(t - s_1) ds_1 \right\|_{\alpha - \gamma t, l - \sigma, \beta_1} \leq \frac{C}{\gamma} \left(\frac{t - s}{s} \right)^\sigma \|f^\epsilon\|_{\alpha, \gamma, l - \sigma, \beta_2, \tau}. \tag{5.16}$$

We then consider $E_6^\epsilon(t)$ in (3.2). It follows from (3.2), (5.12) and Theorem 2.1 that

$$\begin{aligned} & \left| \int_s^t \chi \left(|k| > \frac{\kappa_0}{\epsilon} \right) e^{s_1 \widehat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(t - s_1) ds_1 \right| \\ & \leq C(1 + |p|)^{-(\beta - 1)} e^{-(\alpha - \gamma t)|k|} (1 + |k|)^{-(l - \sigma)} \\ & \quad \times \left\| \Lambda^{-1} Q_0 \Lambda f^\epsilon \right\|_{\alpha, \gamma, l - \sigma, \beta - 1, \tau} \int_s^t e^{-\frac{s_1}{\epsilon} \nu(p)} \frac{\nu(p)}{\epsilon} ds_1 \\ & \leq C \left(\frac{t - s}{s} \right)^\sigma (1 + |p|)^{-(\beta - 1)} e^{-(\alpha - \gamma t)|k|} (1 + |k|)^{-(l - \sigma)} \|f^\epsilon\|_{\alpha, \gamma, l - \sigma, \beta - 1, \tau}. \end{aligned}$$

For $\beta > 5/2$, we have from Theorem 2.1 that

$$\begin{aligned} & \left| \int_s^t \chi \left(|k| > \frac{\kappa_0}{\epsilon} \right) U_2(\epsilon, s_1, k) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(t-s_1) ds_1 \right| \\ & \leq C(1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau} \int_s^t e^{-\sigma_0 \frac{s_1}{\epsilon}} \frac{1}{\epsilon} ds_1 \\ & \leq C \left(\frac{t-s}{s} \right)^\sigma (1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau}. \end{aligned}$$

By these two estimates, we arrive at

$$\left\| \int_s^t E_6^\epsilon(s_1) Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(t-s_1) ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C \left(\frac{t-s}{s} \right)^\sigma \|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau}. \tag{5.17}$$

The term $E_5^\epsilon(t)$ in (3.2) can be treated in the same way. Thus we have from (5.15)-(5.17) that

$$\| \mathcal{F}_x^{-1} I_3 \|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C \left(1 + \frac{1}{\gamma} \right) \left(\frac{t-s}{s} \right)^\sigma \|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau}. \tag{5.18}$$

We then estimate the term I_4 in (5.15). It follows from (3.2), (5.15), (5.14), (5.13) and Theorem 2.1 that

$$\begin{aligned} & \left| \int_0^s \chi \left(|k| < \frac{\kappa_0}{\epsilon} \right) e^{\lambda_j(\epsilon|k|) \frac{s_1}{\epsilon}} P_j(\epsilon k) Q_0 \frac{1}{\epsilon} \Lambda \{ \widehat{f}^\epsilon(t-s_1) - \widehat{f}^\epsilon(s-s_1) \} ds_1 \right| \\ & \leq C(1+|p|)^{-(\beta-1)} (1+|k|)^{-(l-\sigma)} \\ & \quad \times \int_0^s e^{-(\alpha-\gamma(t-s_1))|k|} |k| \|f^\epsilon(t-s_1) - f^\epsilon(s-s_1)\|_{\alpha-\gamma(t-s_1), l-\sigma, \beta} ds_1 \\ & \leq C(1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \int_0^s e^{-\gamma s_1 |k|} \left(\frac{t-s}{s-s_1} \right)^\sigma |k| ds_1 \\ & \leq \frac{C}{\gamma} \left(\frac{t-s}{s} \right)^\sigma (1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}. \end{aligned}$$

By this and (3.2), for any $0 \leq j \leq 4$, we have

$$\begin{aligned} & \left\| \int_0^s E_j^\epsilon(s_1) Q_0 \frac{1}{\epsilon} \Lambda \{ \widehat{f}^\epsilon(t-s_1) - \widehat{f}^\epsilon(s-s_1) \} ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta-1} \\ & \leq \frac{C}{\gamma} \left(\frac{t-s}{s} \right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}. \end{aligned} \tag{5.19}$$

We then consider $E_6^\epsilon(t)$. We have from (3.2), (5.15), (5.14), (5.13) and Theorem 2.1 that

$$\begin{aligned} & \left| \int_0^s \chi\left(|k| > \frac{\kappa_0}{\epsilon}\right) e^{s_1 \widehat{A}^\epsilon(k)} \frac{\nu(p)}{\epsilon} \Lambda^{-1} Q_0 \Lambda \{ \widehat{f}^\epsilon(t-s_1) - \widehat{f}^\epsilon(s-s_1) \} ds_1 \right| \\ & \leq C(1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \\ & \quad \times \int_0^s e^{-\frac{s_1}{\epsilon} \nu(p)} \frac{\nu(p)}{\epsilon} \left\| \Lambda^{-1} Q_0 \Lambda \{ \widehat{f}^\epsilon(t-s_1) - \widehat{f}^\epsilon(s-s_1) \} \right\|_{\alpha-\gamma(t-s_1), l-\sigma, \beta-1} ds_1 \\ & \leq C(1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \int_0^s e^{-\frac{s_1}{\epsilon} \nu(p)} \frac{\nu(p)}{\epsilon} \left(\frac{t-s}{s-s_1} \right)^\sigma ds_1 \\ & \leq C \left(\frac{t-s}{s} \right)^\sigma (1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}. \end{aligned}$$

For $\beta > 5/2$, we have from Theorem 2.1 that

$$\begin{aligned} & \left| \int_0^s \chi\left(|k| > \frac{\kappa_0}{\epsilon}\right) U_2(\epsilon, s_1, k) Q_0 \frac{1}{\epsilon} \Lambda \{ \widehat{f}^\epsilon(t-s_1) - \widehat{f}^\epsilon(s-s_1) \} ds_1 \right| \\ & \leq C(1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \\ & \quad \times \int_0^s e^{-\sigma_0 \frac{s_1}{\epsilon}} \frac{1}{\epsilon} \left\| \widehat{f}^\epsilon(t-s_1) - \widehat{f}^\epsilon(s-s_1) \right\|_{\alpha-\gamma(t-s_1), l-\sigma, \beta} ds_1 \\ & \leq C(1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \int_0^s e^{-\frac{s_1}{\epsilon} \sigma_0} \frac{1}{\epsilon} \left(\frac{t-s}{s-s_1} \right)^\sigma ds_1 \\ & \leq C \left(\frac{t-s}{s} \right)^\sigma (1+|p|)^{-(\beta-1)} e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-(l-\sigma)} \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}. \end{aligned}$$

By this and (3.2), we arrive at

$$\begin{aligned} & \left\| \int_0^s E_6^\epsilon(s_1) Q_0 \frac{1}{\epsilon} \Lambda \{ \widehat{f}^\epsilon(t-s_1) - \widehat{f}^\epsilon(s-s_1) \} ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta_1} \\ & \leq C \left(\frac{t-s}{s} \right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}. \end{aligned} \tag{5.20}$$

The term $E_5^\epsilon(t)$ can be treated in the same way. Thus we have from (5.15), (5.19) and (5.20) that

$$\left\| \mathcal{F}_x^{-1} I_4 \right\|_{\alpha-\gamma t, l-\sigma, \beta-1} \leq C \left(1 + \frac{1}{\gamma} \right) \left(\frac{t-s}{s} \right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}. \tag{5.21}$$

By using (5.15), (3.2), (5.18) and (5.21), we arrive at

$$\begin{aligned} & \|H^\epsilon f^\epsilon(t) - H^\epsilon f^\epsilon(s)\|_{\alpha-\gamma t, l-\sigma, \beta-1} \\ & \leq C \left(1 + \frac{1}{\gamma}\right) \left(\frac{t-s}{s}\right)^\sigma (\|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau} + \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}). \end{aligned}$$

Recalling (5.14), we have from this that

$$\llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta-1}^{\tau, \sigma} \leq C \left(1 + \frac{1}{\gamma}\right) (\|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau} + \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}).$$

For $\beta > 5/2$ and $l > \sigma$, we have from this and (3.12) that

$$\begin{aligned} & \llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta-1}^{\tau, \sigma} + \|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta-1, \tau} \\ & \leq C \left(1 + \frac{1}{\gamma}\right) (\|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau} + \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}). \end{aligned} \tag{5.22}$$

We will make use of the smoothing property of K in (2.4) to recover the loss of the p -weight. Notice that

$$\begin{aligned} \mathcal{F}_x H^\epsilon f^\epsilon(t) &= \int_0^t e^{(t-s_1)\widehat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(s_1) ds_1 \\ & \quad + \int_0^t e^{(t-s_1)\widehat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x H^\epsilon f^\epsilon(s_1) ds_1. \end{aligned} \tag{5.23}$$

We first estimate the first part of (5.23). One has

$$\begin{aligned} & \int_0^t e^{s_1 \widehat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(t-s_1) ds_1 - \int_0^s e^{s_1 \widehat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(s-s_1) ds_1 \\ &= \int_s^t e^{s_1 \widehat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(t-s_1) ds_1 \\ & \quad + \int_0^s e^{s_1 \widehat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda (\widehat{f}^\epsilon(t-s_1) - f^\epsilon(s-s_1)) ds_1 \\ & := I_5 + I_6. \end{aligned} \tag{5.24}$$

For the term I_5 , for $\eta \in (0, 1]$, we have from (5.12) that

$$\begin{aligned} & \left\| \mathcal{F}_x^{-1} \int_s^t e^{s_1 \widehat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda \widehat{f}^\epsilon(t-s_1) ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta} \\ & \leq C \sup_p \int_s^t e^{-\nu(p) \frac{s_1}{\epsilon}} \frac{\nu(p)}{\epsilon} ds_1 \sup_{0 \leq s_1 \leq t} \|\Lambda^{-1} Q_0 \Lambda f^\epsilon(s_1)\|_{\alpha-\gamma s_1, l-\sigma, \beta} \\ & \leq C \left(\frac{t-s}{s}\right)^\sigma \|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau}. \end{aligned}$$

For the term I_6 , we have from (5.12) and (5.14) that

$$\begin{aligned} & \left\| \mathcal{F}_x^{-1} \int_0^s e^{s_1 \hat{A}^\epsilon(k)} Q_0 \frac{1}{\epsilon} \Lambda(\hat{f}^\epsilon(t-s_1) - \hat{f}^\epsilon(s-s_1)) ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta} \\ & \leq C \sup_p \int_0^s e^{-\nu(p) \frac{s_1}{\epsilon}} \frac{\nu(p)}{\epsilon} \left\| \Lambda^{-1} Q_0 \Lambda(\hat{f}^\epsilon(t-s_1) - \hat{f}^\epsilon(s-s_1)) \right\|_{\alpha-\gamma(t-s_1), l-\sigma, \beta} ds_1 \\ & \leq C \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \sup_p \int_0^s e^{-\nu(p) \frac{s_1}{\epsilon}} \frac{\nu(p)}{\epsilon} \left(\frac{t-s}{s-s_1} \right)^\sigma ds_1 \leq C \left(\frac{t-s}{s} \right)^\sigma \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}. \end{aligned}$$

We then estimate the second part of (5.23). One has

$$\begin{aligned} & \int_0^t e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x H^\epsilon f^\epsilon(t-s_1) ds_1 - \int_0^s e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x H^\epsilon f^\epsilon(s-s_1) ds_1 \\ & = \int_s^t e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x H^\epsilon f^\epsilon(t-s_1) ds_1 \\ & \quad + \int_0^s e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x (H^\epsilon f^\epsilon(t-s_1) - H^\epsilon f^\epsilon(s-s_1)) ds_1. \end{aligned} \tag{5.25}$$

We denote the terms in (5.25) by I_7 and I_8 . For the term I_7 , we have from (2.4) and (5.12) that

$$\begin{aligned} & \left\| \mathcal{F}_x^{-1} \int_s^t e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x H^\epsilon f^\epsilon(t-s_1) ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta} \\ & \leq C \int_s^t e^{-\nu_0 \frac{s_1}{\epsilon}} \frac{1}{\epsilon} ds_1 \sup_{0 \leq s_1 \leq t} \|H^\epsilon f^\epsilon(s_1)\|_{\alpha-\gamma s_1, l-\sigma, \beta-\eta} \\ & \leq C \left(\frac{t-s}{s} \right)^\sigma \|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta-\eta, \tau}. \end{aligned}$$

For the term I_8 , we have from (2.4) and (5.13) that

$$\begin{aligned} & \left\| \mathcal{F}_x^{-1} \int_0^s e^{s_1 \hat{A}^\epsilon(k)} \frac{1}{\epsilon} K \mathcal{F}_x (H^\epsilon f^\epsilon(t-s_1) - H^\epsilon f^\epsilon(s-s_1)) ds_1 \right\|_{\alpha-\gamma t, l-\sigma, \beta} \\ & \leq C \int_0^s e^{-\nu_0 \frac{s_1}{\epsilon}} \frac{1}{\epsilon} \|H^\epsilon f^\epsilon(t-s_1) - H^\epsilon f^\epsilon(s-s_1)\|_{\alpha-\gamma(t-s_1), l-\sigma, \beta-\eta} ds_1 \\ & \leq C \llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta-\eta}^{\tau, \sigma} \int_0^s e^{-\nu_0 \frac{s_1}{\epsilon}} \frac{1}{\epsilon} \left(\frac{t-s}{s-s_1} \right)^\sigma ds_1 \\ & \leq C \left(\frac{t-s}{s} \right)^\sigma \llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta-\eta}^{\tau, \sigma}. \end{aligned}$$

By using (5.23)-(5.25) and the above estimates, we arrive at

$$\begin{aligned} & \|H^\epsilon f^\epsilon(t) - H^\epsilon f^\epsilon(s)\|_{\alpha-\gamma t, l-\sigma, \beta} \\ & \leq C \left(\frac{t-s}{s}\right)^\sigma \left(\|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau} + \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} + \|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta-\eta, \tau} + \llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta-\eta}^{\tau, \sigma}\right). \end{aligned}$$

It follows from this and (5.14) that

$$\llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \leq C \left(\|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau} + \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} + \|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta-\eta, \tau} + \llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta-\eta}^{\tau, \sigma}\right).$$

This and (3.16) imply

$$\begin{aligned} & \llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} + \|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau} \\ & \leq C \left(\|f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta, \tau} + \llbracket f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} + \|H^\epsilon f^\epsilon\|_{\alpha, \gamma, l-\sigma, \beta-\eta, \tau} + \llbracket H^\epsilon f^\epsilon \rrbracket_{\alpha, \gamma, l, \beta-\eta}^{\tau, \sigma}\right). \end{aligned}$$

By using (5.22) and (5.23), for $\eta \in (0, 1)$, we iterate this successively in the finite times to get the estimate in this lemma. The proof is complete. \square

Recalling (4.2), for any $\beta > 7/2$ and $l > \sigma$ with $\sigma \in (0, 1)$, we assume that

$$[f^\epsilon]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} = \sup_{0 < \epsilon' < \epsilon, 0 < s \leq \tau} \left(\frac{s^2}{\epsilon'}\right)^{\frac{\sigma}{2}} \|f^{\epsilon'}(s)\|_{\alpha-\gamma s, l-\sigma, \beta-1} < C. \tag{5.26}$$

Lemma 5.3. Assume that (5.26) holds. Recalling (3.3), one has

$$[H^\epsilon f^\epsilon]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \leq C \left(1 + \frac{1}{\gamma}\right) [f^\epsilon]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}. \tag{5.27}$$

Proof. For $0 \leq j \leq 4$ and $t \in [0, \tau]$, we have from (3.6), (3.2), (5.13) and Theorem 2.1 that

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-2} |\mathcal{F}_x H_j^{\epsilon'} f^{\epsilon'}(t)| \\ & \leq e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \int_0^t \left\| \left(|k| \leq \frac{\kappa_0}{\epsilon}\right) e^{\lambda_j(|\epsilon'k|)\frac{(t-s)}{\epsilon'}} P_j(\epsilon'k) Q_0 \frac{1}{\epsilon'} \Lambda \widehat{f}^{\epsilon'}(s) \right\|_{\beta-2} ds \\ & \leq C [f^\epsilon]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \int_0^t e^{-\gamma|k|(t-s)} |k| \left(\frac{s^2}{\epsilon'}\right)^{-\frac{\sigma}{2}} ds \leq \frac{C}{\gamma} \left(\frac{t^2}{\epsilon'}\right)^{-\frac{\sigma}{2}} [f^\epsilon]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}. \end{aligned}$$

For $0 \leq j \leq 4$, one has from this that

$$[H_j^\epsilon f^\epsilon]_{\alpha, \gamma, l, \beta-2}^{\epsilon, \tau, \sigma} \leq \frac{C}{\gamma} [f^\epsilon]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}. \tag{5.28}$$

It follows from (3.6), (3.2) and Theorem 2.1 that

$$\begin{aligned} & \mathcal{F}_x H_5^{\epsilon'} f^{\epsilon'}(t) \\ &= \int_0^t \chi\left(|k| < \frac{\kappa_0}{\epsilon'}\right) \left(e^{(t-s)\widehat{A}^{\epsilon'}(k)} Q(\epsilon'k) + U_1(\epsilon', t-s, k) \right) Q_0 \frac{1}{\epsilon'} \Lambda f^{\epsilon'}(s) ds. \end{aligned} \quad (5.29)$$

For the first term in (5.29), for any $\beta > 7/2$, we have from (5.13) and Theorem 2.1 that

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-2} \\ & \times \left| \int_0^t \chi\left(|k| < \frac{\kappa_0}{\epsilon'}\right) e^{(t-s)\widehat{A}^{\epsilon'}(k)} Q(\epsilon'k) Q_0 \frac{1}{\epsilon'} \Lambda \widehat{f}^{\epsilon'}(s) ds \right| \\ & \leq C e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-2} \\ & \times \int_0^t \chi\left(|k| < \frac{\kappa_0}{\epsilon'}\right) e^{-\frac{\nu(p)}{\epsilon'}(t-s)} \frac{\nu(p)}{\epsilon'} |\Lambda^{-1} Q(\epsilon'k) Q_0 \Lambda \widehat{f}^{\epsilon'}(s)| ds \\ & \leq C [f^{\epsilon}]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \int_0^t e^{-\frac{\nu(p)}{\epsilon'}(t-s)} \frac{\nu(p)}{\epsilon'} \left(\frac{s^2}{\epsilon'}\right)^{-\frac{\sigma}{2}} ds \leq C \left(\frac{t^2}{\epsilon'}\right)^{-\frac{\sigma}{2}} [f^{\epsilon}]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}. \end{aligned}$$

For the second term in (5.29), for any $\beta > 7/2$, we have from (2.1), (5.13) and Theorem 2.1 that

$$\begin{aligned} & e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-2} \left| \int_0^t \chi\left(|k| < \frac{\kappa_0}{\epsilon'}\right) U_1(\epsilon', t-s, k) Q_0 \frac{1}{\epsilon'} \Lambda \widehat{f}^{\epsilon'}(s) ds \right| \\ & \leq e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \int_0^t \chi\left(|k| < \frac{\kappa_0}{\epsilon'}\right) \left\| U_1(\epsilon', t-s, k) Q_0 \frac{1}{\epsilon'} \Lambda \widehat{f}^{\epsilon'}(s) \right\|_{\beta-2} ds \\ & \leq e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \int_0^t e^{-\frac{\sigma_0(t-s)}{\epsilon'}} \frac{1}{\epsilon'} \left(\left\| Q_0 \Lambda \widehat{f}^{\epsilon'}(s) \right\|_{\beta-2-\eta} + \left\| Q_0 \Lambda \widehat{f}^{\epsilon'}(s) \right\| \right) ds \\ & \leq C [f^{\epsilon}]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \int_0^t e^{-\frac{\sigma_0(t-s)}{\epsilon'}} \frac{1}{\epsilon'} \left(\frac{s^2}{\epsilon'}\right)^{-\frac{\sigma}{2}} ds \leq C \left(\frac{t^2}{\epsilon'}\right)^{-\frac{\sigma}{2}} [f^{\epsilon}]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}. \end{aligned}$$

Thus, by using (5.29), (5.26) and the above two estimates, we have

$$[H_5^{\epsilon} f^{\epsilon}]_{\alpha, \gamma, l, \beta-2}^{\epsilon, \tau, \sigma} \leq C [f^{\epsilon}]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}. \quad (5.30)$$

The term $H_6^{\epsilon} f^{\epsilon}$ can be treated in the same way. By (3.6), (5.28) and (5.30), we arrive at

$$[H^{\epsilon} f^{\epsilon}]_{\alpha, \gamma, l, \beta-2}^{\epsilon, \tau, \sigma} \leq C \left(1 + \frac{1}{\gamma}\right) [f^{\epsilon}]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}. \quad (5.31)$$

We will use the smoothing property of K in (2.4) again. It follows from (2.44) and (3.3) that

$$\begin{aligned} \mathcal{F}_x H^{\epsilon'} f^{\epsilon'}(t) &= \int_0^t e^{(t-s)\widehat{A}^{\epsilon'}(k)} Q_0 \frac{1}{\epsilon'} \Lambda \widehat{f}^{\epsilon'}(s) ds \\ &\quad + \int_0^t e^{(t-s)\widehat{A}^{\epsilon'}(k)} \frac{1}{\epsilon'} K \mathcal{F}_x H^{\epsilon'} f^{\epsilon'}(s) ds. \end{aligned} \tag{5.32}$$

For the first term in (5.32), one has

$$\begin{aligned} &e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} \left| \int_0^t e^{(t-s)\widehat{A}^{\epsilon'}(k)} Q_0 \frac{1}{\epsilon'} \Lambda \widehat{f}^{\epsilon'}(s) ds \right| \\ &\leq C e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} \int_0^t e^{-\frac{\nu(p)}{\epsilon'}(t-s)} \frac{\nu(p)}{\epsilon'} |\Lambda^{-1} Q_0 \Lambda \widehat{f}^{\epsilon'}(s)| ds \\ &\leq C [f^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \int_0^t e^{-\frac{\nu(p)}{\epsilon'}(t-s)} \frac{\nu(p)}{\epsilon'} \left(\frac{s^2}{\epsilon'}\right)^{-\frac{\sigma}{2}} ds \leq C \left(\frac{t^2}{\epsilon'}\right)^{-\frac{\sigma}{2}} [f^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma}. \end{aligned}$$

For the second term in (5.32), one has from (2.4) that

$$\begin{aligned} &e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} (1+|p|)^{\beta-1} \left| \int_0^t e^{(t-s)\widehat{A}^{\epsilon'}(k)} \frac{1}{\epsilon'} K \mathcal{F}_x H^{\epsilon'} f^{\epsilon'}(s) ds \right| \\ &\leq C e^{(\alpha-\gamma t)|k|} (1+|k|)^{l-\sigma} \int_0^t e^{-\frac{\nu_0}{\epsilon'}(t-s)} \frac{1}{\epsilon'} \|\mathcal{F}_x H^{\epsilon'} f^{\epsilon'}(s)\|_{\beta-1-\eta} ds \\ &\leq C \left(\frac{t^2}{\epsilon'}\right)^{-\frac{\sigma}{2}} [H^\epsilon f^\epsilon]_{\alpha,\gamma,l,\beta-1-\eta}^{\epsilon,\tau,\sigma}. \end{aligned}$$

Thus, by using (5.32) and the above two estimates, we arrive at

$$[H^\epsilon f^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \leq C [f^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} + C [H^\epsilon f^\epsilon]_{\alpha,\gamma,l,\beta-1-\eta}^{\epsilon,\tau,\sigma}.$$

Noticing that $\eta \in (0,1]$, we iterate the above estimate in finite times to get

$$[H^\epsilon f^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \leq C [f^\epsilon]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} + C [H^\epsilon f^\epsilon]_{\alpha,\gamma,l,\beta-2}^{\epsilon,\tau,\sigma}.$$

Combining this and (5.31), we deduce the desired estimate. This completes the proof of the lemma. □

Proof of Theorem 1.2. For any $\epsilon_0 > 0$ small enough and $f = f^\epsilon$ with the limit f^0 , we introduce the norm

$$\llbracket f \rrbracket = \|f\|_{\alpha,\gamma,l,\beta,\tau} + \sup_{0 < \epsilon \leq \epsilon_0} \left\{ [f^\epsilon - f^0]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} + \llbracket f^\epsilon \rrbracket_{\alpha,\gamma,l,\beta}^{\tau,\sigma} \right\}. \tag{5.33}$$

As in (3.20), we define the nonlinear map N^ϵ by

$$N^\epsilon[f^\epsilon](t) = e^{tB^\epsilon} f_0 + H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon, f^\epsilon). \tag{5.34}$$

For any $0 < s < t \leq \tau$ and $\ell > 3$, we have from (3.18) that

$$\begin{aligned} & \|\Lambda^{-1}(\Gamma(u(t), v(t)) - \Gamma(u(s), v(s)))\|_{\alpha-\gamma t, \ell, \beta} \\ &= \|\Lambda^{-1}(\Gamma(u(t) - u(s), v(t)) - \Gamma(u(s), v(t) - v(s)))\|_{\alpha-\gamma t, \ell, \beta} \\ &\leq C \|u(t) - u(s)\|_{\alpha-\gamma t, \ell, \beta} \|v(t)\|_{\alpha-\gamma t, \ell, \beta} \\ &\quad + C \|v(t) - v(s)\|_{\alpha-\gamma t, \ell, \beta} \|u(s)\|_{\alpha-\gamma t, \ell, \beta}. \end{aligned} \tag{5.35}$$

For any $l > 3 + \sigma$, by using (5.14) and (5.35), we arrive at

$$\|[\Lambda^{-1} \Gamma(f^\epsilon, f^\epsilon)]\|_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \leq C \|f\|_{\alpha, \gamma, l-\sigma, \beta, \tau} \| [f^\epsilon] \|_{\alpha, \gamma, l, \beta}^{\tau, \sigma}. \tag{5.36}$$

Note that $Q_0 \Gamma(f^\epsilon, f^\epsilon) = \Gamma(f^\epsilon, f^\epsilon)$. By using Lemma 5.2, (5.14), (3.18) and this, one has

$$\begin{aligned} & \| [H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon, f^\epsilon)] \|_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \\ &\leq C \left(1 + \frac{1}{\gamma} \right) \left(\| \Lambda^{-1} \Gamma(f^\epsilon, f^\epsilon) \|_{\alpha, \gamma, l-\sigma, \beta, \tau} + \| [\Lambda^{-1} \Gamma(f^\epsilon, f^\epsilon)] \|_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \right) \\ &\leq C \left(1 + \frac{1}{\gamma} \right) \left(\| f \|^2_{\alpha, \gamma, l-\sigma, \beta, \tau} + \| f \|_{\alpha, \gamma, l-\sigma, \beta, \tau} \| [f^\epsilon] \|_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \right). \end{aligned}$$

It follows from Lemma 5.1 that

$$\| [e^{tB^\epsilon} f_0] \|_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \leq C_0 \| f_0 \|_{\alpha, l, \beta}.$$

Since $f = f^\epsilon \in W_{\beta, \tau}^{\alpha, \gamma, l}$ with the limit f^0 ,

$$\| f^\epsilon - f^0 \|_{\dot{Y}_{\beta-1}^{\alpha, \gamma, l}([\zeta, \tau])} \leq \| f \|_{\alpha, \gamma, l, \beta-1, \tau}$$

for any $\epsilon > 0$ small enough and any $\zeta > 0$. Then

$$\| f^0 \|_{\dot{Y}_{\beta-1}^{\alpha, \gamma, l}([\zeta, \tau])} \leq 2 \| f \|_{\alpha, \gamma, l, \beta-1, \tau}$$

for any $\zeta > 0$. Since $f^0 \in \dot{Y}_\beta^{\alpha, \gamma, l}((0, \tau])$, as $\zeta \rightarrow 0$, one has that

$$\| f^0 \|_{\dot{Y}_{\beta-1}^{\alpha, \gamma, l}((0, \tau])} \leq 2 \| f \|_{\alpha, \gamma, l, \beta-1, \tau}.$$

We have from (3.18) that

$$\begin{aligned} & \|\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) - \Lambda^{-1}\Gamma(f^0, f^0)\|_{\alpha-\gamma t, l-\sigma, \beta-1} \\ & \leq C\|f^\epsilon - f^0\|_{\alpha-\gamma t, l-\sigma, \beta-1}\|f^\epsilon + f^0\|_{\alpha-\gamma t, l-\sigma, \beta-1}. \end{aligned}$$

By using these facts and (5.26), one has

$$[\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) - \Lambda^{-1}\Gamma(f^0, f^0)]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \leq C\|f\|_{\alpha, \gamma, l, \beta-1, \tau} [f^\epsilon - f^0]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma}. \quad (5.37)$$

For any $l > 3 + \sigma$, we have from (5.36) that

$$\llbracket \Lambda^{-1}\Gamma(f^{\epsilon'}, f^{\epsilon'}) \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \leq C\|f\|_{\alpha, \gamma, l-\sigma, \beta, \tau} \llbracket f^{\epsilon'} \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma}. \quad (5.38)$$

By using Lemma 4.6(i), (3.18), (5.37) and (5.38), one has

$$\begin{aligned} & [H^\epsilon \Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) - H^0 \Lambda^{-1}\Gamma(f^0, f^0)]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \\ & \leq C \left(1 + \frac{1}{\gamma}\right) \left(\|\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon)\|_{\alpha, \gamma, l, \beta, \tau} + [\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon) - \Lambda^{-1}\Gamma(f^0, f^0)]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \right. \\ & \quad \left. + \sup_{0 < \epsilon' \leq \epsilon} \llbracket \Lambda^{-1}\Gamma(f^{\epsilon'}, f^{\epsilon'}) \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \right) \\ & \leq C \left(1 + \frac{1}{\gamma}\right) \left(\|f\|_{\alpha, \gamma, l, \beta, \tau}^2 + \|f\|_{\alpha, \gamma, l, \beta-1, \tau} [f^\epsilon - f^0]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \right. \\ & \quad \left. + \|f\|_{\alpha, \gamma, l-\sigma, \beta, \tau} \sup_{0 < \epsilon' \leq \epsilon} \llbracket f^{\epsilon'} \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \right). \quad (5.39) \end{aligned}$$

It follows from Lemma 4.1(ii) that

$$[e^{tB^\epsilon} f_0 - E^0(t) f_0]_{\alpha, \gamma, l, \beta-1}^{\epsilon, \tau, \sigma} \leq C\|f_0\|_{\alpha, l, \beta}.$$

It follows from Lemmas 3.1-3.3 that

$$\|e^{tB^\epsilon} f_0\|_{\alpha, \gamma, l, \beta, \tau} \leq C\|f_0\|_{\alpha, l, \beta},$$

and

$$\begin{aligned} \|H^\epsilon \Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon)\|_{\alpha, \gamma, l, \beta, \tau} & \leq C \left(1 + \frac{1}{\gamma}\right) \|\Lambda^{-1}\Gamma(f^\epsilon, f^\epsilon)\|_{\alpha, \gamma, l, \beta, \tau} \\ & \leq C \left(1 + \frac{1}{\gamma}\right) \|f\|_{\alpha, \gamma, l, \beta, \tau}^2. \end{aligned}$$

By using (5.33), (5.34) and the above estimates, we arrive at

$$\|N[f]\| \leq C'_1 \|f_0\|_{\alpha,l,\beta} + C'_2 \left(1 + \frac{1}{\gamma}\right) \|f\|^2. \tag{5.40}$$

Next we prove that N^ϵ in (5.34) is contractive. For any $f = f^\epsilon \in W_{\beta,\tau}^{\alpha,\gamma,l}$ and $h = h^\epsilon \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limits f^0 and h^0 , we set $G_\pm = G_\pm^\epsilon = f^\epsilon \pm h^\epsilon$ and then their limits are $G_\pm^0 = f^0 \pm h^0$. Note that

$$\Gamma(f^\epsilon, f^\epsilon) - \Gamma(h^\epsilon, h^\epsilon) = \Gamma(f^\epsilon + h^\epsilon, f^\epsilon - h^\epsilon) = \Gamma(G_+^\epsilon, G_-^\epsilon). \tag{5.41}$$

By the similar arguments as in Theorem 1.1(ii), we have that $\Lambda^{-1}\Gamma(G_+^\epsilon, G_-^\epsilon) \in W_{\beta,\tau}^{\alpha,\gamma,l}$ with the limit $\Lambda^{-1}\Gamma(G_+^0, G_-^0)$. By using (5.41) and the similar arguments as (5.39), one has

$$\begin{aligned} & \left[H^\epsilon \Lambda^{-1}\Gamma(G_+^\epsilon, G_-^\epsilon) - H^0 \Lambda^{-1}\Lambda^{-1}\Gamma(G_+^0, G_-^0) \right]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \\ & \leq C \left(1 + \frac{1}{\gamma}\right) \left(\|\Lambda^{-1}\Gamma(G_+^\epsilon, G_-^\epsilon)\|_{\alpha,\gamma,l,\beta,\tau} + \|\Lambda^{-1}\Gamma(G_+^\epsilon, G_-^\epsilon) - \Lambda^{-1}\Gamma(G_+^0, G_-^0)\|_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \right. \\ & \quad \left. + \sup_{0 < \epsilon' \leq \epsilon} \|\Lambda^{-1}\Gamma(G_+^{\epsilon'}, G_-^{\epsilon'})\|_{\alpha,\gamma,l,\beta}^{\tau,\sigma} \right) \\ & \leq C \left(1 + \frac{1}{\gamma}\right) \left(\|G_+\|_{\alpha,\gamma,l,\beta,\tau} \|G_-\|_{\alpha,\gamma,l,\beta,\tau} + \|G_-\|_{\alpha,\gamma,l,\beta-1,\tau} [G_+^\epsilon - G_+^0]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \right. \\ & \quad + \|G_+\|_{\alpha,\gamma,l,\beta-1,\tau} [G_-^\epsilon - G_-^0]_{\alpha,\gamma,l,\beta-1}^{\epsilon,\tau,\sigma} \\ & \quad + \|G_+\|_{\alpha,\gamma,l-\sigma,\beta,\tau} \sup_{0 < \epsilon' \leq \epsilon} \|G_-^{\epsilon'}\|_{\alpha,\gamma,l,\beta}^{\tau,\sigma} \\ & \quad \left. + \|G_-\|_{\alpha,\gamma,l-\sigma,\beta,\tau} \sup_{0 < \epsilon' \leq \epsilon} \|G_+^{\epsilon'}\|_{\alpha,\gamma,l,\beta}^{\tau,\sigma} \right). \end{aligned}$$

Here we used the fact that $\|G_+^0\|_{\dot{Y}_{\beta-1}^{\alpha,\gamma,l}((0,\tau))} \leq C \|G_+\|_{\alpha,\gamma,l,\beta-1,\tau}$.

By using Lemma 5.2, (5.14) and (5.35), we have

$$\begin{aligned} & \|\Lambda^{-1}\Gamma(G_+^\epsilon, G_-^\epsilon)\|_{\alpha,\gamma,l,\beta}^{\tau,\sigma} \\ & \leq C \left(1 + \frac{1}{\gamma}\right) \left(\|\Lambda^{-1}\Gamma(G_+^\epsilon, G_-^\epsilon)\|_{\alpha,\gamma,l,\beta,\tau} + \|\Lambda^{-1}\Gamma(G_+^\epsilon, G_-^\epsilon)\|_{\alpha,\gamma,l,\beta}^{\tau,\sigma} \right) \end{aligned}$$

$$\leq C \left(1 + \frac{1}{\gamma} \right) \left(\|G_+ \|_{\alpha, \gamma, l, \beta, \tau} \|G_- \|_{\alpha, \gamma, l, \beta, \tau} + \llbracket G_+^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \|G_- \|_{\alpha, \gamma, l, \beta, \tau} + \llbracket G_-^\epsilon \rrbracket_{\alpha, \gamma, l, \beta}^{\tau, \sigma} \|G_+ \|_{\alpha, \gamma, l, \beta, \tau} \right).$$

It follows from (3.22) that

$$\|H^\epsilon \Lambda^{-1} \Gamma(G_+^\epsilon, G_-^\epsilon)\|_{\alpha, \gamma, l, \beta, \tau} \leq C \left(1 + \frac{1}{\gamma} \right) \|G_+ \|_{\alpha, \gamma, l, \beta, \tau} \|G_- \|_{\alpha, \gamma, l, \beta, \tau}.$$

By using (5.41), (5.33), (5.34) and the above estimates, we arrive at

$$\llbracket N[f] - N[h] \rrbracket \leq C'_2 \left(1 + \frac{1}{\gamma} \right) \llbracket f - h \rrbracket \llbracket f + h \rrbracket. \tag{5.42}$$

By (5.40) and (5.42), it is clear that N^ϵ in (5.34) is contractive on a closed ball $\{f = f^\epsilon \in Z_{\beta, \tau}^{\alpha, \gamma, l} \mid \llbracket f \rrbracket \leq a'_1\}$ provided $\|f_0\|_{\alpha, l, \beta} \leq a'_0$ where a'_0 and a'_1 are as in Theorem 1.2. By (5.33) and (5.26), we obtain the desired estimates. This completes the proof of Theorem 1.2. \square

6 Initial layer of the solution

By Theorem 1.1(ii), the convergence is not uniform near $t=0$ and the initial layer shall appear. However, if the initial data F_0 is itself a local Maxwellian, the convergence becomes uniform and the initial layer disappears. We shall prove this in the following.

We introduce the function space

$$V_{\beta, \tau}^{\alpha, \gamma, l} = \left\{ f = f^\epsilon \in Z_{\beta, \tau}^{\alpha, \gamma, l} \mid \exists f^0 \in \dot{Y}_\beta^{\alpha, \gamma, l}([0, \tau]), \|f^\epsilon - f^0\|_{\dot{Y}_{\beta-1}^{\alpha, \gamma, l}([0, \tau])} \rightarrow 0, \text{ as } \epsilon \rightarrow 0 \right\}. \tag{6.1}$$

Obviously the space $V_{\beta, \tau}^{\alpha, \gamma, l}$ is a subspace of $W_{\beta, \tau}^{\alpha, \gamma, l}$ in (4.1) and so it is a Banach space with the norm $\|\cdot\|_{\alpha, \gamma, l, \beta, \tau}$. We readily have the following lemma.

Lemma 6.1. *Let $f = f^\epsilon(t) \in W_{\beta, \tau}^{\alpha, \gamma, l}$ with the limit $f^0(t)$. Suppose that*

$$\exists f_0 \in \dot{X}_\beta^{\alpha, l}, \quad \sup_{0 \leq s \leq t} \|\mathcal{F}_x^{-1}(e^{-\gamma|k|s} \widehat{f}^\epsilon(t) - \widehat{f}_0)\|_{\alpha, l, \beta-1} \rightarrow 0, \quad \epsilon, t \rightarrow 0, \quad \epsilon, t > 0.$$

Then $f^\epsilon(t) \in V_{\beta, \tau}^{\alpha, \gamma, l}$ if $f^\epsilon(t)$ is extended to $\epsilon = 0$ by $f^0(t)$ for $t > 0$ and by f_0 for $t = 0$.

Given a $f_0 \in \dot{X}_\beta^{\alpha,l}$, $V(f_0)$ denotes a closed subset of $V_{\beta,\tau}^{\alpha,\gamma,l}$ defined as

$$V(f_0) = \left\{ f^\epsilon(t) \in V_{\beta,\tau}^{\alpha,\gamma,l} \mid f^0(0) = f_0 \right\}. \tag{6.2}$$

We now show N^ϵ in (3.20) maps $V(f_0)$ into itself for f_0 satisfying the conditions in Theorem 1.3.

Due to the fact that $f_0 \in \dot{X}_{\beta-1}^{\alpha,l+1} \cap \dot{X}_\beta^{\alpha,l}$, $\hat{p} \cdot \nabla_x f_0 \in \dot{X}_{\beta-1}^{\alpha,l}$, $Lf_0 \in \dot{X}_{\beta-1}^{\alpha,l}$ and $f_0 \in D(B^\epsilon)$. Since $d(e^{tB^\epsilon})/dt = e^{tB^\epsilon} B^\epsilon$ holds on $D(B^\epsilon)$, we have from (2.6) that

$$e^{tB^\epsilon} f_0 = f_0 + \int_0^t e^{sB^\epsilon} (-\hat{p} \cdot \nabla_x f_0) ds + \int_0^t e^{(t-s)B^\epsilon} \frac{1}{\epsilon} Lf_0 ds := f_0 + w_1^\epsilon + w_2^\epsilon. \tag{6.3}$$

By virtue of (4.62), we have that $Lf_0 = -\Gamma(f_0, f_0)$. Thus one has

$$w_2^\epsilon = \int_0^t e^{(t-s)B^\epsilon} \frac{1}{\epsilon} Lf_0 ds = - \int_0^t e^{(t-s)B^\epsilon} Q_0 \frac{1}{\epsilon} \Gamma(f_0, f_0) ds.$$

By this and (3.3), we arrive at

$$H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon - f_0, f^\epsilon + f_0) = H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon, f^\epsilon) + w_2^\epsilon.$$

It follows from this, (3.20) and (6.3) that

$$N^\epsilon[f^\epsilon](t) = f_0 + w_1^\epsilon + H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon - f_0, f^\epsilon + f_0). \tag{6.4}$$

For the term w_1^ϵ , we have from (6.3) and (3.1) that

$$\begin{aligned} |\mathcal{F}_x w_1^\epsilon| &= \left| \int_0^t e^{s\hat{B}^\epsilon(k)} (-i\hat{p} \cdot k \hat{f}_0) ds \right| \\ &\leq \int_0^t e^{-(\alpha-\gamma s)|k|} (1+|k|)^{-l} (1+|p|)^{-(\beta-1)} \|e^{sB^\epsilon} \mathcal{F}_x^{-1}(-i\hat{p} \cdot k \hat{f}_0)\|_{\alpha-\gamma s, l, \beta-1} ds \\ &\leq C t e^{-(\alpha-\gamma t)|k|} (1+|k|)^{-l} (1+|p|)^{-(\beta-1)} \|\mathcal{F}_x^{-1}(-i\hat{p} \cdot k \hat{f}_0)\|_{\alpha, l, \beta-1}. \end{aligned}$$

This gives

$$\|w_1^\epsilon\|_{\alpha-\gamma t, l, \beta-1} \leq C t \|f_0\|_{\alpha, l+1, \beta-1}. \tag{6.5}$$

For any $\beta > 7/2$ and $l > 3$, we have from Lemmas 3.2 and 3.3 that

$$\begin{aligned} &\|H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon - f_0, f^\epsilon + f_0)\|_{\alpha-\gamma t, l, \beta-1} \\ &\leq C \left(1 + \frac{1}{\gamma}\right) \sup_{0 \leq s \leq t} \|\Lambda^{-1} \Gamma(f^\epsilon - f_0, f^\epsilon + f_0)\|_{\alpha-\gamma s, l, \beta-1} \\ &\leq C \left(1 + \frac{1}{\gamma}\right) \sup_{0 \leq s \leq t} (\|f^\epsilon + f_0\|_{\alpha-\gamma s, l, \beta-1} \|f^\epsilon - f_0\|_{\alpha-\gamma s, l, \beta-1}). \end{aligned}$$

By this and the direct calculations, we arrive at

$$\begin{aligned}
& \|H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon - f_0, f^\epsilon + f_0)\|_{\alpha-\gamma t, l, \beta-1} \\
& \leq C \left(1 + \frac{1}{\gamma}\right) \sup_{0 \leq s \leq t} \|f^\epsilon + f_0\|_{\alpha-\gamma s, l, \beta-1} \\
& \quad \times \left(\sup_{0 \leq s \leq t} \|\mathcal{F}_x^{-1}((e^{-\gamma|k|s} - 1)\widehat{f}_0)\|_{\alpha, l, \beta-1} + \sup_{0 \leq s \leq t} \|\mathcal{F}_x^{-1}(e^{-\gamma|k|s}\widehat{f}^\epsilon(s) - \widehat{f}_0)\|_{\alpha, l, \beta-1} \right) \\
& \leq C \left(1 + \frac{1}{\gamma}\right) \sup_{0 \leq s \leq t} \|f^\epsilon + f_0\|_{\alpha-\gamma s, l, \beta-1} \\
& \quad \times \left(t \|f_0\|_{\alpha, l+1, \beta-1} + \sup_{0 \leq s \leq t} \|\mathcal{F}_x^{-1}(e^{-\gamma|k|s}\widehat{f}^\epsilon(s) - \widehat{f}_0)\|_{\alpha, l, \beta-1} \right). \tag{6.6}
\end{aligned}$$

Here we used the fact that, by (4.6)

$$\sup_{0 \leq s \leq t} \|\mathcal{F}_x^{-1}((e^{-\gamma|k|s} - 1)\widehat{f}_0)\|_{\alpha, l, \beta-1} \leq Ct \|f_0\|_{\alpha, l+1, \beta-1}.$$

We also have

$$\begin{aligned}
& \|\mathcal{F}_x^{-1}(e^{-\gamma|k|t}\widehat{N}^\epsilon[f^\epsilon](t) - \widehat{f}_0)\|_{\alpha, l, \beta-1} \\
& \leq \|N^\epsilon[f^\epsilon](t) - f_0\|_{\alpha-\gamma t, l, \beta-1} + \|\mathcal{F}_x^{-1}((e^{-\gamma|k|t} - 1)\widehat{f}_0)\|_{\alpha, l, \beta-1} \\
& \leq \|N^\epsilon[f^\epsilon](t) - f_0\|_{\alpha-\gamma t, l, \beta-1} + Ct \|f_0\|_{\alpha, l+1, \beta-1}. \tag{6.7}
\end{aligned}$$

From the estimate (6.4) to (6.7), we arrive at

$$\begin{aligned}
& \|\mathcal{F}_x^{-1}(e^{-\gamma|k|t}\widehat{N}^\epsilon[f^\epsilon](t) - \widehat{f}_0)\|_{\alpha, l, \beta-1} \\
& \leq Ct \|f_0\|_{\alpha, l+1, \beta-1} + \|w_1^\epsilon\|_{\alpha-\gamma t, l, \beta-1} + \|H^\epsilon \Lambda^{-1} \Gamma(f^\epsilon - f_0, f^\epsilon + f_0)\|_{\alpha-\gamma t, l, \beta-1} \\
& \leq C \left(1 + \frac{1}{\gamma}\right) \left(1 + \sup_{0 \leq s \leq t} \|f^\epsilon + f_0\|_{\alpha-\gamma s, l, \beta-1}\right) \\
& \quad \times \left(t \|f_0\|_{\alpha, l+1, \beta-1} + \sup_{0 \leq s \leq t} \|\mathcal{F}_x^{-1}(e^{-\gamma|k|s}\widehat{f}^\epsilon(s) - \widehat{f}_0)\|_{\alpha, l, \beta-1} \right) \\
& \leq C \left(1 + \frac{1}{\gamma}\right) (1 + \|f\|_{\alpha, \gamma, l, \beta-1, \tau} + \|f_0\|_{\alpha, l, \beta-1}) \\
& \quad \times \left(t \|f_0\|_{\alpha, l+1, \beta-1} + \sup_{0 \leq s \leq t} \|\mathcal{F}_x^{-1}(e^{-\gamma|k|s}\widehat{f}^\epsilon(s) - \widehat{f}_0)\|_{\alpha, l, \beta-1} \right).
\end{aligned}$$

By the similar arguments as the above, for any $t_1 \in [0, t]$, we also have

$$\begin{aligned} & \|\mathcal{F}_x^{-1}(e^{-\gamma|k|t_1}\widehat{N}^\epsilon[f^\epsilon](t_1) - \widehat{f}_0)\|_{\alpha, l, \beta-1} \\ & \leq C \left(1 + \frac{1}{\gamma}\right) (1 + \|f\|_{\alpha, \gamma, l, \beta-1, \tau} + \|f_0\|_{\alpha, l, \beta-1}) \\ & \quad \times \left(t\|f_0\|_{\alpha, l+1, \beta-1} + \sup_{0 \leq s \leq t} \|\mathcal{F}_x^{-1}(e^{-\gamma|k|s}\widehat{f}^\epsilon(s) - \widehat{f}_0)\|_{\alpha, l, \beta-1}\right). \end{aligned}$$

Thus we arrive at

$$\begin{aligned} & \sup_{0 \leq t_1 \leq t} \|\mathcal{F}_x^{-1}(e^{-\gamma|k|t_1}\widehat{N}^\epsilon[f^\epsilon](t_1) - \widehat{f}_0)\|_{\alpha, l, \beta-1} \\ & \leq C \left(1 + \frac{1}{\gamma}\right) (1 + \|f\|_{\alpha, \gamma, l, \beta-1, \tau} + \|f_0\|_{\alpha, l, \beta-1}) \\ & \quad \times \left(t\|f_0\|_{\alpha, l+1, \beta-1} + \sup_{0 \leq s \leq t} \|\mathcal{F}_x^{-1}(e^{-\gamma|k|s}\widehat{f}^\epsilon(s) - \widehat{f}_0)\|_{\alpha, l, \beta-1}\right). \end{aligned}$$

If $f = f^\epsilon \in V(f_0)$ and $f_0 \in \dot{X}_{\beta-1}^{\alpha, l+1} \cap \dot{X}_\beta^{\alpha, l}$, the last line in the above tends to 0 as $t, \epsilon \rightarrow 0^+$. Thus we prove that N^ϵ in (3.20) maps $V(f_0)$ into itself if f_0 satisfying the conditions in Theorem 1.3.

By Theorems 3.1 and 3.2, it is clear that N^ϵ in (3.20) is contraction on $V(f_0) \cap Z_0$ so that $f^\epsilon(t)$ of Theorem 1.1 is in $V(f_0) \cap Z_0$. This completes the proof of Theorem 1.3.

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