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A Note on Asymptotic Stability of Rarefaction Wave of the Impermeable Problem for Radiative Euler Flows

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Abstract. This paper is devoted to studying the initial-boundary value problem for the radiative full Euler equations, which are a fundamental system in the radiative hydrodynamics with many practical applications in astrophysical and nuclear phenomena, with the slip boundary condition on an impermeable wall. Different from our recent paper named "Asymptotic stability of rarefaction wave with slip boundary condition for radiative Euler flow", in this paper we study the initial-boundary value problem with the Neumann boundary condition instead of the Dirichlet boundary on the temperature. Based on the Neumann boundary condition on the temperature, we obtain that the pressure also satisfies the Neumann boundary condition. This observation allows us to establish the local existence and a priori estimates more easily than the case of the Dirichlet boundary condition which is studied in the mentioned paper. Since for the impermeable problem, there are quite a few results available for the Navier-Stokes equations and the radiative Euler equations, it will contribute a lot to our systematical study on the asymptotic behaviors of the rarefaction wave with the radiative effect and different boundary conditions such as the inflow/outflow problem and the impermeable boundary problem in our series papers.

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1 Introduction

The radiative full Euler equations are a fundamental system to describe the motion of the compressible gas with the radiative heat transfer phenomena, which has many applications in astrophysics and nuclear explosions. Mathematically, the one-dimensional radiative full Euler equations in the Eulerian coordinates are a hyperbolic-elliptic coupled system of the following form:

$$\begin{cases} \rho_t+(\rho u)_x=0, & \text{(1.1a)}\\ (\rho u)_t+(\rho u^2+p)_x=0, & \text{(1.1b)}\\ \left\{\rho\left(e+\dfrac{u^2}{2}\right)\right\}_t+\left\{\rho u\left(e+\dfrac{u^2}{2}\right)+pu\right\}_x+q_x=0, & \text{(1.1c)}\\ -q_{xx}+aq+b(\theta^4)_x=0, & \text{(1.1d)} \end{cases}$$

where ρ,u,p,e and θ are respectively the density, velocity, pressure, internal energy and absolute temperature of the gas, and q is the radiative heat flux. Positive constants a and b depend only on the gas itself. Like the classic compressible Euler equations, the Eqs. (1.1a)-(1.1c) stand for the conservation of the mass, momentum and energy respectively. The Eq. (1.1d) is related to the radiative heat transfer phenomenon, and one can refer [1,12,23,29,36,40] for more details. System (1.1) can also be derived by the non-relativistic limit (speed of light tending to $+\infty$) from a hyperbolic-kinetic system, and rigorous mathematical derivation can be found in [16]. Throughout this paper, we will concentrate on the ideal polytropic gas

$$p=R\rho\theta,\quad e=C_v\theta,\quad C_v=\frac{R}{\gamma-1}, \tag{1.2}$$

where $\gamma>1$ is the adiabatic exponent and $R>0$ is the specific gas constant.

In this paper, we will investigate the initial-boundary value problem of system (1.1) on $0\le x<+\infty$ and $0\le t<+\infty$ with the initial data

$$(\rho,u,\theta)(x,0)=(\rho_0,u_0,\theta_0)(x)\quad \text{for}\quad x\ge 0,\quad \text{and}\quad \inf_{x\in\mathbb{R}^+}(\rho_0,\theta_0)(x)>0, \tag{1.3}$$

the asymptotic boundary condition at the far field $x=+\infty$

$$(\rho,u,\theta,q)(+\infty,t)=(\rho_+,u_+,\theta_+,0), \quad t\geq 0, \tag{1.4}$$

and the Dirichlet-Neumann boundary conditions on the boundary $x=0$

$$u(0,t)=0, \quad \theta_x(0,t)=\theta_-, \quad q(0,t)=0, \quad t\geq 0, \tag{1.5}$$

where $\rho_+>0, u_+, \theta_\pm>0$ are given constants.

The slip boundary condition $u(0,t)=0$ means the wall is impermeable, so we call the initial-boundary value problem (1.1)-(1.5) the impermeable wall problem. We will consider the asymptotic stability of the 3-rarefaction wave of the impermeable wall problem (1.1)-(1.5). As far as we know, so far there is one rigorous result obtained recently in [9] on the global-in-time solutions of the impermeable wall problem for the radiative Euler equations, and most of the existing results are on the global-in-time existence and stability of the elementary wave of the Cauchy problem or the initial-boundary value (inflow/outflow) problem for the one-dimensional radiative full Euler equations (1.1). Actually, due to the difficulty that the velocity vanishes on the boundary, even for some strong dissipative systems such as Navier-Stokes equations, there are quite few results (see [25] for traveling wave and [28] for rarefaction wave) on the impermeable wall problem.

To study this problem, lots of additional boundary estimates on the perturbation of the velocity is needed (see (4.23), (4.47), (4.51), (4.75)). However, we do not need the estimates on the time-derivatives thanks for the Neumann boundary condition on the temperature considered in this paper. It is very different from those estimates in [9]. In fact, the combination of the condition $\theta_x(0,t)=0$ with $u(0,t)=0$ implies $p_x(0,t)=0$, which greatly simplifies the estimates on the boundary.

For the Cauchy problem, the global-in-time existence of solutions around a constant state was shown in [17]. If the initial data is a small perturbation of a given rarefaction wave with small strength, it was proved in [20] that the solutions converge to the rarefaction wave as $t\to+\infty$. Then in [14], the authors showed that when the absorption coefficient α tends to $+\infty$, the solutions converge to the rarefaction wave with the convergence rate $\alpha^{-1/3}|\ln\alpha|^2$, where the absorption coefficient α is defined by the relationship $a=3\alpha^2$ and $b=4\alpha\sigma$ for positive constants a,b and the Stefan-Boltzmann constant σ. The asymptotic stability of a single viscous contact wave was proved in [37,38]. The existence and stability for zero mass perturbation of the small amplitude shock profile were respectively studied in [21, 22]. The authors in [30] showed the nonlinear orbital asymptotic stability of small amplitude shock profiles for general hyperbolic-elliptic coupled systems of the type modeling the radiative gas. Analysis of large amplitude shock profiles was given in [2,24]. Finally, for the case of composite waves, the stability of rarefaction waves and a viscous contact wave was investigated in [31,39]. The unique global-in-time existence and the asymptotic stability of two viscous shock waves were studied in [4] by employing the anti-derivative method.

We initiated the research of the initial-boundary value problem on a half line for the radiative full Euler equations (1.1) in [5], where the asymptotic stability of rarefaction wave for the inflow problem was established. Then the asymptotic stability of rarefaction wave for the outflow problem was established in [7]. Recently the asymptotic stability of rarefaction wave for the impermeable wall problem was established in [9]. In addition, the asymptotic stability of viscous contact wave for the inflow problem and the asymptotic stability of shock wave for the outflow problem were established in [6,8], respectively.

In order to systematically study the behaviour of rarefaction wave with the radiative effect and of different-type boundary conditions such as the inflow/outflow problems [5,7], it is natural to consider the impermeable wall problem.

We need to mention that we are also motivated by the related investigations on the simplified radiative Euler model (Hamer model), which gives a good approximation to the fundamental system in a certain physical situation, c.f. [13,19]. The investigations on the simplified model provide a good understanding on the radiative effect. The exhaustive literature list is beyond the scope of the paper, and thus, only few closely related results on the rarefaction waves are mentioned, c.f. [3,10,11,18,32–34]. Interested readers can refer to them and references therein.

The rest of the paper is organized as follows. In Section 2, the smooth rarefaction wave is constructed based on the Riemann problem of the full Euler equations. Properties of smooth rarefaction waves which will be frequently used in this paper and the main theorem of this paper are given. In Section 3, we reformulate the system and establish the local existence of the reformulated problem. Then series of a priori estimates are established in Sections 4-5.

2 Construction on rarefaction wave and main results

In this section, we will introduce the smooth rarefaction wave which is the asymptotic profile considered in this paper. Then several properties of the smooth rarefaction wave and main theorem of this paper will be given.

2.1 Construction on rarefaction wave

It is well known that the 3-rarefaction wave curve through the right-hand side state (ρ_+,u_+,θ_+) is

$$R_3(\rho_+,u_+,\theta_+):=\Big\{(\rho^r,u^r,\theta^r):0<\rho^r<\rho_+,(\rho^r)^{1-\gamma}\theta^r=\rho_+^{1-\gamma}\theta_+,$$
$$u^r=u_++\frac{2}{\gamma-1}\sqrt{R\gamma\rho_+^{1-\gamma}\theta_+}\Big[(\rho^r)^{\frac{\gamma-1}{2}}-\rho_+^{\frac{\gamma-1}{2}}\Big]\Big\}. \tag{2.1}$$

In particular, there exist an unique pair (ρ_-,θ_-) such that $(\rho_-,0,\theta_-)\in R_3(\rho_+,u_+,\theta_+)$. The 3-rarefaction wave $(\rho^r,u^r,\theta^r)(x/t)$ connecting $(\rho_-,0,\theta_-)$ and (ρ_+,u_+,θ_+) is a global-in-time weak solution to the following Riemann problem of Euler system:

$$\begin{cases}\rho^r_t+(\rho^r u^r)_x=0,\\(\rho^r u^r)_t+[\rho^r(u^r)^2+p^r]_x=0,\\\left\{\rho\left[e^r+\frac{(u^r)^2}{2}\right]\right\}_t+\left\{\rho^r u^r\left(e^r+\frac{(u^r)^2}{2}\right)+p^r u^r\right\}_x=0,\\(\rho^r,u^r,\theta^r)(x,0)=\begin{cases}(\rho_-,0,\theta_-), & x<0,\\(\rho_+,u_+,\theta_+), & x>0.\end{cases}\end{cases} \tag{2.2}$$

In addition, q^r is defined by

$$q^r=-\frac{b}{a}\{(\theta^r)^4\}_x. \tag{2.3}$$

Next, in order to give the details of the large-time behavior of the solutions to the impermeable problem, it is necessary to construct a smooth approximation solution $(\widetilde{\rho},\widetilde{u},\widetilde{\theta})(x,t)$ from $(\rho^r,u^r,\theta^r)(x/t)$. As done in [15], firstly let us define $\tilde{w}(x/t)$ to be the solution of

$$\begin{aligned}&\tilde{w}_t+\tilde{w}\tilde{w}_x=0,\\&\tilde{w}(x,0)=\tilde{w}_0(x)=\frac{1}{2}(w_++w_-)+\bar{w}K_\nu\int_0^{\varepsilon x}\frac{dy}{(1+y^2)^\nu},\end{aligned} \tag{2.4}$$

where $\bar{w}=(w_+-w_-)/2>0,\varepsilon>0$ and K_ν is a constant such that

$$K_\nu\int_{-\infty}^{\infty}\frac{dy}{(1+y^2)^\nu}=1$$

for $\nu>3/2$. The properties of the solution $\tilde{w}$ to the regularized problem (2.4) were given in Lemma 2.1 of [26,27,35] as follows.

Lemma 2.1 ([26,27,35])**.** *The regularized problem* (2.4) *admits a unique global smooth solution $\tilde{w}(x,t)$ satisfying the following properties:*

(i) $w_- < \tilde{w}(x,t) < w_+, \tilde{w}_x(x,t) > 0$ *for each* $(x,t) \in \mathbb{R} \times [0,\infty)$.

(ii) For any p *with* $1 \le p \le \infty$, *there exists a constant* $C_{p,\nu}$ *depending on* p *and* ν *such that*

$$\|\tilde{w}_x(t)\|_{L^p}^p \le C_{p,\nu} \min(\varepsilon^{p-1}\varpi^p, \varpi t^{-p+1}),$$
$$\|\tilde{w}_{xx}(t)\|_{L^p}^p \le C_{p,\nu} \min\left(\varepsilon^{2p-1}\varpi^p, \varepsilon^{(p-1)(1-\frac{1}{2\nu})}\varpi^{-\frac{p-1}{2\nu}} t^{-p-\frac{p-1}{2\nu}}\right).$$

(iii) There exists a constant C_ν *depending on* ν *such that*

$$\int_{\mathbb{R}} \left|\frac{\tilde{w}_{xx}^2}{\tilde{w}_x}\right| dx = \left\|\frac{\tilde{w}_{xx}^2}{\tilde{w}_x}\right\|_{L^1} \le C_\nu \min\left(\varepsilon^2 \varpi, \varepsilon^{1-\frac{1}{2\nu}} \varpi^{-\frac{1}{2\nu}} t^{-1-\frac{1}{2\nu}}\right).$$

(iv) $|\partial_t^l \partial_x^k \tilde{w}|_\infty \le C|w_+ - w_-|^{l+k+1}, l,k \ge 0, l+k \le 4.$

(v) $\sup_{\mathbb{R}} |\tilde{w}(x,t) - w^R(x/t)| \to 0$, *as* $t \to +\infty$, *where* $w^R(x/t)$ *is a classic rarefaction wave connecting* w_- *and* w_+.

Let $w_- = \lambda_3(\rho_-,0,\theta_-)$ and $w_+ = \lambda_3(\rho_+,u_+,\theta_+)$. Here λ_3 is the third eigenvalue of the full Euler system. Then the smooth approximated solution

$$\widetilde{z}(x,t) = \left(\tilde{\rho}(x,t), \tilde{u}(x,t), \tilde{\theta}(x,t)\right)$$

is constructed by solving the following equations:

$$\begin{aligned} &S^r(\widetilde{\rho},\widetilde{u},\widetilde{\theta})(x,t) = S^r(\rho_+,u_+,\theta_+), \\ &\lambda_3(\widetilde{\rho},\widetilde{u},\widetilde{\theta})(x,t) = \widetilde{w}(x,1+t), \\ &\widetilde{u} = u_+ - \int_{v_+}^{\widetilde{v}} \lambda_3(\mu, S_+^r) d\mu, \end{aligned} \tag{2.5}$$

where

$$S^r(\widetilde{\rho},\widetilde{u},\widetilde{\theta}) = R\widetilde{\theta}\widetilde{\rho}^{1-\gamma}, \quad S_+^r = S^r(\rho_+,u_+,\theta_+) = R\theta_+\rho_+^{1-\gamma}.$$

It is easy to check that

$$\begin{cases} \widetilde{\rho}_t + (\widetilde{\rho}\widetilde{u})_x = 0, \\ (\widetilde{\rho}\widetilde{u})_t + \left(\widetilde{\rho}\widetilde{u}^2 + \widetilde{p}\right)_x = 0, \\ \left\{\widetilde{\rho}\left(\dfrac{R}{\gamma-1}\widetilde{\theta} + \dfrac{\widetilde{u}^2}{2}\right)\right\}_t + \left\{\widetilde{\rho}\widetilde{u}\left(\dfrac{R}{\gamma-1}\widetilde{\theta} + \frac{\widetilde{u}^2}{2}\right) + \widetilde{p}\widetilde{u}\right\}_x = 0, \\ (\widetilde{\rho},\widetilde{u},\widetilde{\theta})(0,t) = (\rho_-,0,\theta_-), \\ (\widetilde{\rho},\widetilde{u},\widetilde{\theta})(x,0) \ \to \ \begin{cases} (\rho_-,0,\theta_-), & x \ \to \ 0^+, \\ (\rho_+,u_+,\theta_+), & x \ \to \ +\infty. \end{cases} \end{cases} \tag{2.6}$$

In addition, $\tilde{q}$ is defined by

$$\widetilde{q} = -\frac{b}{a}(\widetilde{\theta}^4)_x. \tag{2.7}$$

The following properties are satisfied by the smooth rarefaction wave $\tilde{z}$ and $\tilde{q}$.

Lemma 2.2 (Property of Smooth Rarefaction Wave). *Smooth rarefaction wave $\tilde{z}(x,t)$ obtained via* (2.4) *and* (2.5) *satisfies*

(1) $\tilde{u}_x \geq 0$ *for* $x>0, t>0$.

(2) For any p $(1\leq p\leq +\infty)$, *there exists a constant* C *such that*

$$\begin{aligned} &\left\|\left(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x\right)(t)\right\|_{L^p} \leq C_{p,\nu}\min\left\{\epsilon^{1-\frac{1}{p}},(1+t)^{-1+\frac{1}{p}}\right\}, \\ &\left\|\left(\tilde{\rho}_{xx},\tilde{u}_{xx},\tilde{\theta}_{xx}\right)(t)\right\|_{L^p} \leq C_{p,\nu}\min\left\{\epsilon^{2-\frac{1}{p}},(1+t)^{-1-\frac{p-1}{2p}}\right\}. \end{aligned} \tag{2.8}$$

(3) $\lim_{t\to+\infty}\sup_{x\in\mathbb{R}^+}|(\tilde{\rho},\tilde{u},\tilde{\theta})(x,t)-(\rho^r,u^r,\theta^r)(x/t)|=0.$

(4) In particular, for $p=2$,

$$\begin{aligned} &\|(\widetilde{v}_x,\widetilde{u}_x,\widetilde{\theta}_x)(t)\|^2 \lesssim \epsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}, \quad \|\widetilde{q}_x(t)\|^2 \lesssim \epsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{4}}, \\ &\|(\widetilde{v}_{xx},\widetilde{u}_{xx},\widetilde{\theta}_{xx})(t)\| \lesssim \epsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}. \end{aligned} \tag{2.9}$$

(5)

$$\int_{\mathbb{R}^+}\left(\frac{\widetilde{\theta}_{xx}^2}{\widetilde{u}_x}+\frac{\widetilde{\theta}_x^4}{\widetilde{u}_x}\right)(x,t)dx \lesssim \epsilon^{\frac{1}{8}}(1+t)^{-\frac{9}{8}}. \tag{2.10}$$

The proof of Lemma 2.2 can be found elsewhere such as [7,35].

In this paper, we will use $(\widetilde{\rho},\widetilde{u},\widetilde{\theta},\widetilde{q})(x,t)$ to represent $(\widetilde{\rho},\widetilde{u},\widetilde{\theta},\widetilde{q})(x,t)|_{x\geq 0}$ for the notational simplicity.

2.2 Main results

In this subsection, we will reformulate the impermeable problem mathematically by introducing the difference of the solutions and the smooth rarefaction wave defined by (2.5) and (2.7)

$$(\phi,\psi,\xi,w) = (\rho,u,\theta,q) - (\widetilde{\rho},\widetilde{u},\widetilde{\theta},\widetilde{q}). \tag{2.11}$$

Then (ϕ,ψ,ξ,w) satisfies the following equations:

$$\phi_t+u\phi_x+\rho\psi_x=h_1, \tag{2.12a}$$
$$\rho(\psi_t+u\psi_x)+(p-\widetilde{p})_x=h_2, \tag{2.12b}$$
$$C_v\rho(\xi_t+u\xi_x)+p\psi_x+w_x=h_3, \tag{2.12c}$$
$$-w_{xx}+aw+4b\theta^3\xi_x+4b\widetilde{\theta}_x\xi\big(\theta^2+\theta\widetilde{\theta}+\widetilde{\theta}^2\big)=\widetilde{q}_{xx}, \tag{2.12d}$$

where

$$\begin{aligned} h_1&:=-\widetilde{\rho}_x\psi-\widetilde{u}_x\phi=\mathcal{O}(1)\big|(\widetilde{\rho}_x,\widetilde{u}_x)\big|\big|(\phi,\psi)\big|,\\ h_2&:=-\rho\widetilde{u}_x\psi+\frac{\widetilde{p}_x}{\widetilde{\rho}}\phi=\mathcal{O}(1)\big|(\widetilde{\rho}_x,\widetilde{u}_x,\widetilde{\theta}_x)\big|\big|(\phi,\psi)\big|,\\ h_3&:=-R\rho\xi\widetilde{u}_x-C_v\widetilde{\theta}_x\rho\psi-\widetilde{q}_x=\mathcal{O}(1)\big|(\widetilde{u}_x,\widetilde{\theta}_x)\big|\big|(\psi,\xi)\big|+\widetilde{q}_x \end{aligned} \tag{2.13}$$

with the initial-boundary conditions

$$\begin{cases} (\phi,\psi,\xi)(x,0)=(\phi_0,\psi_0,\xi_0)(x)\to(0,0,0) \quad \text{as} \quad x\to+\infty,\\ \psi(0,t)=0, \quad \xi_x(0,t)=-\widetilde{\theta}_x(0,t), \quad w(0,t)=-\widetilde{q}(0,t)=\dfrac{4b}{a}\theta^3\widetilde{\theta}_x(0,t). \end{cases} \tag{2.14}$$

We are ready to introduce the main result of this paper in this subsection. First, we define the solution space as

$$\begin{aligned} \mathbb{X}_M(0,t):=\Big\{&(\phi,\psi,\xi)\in C\big([0,t];H^2(\mathbb{R}^+)\big),w\in C\big([0,t];H^3(\mathbb{R}^+)\big),\\ &w_t\in C\big(0,t;H^2(\mathbb{R}^+)\big),(\phi,\psi,\xi)_x\in L^2\big(0,t;H^1(\mathbb{R}^+)\big),\\ &w\in L^2\big([0,t];H^3(\mathbb{R}^+)\big),w_t\in L^2\big(0,t;H^2(\mathbb{R}^+)\big),\\ &\sup_{\tau\in[0,t]}\big\{\|(\phi,\psi,\xi)(\tau)\|_2+\|w(\tau)\|_3+\|w_t(\tau)\|_2\big\}\le M\Big\}. \end{aligned} \tag{2.15}$$

Now we turn to state our main result that the smooth rarefaction wave constructed in (2.5) and (2.7) is globally stable.

Theorem 2.1. *Assume* $(\rho_-,0,\theta_-)\in R_3(\rho_+,u_+,\theta_+)$. *Suppose the initial data* (1.3) *and the boundary data* (1.5) *satisfy the compatibility condition* $u_0(0)=0$, *and the initial data satisfy*

$$\big(\rho_0-\widetilde{\rho}_0,u_0-\widetilde{u}_0,\theta_0-\widetilde{\theta}_0\big)\in(H^2\cap L^1)[0,+\infty). \tag{2.16}$$

If there exist constants $\epsilon_0>0$ *and* $\eta_0>0$ *suitably small such that* $\epsilon\lesssim\epsilon_0$ *and*

$$\big\|\big(\rho_0-\widetilde{\rho}_0,u_0-\widetilde{u}_0,\theta_0-\widetilde{\theta}_0\big)\big\|_2\lesssim\eta_0, \tag{2.17}$$

then the impermeable problem (1.1)-(1.5) *admits a unique solution* $(\rho,u,\theta,q)(x,t)$ *satisfying*

$$(\rho-\widetilde{\rho},u-\widetilde{u},\theta-\widetilde{\theta},q-\widetilde{q})(x,t)\in\mathbb{X}_M[0,+\infty). \tag{2.18}$$

Furthermore, it holds

$$\sup_{x\geq 0}\left|(\rho,u,\theta,q)(x,t)-(\rho^r,u^r,\theta^r,q^r)\left(\frac{x}{t}\right)\right| \to 0 \quad as \quad t\to +\infty. \tag{2.19}$$

3 The local-in-time existence

In this section, we will prove the local-in-time existence of the initial-boundary value problem (2.12)-(2.14), which is stated as follows.

Proposition 3.1 (Local-in-Time Existence)**.** *There exist positive constants ϵ_1,η_1 and $\overline{C}$ ($\overline{C}\eta_1\leq\eta_0$) such that if $\eta\lesssim\eta_1$ and $\epsilon\lesssim\epsilon_1$, then for any constant $M\in(0,\eta_1)$, there exists a positive constant $t_0=t_0(M)$, which does not depend on τ such that if*

$$\|(\phi,\psi,\xi,w)(\tau)\|_2\leq M,$$

then problem (2.12)-(2.14) *admits a unique solution* $(\phi,\psi,\xi,w)(x,t)\in\mathbb{X}_{\bar{C}M}(\tau,\tau+t_0)$.

Proof. We will extend the initial data from being defined on $\{\tau\}\times\mathbb{R}^+$ to $\{\tau\}\times\mathbb{R}$ to show the local-in-time existence. In fact, by the boundary condition (1.5), we extend (u,θ,q) by

$$\big(u(-x,\tau),\theta(-x,\tau),q(-x,\tau)\big):=\big(-u(x,\tau),\theta(x,\tau),-q(x,\tau)\big)\quad\text{for}\quad x\geq 0.$$

Moreover, by the Eq. (1.1b), $p_x(0,t)=0$. So, by the first identity in (1.2), we know that $\rho_x(0,t)=0$. Then we extend ρ by

$$\rho(-x,\tau):=\rho(x,\tau)\quad\text{for}\quad x\geq 0.$$

Then define the initial data (ϕ,ψ,ξ,w) on $\{\tau\}\times\mathbb{R}$ by (2.11), it follows from $\|(\phi,\psi,\xi,w)(\tau)\|_2\leq M$ that the extended functions (ϕ,ψ,ξ,w) also satisfy the same estimate. Now, following the argument in the proof of [4, Theorem 4.1], we know there exists a unique H^2-solution (ϕ,ψ,ξ,w) of Eqs. (2.12) on $(\tau,\tau+t_0)\times\mathbb{R}$ with the initial condition (2.14) and estimate $(\phi,\psi,\xi,w)(x,t)\in\mathbb{X}_{\bar{C}M}(\tau,\tau+t_0)$. Moreover, for a given solution $(\rho(x,t),u(x,t),\theta(x,t),q(x,t))$ of Eqs. (1.1), it is easy to see that $(\rho(-x,t),-u(-x,t),\theta(-x,t),-q(-x,t))$ is also a solution of Eqs. (1.1). Because the two solutions satisfy the same boundary condition due to the fact that

$$\big(\rho(-x,\tau),u(-x,\tau),\theta(-x,\tau),q(-x,\tau)\big)=\big(\rho(x,\tau),-u(x,\tau),\theta(x,\tau),-q(x,\tau)\big),$$

we have

$$\big(\rho(-x,t),u(-x,t),\theta(-x,t),q(-x,t)\big)=\big(\rho(x,t),-u(x,t),\theta(x,t),-q(x,t)\big).$$

So on the boundary $x=0$, we have the solution satisfies the boundary condition (1.5). Finally, assume there are two solutions $(\rho_1(x,t),u_1(x,t),\theta_1(x,t),q_1(x,t))$ and $(\rho_2(x,t),u_2(x,t),\theta_2(x,t),q_2(x,t))$ of Eqs. (1.1). For $i=1$ or 2, define

$$\big(\rho_i(-x,t),u_i(-x,t),\theta_i(-x,t),q_i(-x,t)\big)=\big(\rho_i(x,t),-u_i(x,t),\theta_i(x,t),-q_i(x,t)\big).$$

Then they both are the solutions of Eqs. (1.1) with the same initial data. So, by the uniqueness, we know that

$$\big(\rho_1(x,t),u_1(x,t),\theta_1(x,t),q_1(x,t)\big)=\big(\rho_2(x,t),u_2(x,t),\theta_2(x,t),q_2(x,t)\big).$$

Therefore, problem (2.12)-(2.14) admits a unique solution

$$(\phi,\psi,\xi,w)(x,t)\in\mathbb{X}_{\bar{C}M}(\tau,\tau+t_0).$$

The proof is complete. □

4 Energy estimates on fluid perturbation parts

Based on Proposition 3.1, the global-in-time existence can be established with a priori estimates obtained in this section. Suppose that solutions $(\phi,\psi,\xi,w)(x,t)$ of problem (2.12)-(2.14) has been extended to the time $T>t$, we will derive the following a priori estimates.

Proposition 4.1 (A Priori Estimates)**.** *Under the assumptions of Theorem* 2.1*, there exist positive constants $\eta_2\le\eta_1,\epsilon_2\le\min\{\epsilon_1,1\}$ and C such that for any $t<T$, if $(\phi,\psi,\xi,w)\in X([0,t])$ with satisfying $\epsilon\le\epsilon_2$ and*

$$N(t):=\sup_{0\le\tau\le t}\big\{\|(\phi,\psi,\xi)(\tau)\|_2+\|w(\tau)\|_3+\|w_t(\tau)\|_2\big\}\lesssim\eta_2, \tag{4.1}$$

then it holds the estimate that

$$\begin{aligned}&\sup_{0\le\tau\le t}\Big\{\|(\phi,\psi,\xi)(\tau)\|_2^2+\|w(\tau)\|_3^2+\|w_t(\tau)\|_2^2\Big\}\\&\quad+\int_0^t|(\phi_x,\psi_x,w_x,\psi_{xx},\phi_{tx},w_{tx},w_{txx})|^2(0,\tau)d\tau\\&\quad+\int_0^t\big(\|(\phi_x,\psi_x,\xi_x)(\tau)\|_1^2+\|w(\tau)\|_3^2+\|w_t(\tau)\|_2^2\big)d\tau\\&\lesssim\|(\phi_0,\psi_0,\xi_0)\|_2^2+\epsilon^{\frac{1}{8}}.\end{aligned}\tag{4.2}$$

Here both η_1 and ϵ_1 are the same positive constants as in Proposition 3.1.

Once Proposition 4.1 is proved, we can extend the local solution $(\phi,\psi,\xi,w)(x,t)$, obtained in Proposition 3.1 to the time $t=+\infty$ by the standard continuation argument. Moreover, the estimate (4.2) with passing the limit $t\to\infty$ implies that

$$\int_0^\infty \left(\|(\phi_x,\psi_x,\xi_x,w_x)(\tau)\|^2 + \frac{d}{dt}\|(\phi_x,\psi_x,\xi_x,w_x)(\tau)\|^2 \right) d\tau < +\infty.$$

Combining the Sobolev inequality, we can easily get the asymptotic behavior (2.19), that concludes the proof of Theorem 2.1. Therefore, the remaining task is to show the a priori estimate in Proposition 4.1.

At first, let

$$E = R\widetilde{\theta}\omega\left(\frac{\widetilde{\rho}}{\rho}\right) + \frac{\psi^2}{2} + C_v\widetilde{\theta}\omega\left(\frac{\theta}{\widetilde{\theta}}\right), \quad \omega(s) = s - 1 - \ln s. \tag{4.3}$$

By the definition of ω, we see that there exists a positive continuous function $C(s)$ such that

$$C(s)^{-1}(s-1)^2 \le \omega(s) \le C(s)(s-1)^2.$$

In addition, by direct calculations, one has

$$\omega(s) \ge \frac{1}{3}\ln^2 s \quad \text{as} \quad |s-1| \le \frac{1}{4}. \tag{4.4}$$

Following the almost same calculations as in [9, Lemma 4.1], we have the basic energy estimate on the fluid perturbation part $(\phi,\psi,\xi)(x,t)$ as follows.

Lemma 4.1. *Under the assumptions of Proposition* 4.1, *if* ϵ *and* $N(t)$ *are suitably small, it holds*

$$\begin{aligned} &\|(\phi,\psi,\xi)(t)\|^2 + \int_0^t \left(\|\sqrt{\widetilde{u}_x}(\phi,\psi,\xi)(\tau)\|^2 + \|w(\tau)\|_1^2 \right) d\tau \\ &\lesssim \|(\phi_0,\psi_0,\xi_0)\|^2 + \epsilon^{\frac{1}{8}} + N(t)\int_0^t \|(\xi_x,w_{xx})(\tau)\|^2 d\tau. \end{aligned} \tag{4.5}$$

4.1 First-order energy estimates

In this subsection, we will show the first-order energy estimates on the fluid perturbation part $(\phi,\psi,\xi)(x,t)$. Due to different boundary conditions from [9], here differentiate (2.12) with respect to x instead of t, the reformed equations can be written as

$$\phi_{tx} + u\phi_{xx} + \rho\psi_{xx} = H_1, \tag{4.6a}$$

$$\rho(\psi_{tx}+u\psi_{xx})+R\rho\xi_{xx}+R\theta\phi_{xx}=H_2, \tag{4.6b}$$
$$C_v\rho(\xi_{tx}+u\xi_{xx})+p\psi_{xx}+w_{xx}=H_3, \tag{4.6c}$$
$$-w_{xxx}+aw_x+4b\theta^3\xi_{xx}=H_4, \tag{4.6d}$$

where

$$\begin{aligned} &H_1:=h_{1x}-u_x\phi_x-\rho_x\psi_x,\\ &H_2:=h_{2x}-\rho_x(\psi_t+u\psi_x)-\rho u_x\psi_x\\ &\qquad -2R\left(\phi_x\xi_x+\widetilde{\theta}_x\phi_x+\widetilde{\rho}_x\xi_x\right)-R\widetilde{\rho}_{xx}\xi-R\widetilde{\theta}_{xx}\phi,\\ &H_3:=h_{3x}-C_v\rho_x(\xi_t+u\xi_x)-C_v\rho u_x\xi_x-p_x\psi_x,\\ &H_4:=\widetilde{q}_{xxx}-12b\theta^2\theta_x\xi_x-4b\left[\widetilde{\theta}_x\xi\left(\theta^2+\theta\widetilde{\theta}+\widetilde{\theta}^2\right)\right]_x. \end{aligned} \tag{4.7}$$

Then we have the following first-order energy estimate.

Lemma 4.2. *Under the assumptions of Proposition* 4.1*, if ϵ and $N(t)$ are suitably small, then it holds for $t\in[0,T]$,*

$$\begin{aligned} &\|(\phi_x,\psi_x,\xi_x)(t)\|^2+\int_0^t\left(\|(\xi_x,w_x,w_{xx})(\tau)\|^2+|(w_x,w_{xx})|^2(0,\tau)\right)d\tau\\ \lesssim&\|(\phi_0,\psi_0,\xi_0)\|_1^2+\epsilon^{\frac{1}{8}}+\left(\epsilon+N(t)\right)\int_0^t\|(\phi_x,\psi_x,\psi_{xx})(\tau)\|^2d\tau. \end{aligned} \tag{4.8}$$

Proof. **Step 1.** Multiplying (4.6a) by $R\theta\phi_x/\rho^2$, we get

$$\begin{aligned} &\left(\frac{R\theta}{2\rho^2}\phi_x^2\right)_t+\left(\frac{R\theta u\phi_x^2}{2\rho^2}\right)_x+\frac{R\theta}{\rho}\phi_x\phi_{xx}\\ =&\frac{R\theta}{\rho^2}\phi_xH_1+\left[\left(\frac{R\theta}{2\rho^2}\right)_t+\left(\frac{R\theta u}{2\rho^2}\right)_x\right]\phi_x^2. \end{aligned} \tag{4.9}$$

Multiplying (4.6b) by ψ_x/ρ, we have

$$\left(\frac{1}{2}\psi_x^2\right)_t+\left(\frac{u\psi_x^2}{2}\right)_x+\frac{R\theta}{\rho}\psi_x\phi_{xx}+R\psi_x\xi_{xx}=\frac{u_x}{2}\psi_x^2+\frac{\psi_x}{\rho}H_2. \tag{4.10}$$

Multiplying (4.6c) by $\xi_x/(\rho\theta)$, we obtain

$$\begin{aligned} &\left(\frac{C_v}{2\theta}\xi_x^2\right)_t+\left(\frac{C_vu}{2\theta}\xi_x^2\right)_x+R\psi_x\xi_{xx}+\frac{\xi_x}{\rho\theta}w_{xx}\\ =&\frac{\xi_x}{\rho\theta}H_3+\left[\left(\frac{C_v}{2\theta}\right)_t+\left(\frac{C_vu}{2\theta}\right)_x\right]\xi_x^2. \end{aligned} \tag{4.11}$$

Multiplying (4.6d) by $w_x/(4b\rho\theta^4)$, we get

$$-\left(\frac{w_x}{4b\rho\theta^4}w_{xx}\right)_x+\left(\frac{w_x}{4b\rho\theta^4}\right)_x w_{xx}+\frac{aw_x^2}{4b\rho\theta^4}+\frac{w_x\xi_{xx}}{\rho\theta}=\frac{w_xH_4}{4b\rho\theta^4}. \tag{4.12}$$

Combining (4.9)-(4.12), one has

$$\begin{aligned}&\left(\frac{\phi_x^2}{2}+\frac{\psi_x^2}{2}+\frac{C_v}{2\theta}\xi_x^2\right)_t+\left(\frac{w_x}{4b\rho\theta^4}\right)_x w_{xx}+\frac{aw_x^2}{4b\rho\theta^4}+I_{2x}\\ =&\frac{R\theta}{\rho^2}\phi_xH_1+\frac{\psi_x}{\rho}H_2+\frac{u_x}{2}\psi_x^2+\left[\left(\frac{R\theta}{2\rho^2}\right)_t+\left(\frac{R\theta u}{2\rho^2}\right)_x\right]\phi_x^2+\left(\frac{R\theta}{\rho}\right)_x\psi_x\phi_x\\ &+\frac{\xi_x}{\rho\theta}H_3+\left[\left(C_v\frac{C_v}{2\theta}\right)_t+\left(\frac{C_vu}{2\theta}\right)_x\right]\xi_x^2+\frac{w_xH_4}{4b\rho\theta^4}+\left(\frac{1}{\rho\theta}\right)_x w_x\xi_x,\end{aligned} \tag{4.13}$$

where

$$\begin{aligned}I_2:=&\frac{R\theta u\phi_x^2}{2\rho^2}+\frac{u\psi_x^2}{2}+\frac{C_vu}{2\theta}\xi_x^2+\frac{R\theta}{\rho}\phi_x\psi_x\\ &+R\psi_x\xi_x+\frac{\xi_x}{\rho\theta}w_x-\frac{w_x}{4b\rho\theta^4}w_{xx}.\end{aligned} \tag{4.14}$$

Integrating (4.13) over $\mathbb{R}^+\times[0,t]$, choosing ϵ and $N(t)$ suitable small, we obtain

$$\begin{aligned}&\|(\phi_x,\psi_x,\xi_x)(t)\|^2+\int_0^t\|(w_x,w_{xx})(\tau)\|_1^2d\tau\\ \lesssim&\|(\phi_0,\psi_0,\xi_0)\|_1^2+\big(\epsilon+N(t)\big)\int_0^t\|(\phi_x,\psi_x,\xi_x)(\tau)\|^2d\tau\\ &+\int_0^t\int_{\mathbb{R}^+}\big|\big(\widetilde{\theta}_x,\widetilde{\theta}_{xx}\big)\big|^2|(\phi,\psi,\xi)|^2dxd\tau+\int_0^t I_2(0,\tau)d\tau.\end{aligned} \tag{4.15}$$

Step 2. Estimates of the boundary integral $\int_0^t I_2(0,\tau)d\tau$.
Firstly, we see from $u(0,t)=\psi(0,t)=0,\xi_x(0,t)=-\widetilde{\theta}_x(0,t)$ that

$$I_2(0,t)=\frac{1}{\rho}\left(R\theta\phi_x\psi_x+R\rho\psi_x\xi_x+\frac{\xi_x}{\theta}w_x-\frac{w_x}{4b\theta^4}w_{xx}\right)(0,t). \tag{4.16}$$

From the Eq. (2.12d), it holds

$$w_{xx}(0,t)=aw(0,t)+4b\theta^3\xi_x(0,t)+12b\theta^2\widetilde{\theta}_x\xi(0,t)-\widetilde{q}_{xx}(0,t). \tag{4.17}$$

That is

$$\Big(\frac{w_x}{4b\theta^4}w_{xx}-\frac{w_x}{\theta}\xi_x\Big)(0,t)=\frac{w_x}{4b\theta^4}(aw-\widetilde{q}_{xx})(0,t)+\frac{3w_x}{\theta^2}\widetilde{\theta}_x\xi(0,t). \tag{4.18}$$

Hence, we obtain

$$\begin{aligned}
&\int_0^t\Big(\frac{w_x}{\theta}\xi_x-\frac{w_x}{4b\theta^4}w_{xx}\Big)(0,\tau)d\tau\\
\lesssim&\int_0^t\big|w_x\big(\widetilde{\theta}_x,\widetilde{q}_{xx},\widetilde{\theta}_x\xi\big)\big|(0,\tau)d\tau\\
\lesssim&\int_0^t\|w_x(\tau)\|_\infty\|\big(\widetilde{\theta}_x,\widetilde{q}_{xx},\widetilde{\theta}_x\xi\big)(\tau)\|_\infty d\tau\\
\lesssim&\frac{1}{8}\int_0^t\|w_{xx}(\tau)\|^2d\tau+\int_0^t\|w(\tau)\|_1^2d\tau+\epsilon^{\frac{1}{8}}.
\end{aligned} \tag{4.19}$$

On the other hand, we get from (2.12b) and boundary conditions that

$$(p-\widetilde{p})_x(0,t)=0, \tag{4.20}$$

$$(R\theta\phi_x+R\rho\xi_x)(0,t)=-R(\widetilde{\rho}_x\xi+\widetilde{\theta}_x\phi)(0,t). \tag{4.21}$$

Since $\|(\tilde{\rho}_x,\tilde{u}_x,\tilde{\theta}_x)(t)\|_\infty\lesssim\min\{\epsilon,(1+t)^{-1}\}$, it yields

$$\begin{aligned}
&\int_0^t(R\theta\phi_x+R\rho\xi_x)\psi_x(0,\tau)d\tau\lesssim\int_0^t\big|(\widetilde{\rho}_x,\widetilde{\theta}_x)(\phi,\xi)\psi_x\big|(0,\tau)d\tau\\
\lesssim&\int_0^t\|(\phi,\xi)\psi_x(\tau)\|^{\frac{1}{2}}\|(\phi_x,\xi_x)\psi_{xx}(\tau)\|^{\frac{1}{2}}\big\|\big(\widetilde{\rho}_x,\widetilde{\theta}_x\big)(\tau)\big\|_\infty d\tau\\
\lesssim&N(t)\int_0^t\|(\phi_x,\xi_x)\psi_{xx}(\tau)\|d\tau+\int_0^t\|\psi_x(\tau)\|\big\|\big(\widetilde{\rho}_x,\widetilde{\theta}_x\big)(\tau)\big\|_\infty^2d\tau\\
\lesssim&\big(N(t)+\epsilon\big)\int_0^t\|(\phi_x,\psi_x,\xi_x,\psi_{xx})(\tau)\|^2d\tau+\epsilon^{\frac{1}{8}}.
\end{aligned} \tag{4.22}$$

Furthermore, we get

$$\begin{aligned}
\int_0^t I_2(0,\tau)d\tau\lesssim&\int_0^t(R\theta\phi_x+R\rho\xi_x)\psi_x(0,\tau)d\tau\\
&+\int_0^t\Big(\frac{w_x}{\theta}\xi_x-\frac{w_x}{4b\theta^4}w_{xx}\Big)(0,\tau)d\tau\\
\lesssim&\big(N(t)+\epsilon\big)\int_0^t\|(\phi_x,\psi_x,\xi_x,\psi_{xx})(\tau)\|^2d\tau\\
&+\frac{1}{8}\int_0^t\|w_{xx}(\tau)\|^2d\tau+\int_0^t\|w(\tau)\|_1^2d\tau+\epsilon^{\frac{1}{8}}.
\end{aligned} \tag{4.23}$$

Based on Lemma 4.1, we can control the boundary term and obtain

$$
\begin{aligned}
&\|(\phi_x,\psi_x,\xi_x)(t)\|^2+\int_0^t\|w_x(\tau)\|_1^2d\tau\\
\lesssim&\|(\phi_0,\psi_0,\xi_0)\|_1^2+\epsilon^{\frac18}+\big(\epsilon+N(t)\big)\int_0^t\|(\phi_x,\psi_x,\xi_x,\psi_{xx})(\tau)\|^2d\tau.
\end{aligned}\tag{4.24}
$$

Furthermore, we see that

$$
\begin{aligned}
\int_0^t w_x^2(0,\tau)d\tau\lesssim&\int_0^t\|w_x(\tau)\|_\infty^2d\tau\lesssim\int_0^t\|w_x(\tau)\|\|w_{xx}(\tau)\|d\tau\\
\lesssim&\frac18\int_0^t\|w_{xx}(\tau)\|^2d\tau+\int_0^t\|w_x(\tau)\|^2d\tau.
\end{aligned}\tag{4.25}
$$

By (4.17), it holds

$$
\begin{aligned}
\int_0^t w_{xx}^2(0,\tau)d\tau\lesssim&\int_0^t\big|\big(\widetilde{\theta}_x,\widetilde{\theta}_x\xi,\widetilde{q}_{xx}\big)(0,\tau)\big|^2d\tau\\
\lesssim&\int_0^t\big\|\big(\widetilde{\theta}_x,\widetilde{\theta}_x\xi,\widetilde{q}_{xx}\big)(\tau)\big\|_\infty^2d\tau\lesssim\epsilon^{\frac18}.
\end{aligned}\tag{4.26}
$$

At last, by (2.12d), it holds

$$
\int_0^t\|\xi_x(\tau)\|^2d\tau\lesssim\int_0^t\|(w,w_{xx})(\tau)\|^2d\tau+\int_0^t\int_{\mathbb{R}^+}\big(\widetilde{\theta}_x^2\xi^2+\widetilde{q}_{xx}^2\big)dxd\tau. \tag{4.27}
$$

Thus, we get (4.8). This completes the proof. □

Now we will deal with the term $\int_0^t\|(\phi_x,\psi_x)(\tau)\|^2d\tau$.

Lemma 4.3. *Under the assumptions of Proposition* 4.1, *if* ϵ *and* $N(t)$ *are suitably small, it holds*

$$
\int_0^t\|(\phi_x,\psi_x)(\tau)\|^2d\tau\lesssim\|(\phi_0,\psi_0,\xi_0)\|_1^2+\epsilon^{\frac18}+\big(\epsilon+N(t)\big)\int_0^t\|\psi_{xx}(\tau)\|^2d\tau. \tag{4.28}
$$

Proof. (2.12b) can be written as

$$
\psi_t+u\psi_x+\frac{R\theta}{\rho}\phi_x+R\xi_x=\frac{h_2}{\rho}-\frac{R}{\rho}\big(\widetilde{\theta}_x\phi+\widetilde{\rho}_x\xi\big). \tag{4.29}
$$

Multiplying (4.29) by $R\theta\phi_x/2$, we get

$$
\left(\frac{R\theta}{2}\phi_x\psi\right)_t+\left(\frac{R}{2}\theta u\phi_x\psi+\frac{p}{2}\psi\psi_x\right)_x+\frac{R^2\theta^2}{2\rho}\phi_x^2-\frac{p}{2}\psi_x^2
$$

$$
\begin{aligned}
&=-R\xi_x R\theta\frac{\phi_x}{2}+\frac{R}{2}(\theta_t-u\theta_x+u_x\theta)\phi_x\psi-\frac{R}{2}\theta\psi(u_x\phi_x+\rho_x\psi_x)\\
&\quad+\frac{1}{2}p_x\psi\psi_x+\frac{R}{2}\psi h_{1x}+\frac{R\theta}{2\rho}\phi_x h_2-\frac{R^2\theta}{2\rho}\phi_x\big(\widetilde{\theta}_x\phi+\widetilde{\rho}_x\xi\big).
\end{aligned}\tag{4.30}
$$

Multiplying (2.12c) by ψ_x, we have

$$
\begin{aligned}
&(C_v\rho\xi\psi_x)_t-(C_v\rho\xi\psi_t)_x+p\psi_x^2+C_v\rho u\xi_x\psi_x\\
&\quad+C_v(\rho u)_x\xi\psi_x+C_v(\rho\xi)_x\psi_t+w_x\psi_x=h_3\psi_x.
\end{aligned}\tag{4.31}
$$

Combining (4.30) with (4.31), we obtain

$$
\begin{aligned}
&\left(\frac{R\theta}{2}\phi_x\psi+C_v\rho\xi\psi_x\right)_t+\left(\frac{R}{2}\theta u\phi_x\psi+\frac{p}{2}\psi\psi_x-C_v\rho\xi\psi_t\right)_x+\frac{R^2\theta^2}{2\rho}\phi_x^2+\frac{p}{2}\psi_x^2\\
=&-R\theta^2\xi_x\frac{\phi_x}{2}+\frac{R}{2}(\theta_t-u\theta_x+u_x\theta)\phi_x\psi-\frac{R}{2}\theta\psi(u_x\phi_x+\rho_x\psi_x)\\
&+\frac{1}{2}p_x\psi\psi_x-\frac{R^2\theta}{2\rho}\phi_x\big(\widetilde{\theta}_x\phi+\widetilde{\rho}_x\xi\big)-w_x\psi_x+\frac{R}{2}\psi h_{1x}+\frac{R\theta}{2\rho}\phi_x h_2+h_3\psi_x\\
=&\big(\epsilon+N(t)\big)|(\phi_x,\psi_x,\xi_x)|^2+\mathcal{O}(1)\Big(\big|(\widetilde{\theta}_x,\widetilde{\theta}_{xx})\big||(\phi,\psi,\xi)|^2+|\xi_x\phi_x+w_x\psi_x|\Big).
\end{aligned}\tag{4.32}
$$

Integrating (4.32) over $\mathbb{R}^+\times[0,t]$ and using boundary conditions $(\psi,\psi_t)(0,t)=(0,0)$, one has

$$
\begin{aligned}
\int_0^t\|(\phi_x,\psi_x)(\tau)\|^2d\tau\lesssim&\|(\phi_0,\psi_0,\xi_0)\|_1^2+\left(\frac{1}{8}+\epsilon+N(t)\right)\int_0^t\|(\phi_x,\psi_x)(\tau)\|^2d\tau\\
&+\int_0^t\int_{\mathbb{R}^+}\Big(\big|(\widetilde{\theta}_x,\widetilde{\theta}_{xx})\big|^2|(\phi,\psi,\xi)|^2+\xi_x^2+w_x^2\Big)dxd\tau.
\end{aligned}\tag{4.33}
$$

Using Lemmas 4.1 and 4.2, it yields (4.28). The proof is complete. □

Finally, we get H^1-estimate on the solution (ϕ,ψ,ξ) by using Lemmas 4.1-4.3

$$
\begin{aligned}
&\|(\phi,\psi,\xi)(t)\|_1^2+\int_0^t|(w_x,w_{xx})|^2(0,\tau)d\tau\\
&\quad+\int_0^t\Big(\big\|\sqrt{\widetilde{u}_x}(\phi,\psi,\xi)(\tau)\big\|^2+\|(\phi_x,\psi_x,\xi_x)(\tau)\|^2+\|w(\tau)\|_2^2\Big)d\tau\\
\lesssim&\|(\phi_0,\psi_0,\xi_0)\|_1^2+\epsilon^{\frac{1}{8}}+\big(\epsilon+N(t)\big)\int_0^t\|\psi_{xx}(\tau)\|^2d\tau.
\end{aligned}\tag{4.34}
$$

4.2 Second-order energy estimates

In this subsection, we will show the second-order energy estimates on the fluid perturbation part $(\phi,\psi,\xi)(x,t)$. To do this, we first estimate the radial derivative. Differentiate (2.12) with respect to t, then the reformed equations can be written as

$$\phi_{tt}+u\phi_{tx}+\rho\psi_{tx}=\widetilde{h}_1, \tag{4.35a}$$
$$\rho(\psi_{tt}+u\psi_{tx})+(p-\widetilde{p})_{tx}=\widetilde{h}_2, \tag{4.35b}$$
$$C_v\rho(\xi_{tt}+u\xi_{tx})+p\psi_{xt}+w_{tx}=\widetilde{h}_3, \tag{4.35c}$$
$$-w_{txx}+aw_t+4b\theta^3\xi_{tx}=\widetilde{h}_4, \tag{4.35d}$$

where

$$\begin{aligned}
&\widetilde{h}_1:=h_{1t}-u_t\phi_x-\rho_t\psi_x,\\
&\widetilde{h}_2:=h_{2t}-\rho_t(\psi_t+u\psi_x)-\rho u_t\psi_x,\\
&\widetilde{h}_3:=h_{3t}-C_v\rho_t(\xi_t+u\xi_x)-C_v\rho u_t\xi_x-p_t\psi_x,\\
&\widetilde{h}_4:=\widetilde{q}_{txx}-12b\theta^2\theta_t\xi_x-4b\left[\widetilde{\theta}_x\xi\big(\theta^2+\theta\widetilde{\theta}+\widetilde{\theta}^2\big)\right]_t.
\end{aligned} \tag{4.36}$$

First, we establish the space-time estimates as follow. We remark that the boundary estimates are very different from [9].

Lemma 4.4. *Under the assumptions of Proposition* 4.1*, if ϵ and $N(t)$ are suitably small, it holds*

$$\begin{aligned}
&\|(\phi_{tx},\psi_{tx},\xi_{tx})(t)\|^2+\int_0^t\big(|(\phi_{tx},w_{tx},w_{txx})|^2(0,\tau)+\|(w_{tx},w_{txx})(\tau)\|^2\big)d\tau\\
\lesssim&\|(\phi_0,\psi_0,\xi_0)\|_2^2+\epsilon^{\frac18}+\big(\epsilon+N(t)\big)\int_0^t\|(\phi_{xx},\psi_{xx},\xi_{xx})(\tau)\|^2d\tau.
\end{aligned} \tag{4.37}$$

Proof. Multiplying (4.35b) by ψ_{tx}/ρ, we get

$$\begin{aligned}
&\left(\frac{\psi_{xt}^2}{2}\right)_t+\left(\frac{u}{2}\psi_{xt}^2\right)_x+R\xi_{txx}\psi_{tx}+\frac{R\theta}{\rho}\phi_{txx}\psi_{tx}\\
=&-\Big\{R\theta_t\phi_{xx}+R\rho_t\xi_{xx}+(R\theta_x\phi_x+R\rho_x\xi_x)_t+R\big(\widetilde{\rho}_x\xi+\widetilde{\theta}_x\phi\big)_{tx}\Big\}\frac{\psi_{tx}}{\rho}\\
&+\big\{\widetilde{h}_{2x}-\rho_x\psi_{tt}-(\rho u)_x\psi_{tx}\big\}\frac{\psi_{tx}}{\rho},
\end{aligned} \tag{4.38}$$

since

$$\begin{aligned}
(p-\widetilde{p})_{txx}=&R\theta\phi_{txx}+R\rho\xi_{txx}+\big\{R\big(\widetilde{\theta}_x\phi+\widetilde{\rho}_x\xi\big)\big\}_{tx}\\
&+R\theta_t\phi_{xx}+R\rho_t\xi_{xx}+(R\theta_x\phi_x+R\rho_x\xi_x)_t.
\end{aligned}$$

Multiplying (4.35a) by $R\theta\phi_{xt}/\rho^2$, we have

$$\begin{aligned}&\left(\frac{R\theta}{2\rho^2}\phi_{xt}^2\right)_t+\left(\frac{R\theta u}{2\rho^2}\phi_{xt}^2\right)_x+\frac{R\theta}{\rho}\phi_{xt}\psi_{txx}\\ =&\left\{\left(\frac{R\theta}{2\rho^2}\right)_t+\left(\frac{R\theta u}{2\rho^2}\right)_x\right\}\phi_{xt}^2+\frac{R\theta}{\rho^2}\phi_{xt}\left(\widetilde{h}_{1x}-u_x\phi_{xt}-\rho_x\psi_{tx}\right).\end{aligned}\tag{4.39}$$

Multiplying (4.6c) by $\xi_{tx}/(\rho\theta)$, we obtain

$$\begin{aligned}&\left(\frac{C_v}{2\theta}\xi_{tx}^2\right)_t+\left(\frac{C_v u}{2\theta}\xi_{tx}^2\right)_x+R\xi_{tx}\psi_{txx}+\frac{\xi_{tx}}{\rho\theta}w_{txx}\\ =&\left\{\left(\frac{C_v}{2\theta}\right)_t+\left(\frac{C_v u}{2\theta}\right)_x\right\}\xi_{tx}^2+\frac{\xi_{tx}}{\rho\theta}\left\{\widetilde{h}_{3x}-C_v\rho_x\xi_{tt}-C_v(\rho u)_x\xi_{tx}-R(\rho\theta)_x\psi_{tx}\right\}.\end{aligned}\tag{4.40}$$

Multiplying (4.6d) by $w_{tx}/(4b\rho\theta^4)$, we get

$$\begin{aligned}&\left(-\frac{w_{tx}}{4b\rho\theta^4}w_{txx}\right)_x+\left(\frac{w_{tx}}{4b\rho\theta^4}\right)_x w_{txx}+\frac{aw_{tx}^2}{4b\rho\theta^4}+\frac{w_{tx}}{\rho\theta}\xi_{txx}\\ =&\frac{w_{tx}}{4\rho\theta^4}\widetilde{h}_{4x}-\frac{3\theta_x}{\rho\theta^2}\xi_{tx}w_{tx}.\end{aligned}\tag{4.41}$$

Combining (4.38)-(4.41), one has

$$\begin{aligned}&\left(\frac{R\theta}{2\rho^2}\phi_{xt}^2+\frac{\psi_{xt}^2}{2}+\frac{C_v}{2\theta}\xi_{tx}^2\right)_t+\left(\frac{w_{tx}}{4b\rho\theta^4}\right)_x w_{txx}+\frac{aw_{tx}^2}{4b\rho\theta^4}+I_{3x}\\ =&\left(\frac{1}{\rho\theta}\right)_x w_{tx}\xi_{tx}+\left(\frac{R\theta}{\rho}\right)_x\phi_{tx}\psi_{tx}+\frac{w_{tx}}{4\rho\theta^4}\widetilde{h}_{4x}-\frac{3\theta_x}{\rho\theta^2}\xi_{tx}w_{tx}\\ &+\left\{\left(\frac{R\theta}{2\rho^2}\right)_t+\left(\frac{R\theta u}{2\rho^2}\right)_x\right\}\phi_{xt}^2+\frac{R\theta}{\rho^2}\phi_{xt}\left(\widetilde{h}_{1x}-u_x\phi_{xt}-\rho_x\psi_{tx}\right)\\ &-\left\{R\theta_t\phi_{xx}+R\rho_t\xi_{xx}+(R\theta_x\phi_x+R\rho_x\xi_x)_t+R\left(\widetilde{\rho}_t\xi+\widetilde{\theta}_t\phi\right)_{tx}\right\}\frac{\psi_{tx}}{\rho}\\ &+\left\{\widetilde{h}_{2x}-\rho_x\psi_{tt}-(\rho u)_x\psi_{xx}\right\}\frac{\psi_{xt}}{\rho}+\left\{\left(\frac{C_v}{2\theta}\right)_t+\left(\frac{C_v u}{2\theta}\right)_x\right\}\xi_{tx}^2\\ &+\frac{\xi_{tx}}{\rho\theta}\left\{\widetilde{h}_{3x}-C_v\rho_x\xi_{tt}-C_v(\rho u)_x\xi_{tx}-R(\rho\theta)_x\psi_{tx}\right\},\end{aligned}\tag{4.42}$$

where

$$I_3:=\frac{R\theta u}{2\rho^2}\phi_{xt}^2+\frac{u}{2}\psi_{xt}^2+\frac{C_v u}{2\theta}\xi_{tx}^2+R\xi_{tx}\psi_{tx}+\frac{R\theta}{\rho}\phi_{tx}\psi_{tx}-\frac{w_{tx}}{4b\rho\theta^4}w_{txx}+\frac{w_{tx}}{\rho\theta}\xi_{tx}.$$

Integrating (4.42) over $\mathbb{R}^+\times[0,t]$, we have

$$\begin{aligned}
&\|(\phi_{xt},\psi_{xt},\xi_{tx})(t)\|^2+\int_0^t\|(w_{txx},w_{tx})(\tau)\|^2d\tau\\
\lesssim&\|(\phi_0,\psi_0,\xi_0)\|_2^2+\|(\phi_t,\psi_t,\xi_t)(0)\|_1^2+\epsilon^{\frac{1}{8}}+\int_0^t I_3(0,\tau)d\tau\\
&+\big(\epsilon+N(t)\big)\int_0^t\|(\phi_x,\psi_x,\xi_x)(\tau)\|_1^2d\tau\\
&+\int_0^t\int_{\mathbb{R}^+}\big|\big(\widetilde{\theta}_x,\widetilde{\theta}_{xx},\widetilde{\theta}_{txx}\big)\big|^2|(\phi,\psi,\xi)|^2dxd\tau. \qquad (4.43)
\end{aligned}$$

Similarly, using the boundary condition $u(0,t)=0$, we get

$$I_3(0,t)=\frac{1}{\rho}(R\rho\xi_{tx}\psi_{tx}+R\theta\phi_{tx}\psi_{tx})(0,t)+\frac{1}{\rho}\Big(\frac{w_{tx}}{\theta}\xi_{tx}-\frac{w_{tx}}{4b\theta^4}w_{txx}\Big)(0,t). \qquad (4.44)$$

Recall that $(p-\widetilde{p})_{tx}(0,t)=0$ and

$$\begin{aligned}
(p-\widetilde{p})_{tx}=&R\rho\xi_{tx}+R\theta\phi_{tx}+R\widetilde{\rho}_{tx}\xi+R\widetilde{\theta}_{tx}\phi\\
&+R\widetilde{\rho}_t\xi_x+R\widetilde{\theta}_t\phi_x+R\rho_x\xi_t+R\theta_x\phi_t. \qquad (4.45)
\end{aligned}$$

Thus, it holds by $w(0,t)=-\widetilde{q}(0,t),\partial_t^i\psi(0,t)=0,i=0,1,2,$

$$\begin{aligned}
(R\rho\xi_{tx}+R\theta\phi_{tx})\psi_{tx}(0,t)=&-R(\widetilde{\rho}_{tx}\xi+\widetilde{\theta}_{tx}\phi+\widetilde{\rho}_t\xi_x+\widetilde{\theta}_t\phi_x)\psi_{tx}(0,t)-R\rho_x\xi_t\psi_{tx}(0,t)\\
=&-R\{(\widetilde{\rho}_{tx}\xi+\widetilde{\theta}_{tx}\phi+\widetilde{\rho}_t\xi_x+\widetilde{\theta}_t\phi_x)\psi_x\}_t(0,t)\\
&+R(\widetilde{\rho}_{tx}\xi+\widetilde{\theta}_{tx}\phi+\widetilde{\rho}_t\xi_x+\widetilde{\theta}_t\phi_x)_t\psi_x(0,t)\\
&+\frac{R\rho_x}{C_v\rho}\big(p\psi_x+R\rho\xi\widetilde{u}_x+C_v\widetilde{\theta}_x\rho\psi\big)\psi_{tx}(0,t)\\
=&-R\{(\widetilde{\rho}_{tx}\xi+\widetilde{\theta}_{tx}\phi+\widetilde{\rho}_t\xi_x+\widetilde{\theta}_t\phi_x)\psi_x\}_t(0,t)\\
&+R(\widetilde{\rho}_{tx}\xi+\widetilde{\theta}_{tx}\phi+\widetilde{\rho}_t\xi_x+\widetilde{\theta}_t\phi_x)_t\psi_x(0,t)\\
&+R(\gamma-1)\theta\rho_x\psi_x\psi_{tx}(0,t)\\
&+R\rho_x[(\gamma-1)\xi\widetilde{u}_x+\widetilde{\theta}_x\psi]\psi_{tx}(0,t)\\
=&-R\{(\widetilde{\rho}_{tx}\xi+\widetilde{\theta}_{tx}\phi+\widetilde{\rho}_t\xi_x+\widetilde{\theta}_t\phi_x)\psi_x\}_t(0,t)\\
&+R(\widetilde{\rho}_{tx}\xi+\widetilde{\theta}_{tx}\phi+\widetilde{\rho}_t\xi_x+\widetilde{\theta}_t\phi_x)_t\psi_x(0,t)\\
&+R(\gamma-1)\left(\theta\rho_x\frac{\psi_x^2}{2}+\rho_x\xi\widetilde{u}_x\psi_x\right)_t(0,t)\\
&-R(\gamma-1)\left\{(\theta\rho_x)_t\frac{\psi_x^2}{2}+(\rho_x\xi\widetilde{u}_x)_t\psi_x\right\}(0,t), \qquad (4.46)
\end{aligned}$$

which implies

$$\begin{aligned}&\int_0^t |(R\rho\xi_{tx}+R\theta\phi_{tx})\psi_{tx}|(0,\tau)d\tau\\ \lesssim&\|(\phi,\psi,\xi)(t)\|_1^2+\|(\phi,\psi,\xi)(0)\|_1^2+\epsilon^{\frac{1}{8}}\\ &+\big(\epsilon+N(t)\big)\int_0^t |(\phi_x,\psi_x,\phi_{tx})|^2(0,\tau)d\tau. \end{aligned}\tag{4.47}$$

In addition, we have

$$\begin{aligned}\int_0^t \xi_{tx}^2(0,\tau)d\tau&=\int_0^t \widetilde{\theta}_{tx}^2(0,\tau)d\tau=\int_0^t \|\widetilde{\theta}_{tx}(\tau)\|_\infty^2 d\tau\\ &\lesssim\epsilon^{\frac{1}{4}}\int_0^t (1+\tau)^{-\frac{7}{4}}d\tau\lesssim\epsilon^{\frac{1}{4}}, \end{aligned}\tag{4.48}$$

$$\begin{aligned}\int_0^t w_{tx}^2(0,\tau)d\tau&\lesssim\int_0^t \|w_{tx}(\tau)\|_\infty^2 d\tau\\ &\lesssim\frac{1}{8}\int_0^t \|w_{txx}(\tau)\|^2 d\tau+\int_0^t \|w_{tx}(\tau)\|^2 d\tau. \end{aligned}\tag{4.49}$$

Since $w_{txx}=4b\theta^3\xi_{tx}-\widetilde{h}_4+aw_t$, we have

$$\int_0^t w_{txx}^2(0,\tau)d\tau\lesssim\int_0^t |(\xi_{tx},\widetilde{h}_4,w_t)(0,\tau)|^2 d\tau\lesssim\epsilon^{\frac{1}{8}}. \tag{4.50}$$

In summary, we get

$$\begin{aligned}\int_0^t I_3(0,\tau)d\tau\lesssim&\|(\phi,\psi,\xi)(t)\|_1^2+\|(\phi,\psi,\xi)(0)\|_1^2+\epsilon^{\frac{1}{8}}\\ &+\big(\epsilon+N(t)\big)\int_0^t |(\phi_x,\psi_x,\phi_{tx})|^2(0,\tau)d\tau. \end{aligned}\tag{4.51}$$

At last, (4.45) implies

$$\begin{aligned}\int_0^t \phi_{tx}^2(0,\tau)d\tau\lesssim&\int_0^t \xi_{tx}^2(0,\tau)d\tau+\int_0^t |(\widetilde{\rho}_{tx},\widetilde{\theta}_{tx})(\phi,\xi)|^2(0,\tau)d\tau\\ &+\int_0^t |(\widetilde{\rho}_t,\widetilde{\theta}_t)(\phi_x,\xi_x)|^2(0,\tau)d\tau\\ &+\int_0^t |\rho_x(\psi_x,w_x,h_3)|(0,\tau)d\tau\\ \lesssim&\int_0^t |(\phi_x,\psi_x)|^2(0,\tau)d\tau+\epsilon^{\frac{1}{8}} \end{aligned}$$

$$\begin{aligned}&\lesssim \int_0^t \|(\phi_x,\psi_x)(\tau)\|_\infty^2 d\tau + \epsilon^{\frac{1}{8}}\\&\lesssim \int_0^t \|(\phi_x,\psi_x)(\tau)\|_1^2 d\tau + \epsilon^{\frac{1}{8}},\end{aligned}\tag{4.52}$$

$$\begin{aligned}\int_0^t I_3(0,\tau)d\tau \lesssim{}& \|(\phi,\psi,\xi)(t)\|_1^2 + \|(\phi,\psi,\xi)(0)\|_1^2 + \epsilon^{\frac{1}{8}}\\&+\big(\epsilon+N(t)\big)\int_0^t \|(\phi_{xx},\psi_{xx})(\tau)\|^2 d\tau.\end{aligned}\tag{4.53}$$

Combining (4.53) and (4.43), we obtain (4.37). The proof is complete. □

Now we establish the estimate on the second-order derivatives as follow.

Lemma 4.5. *Under the same assumptions listed in Proposition* 4.1*, if* ϵ *and* $N(t)$ *are suitably small, it holds*

$$\begin{aligned}&\|(\phi_{xx},\psi_{xx},\xi_{xx})(t)\|^2 + \int_0^t \big(|(\psi_{xx},\psi_{txx})|^2(0,\tau) + \|(\xi_{xx},w_{xx},w_{xxx})(\tau)\|^2\big)d\tau\\&\lesssim \|(\phi_0,\psi_0,\xi_0)\|_2^2 + \epsilon^{\frac{1}{8}} + \big(\epsilon+N(t)\big)\int_0^t \|(\phi_{xx},\psi_{xx})(\tau)\|^2 d\tau.\end{aligned}\tag{4.54}$$

Proof. **Step 1.** Multiplying (4.6a) by $R\theta\phi_{xx}/\rho^2$, we get

$$\begin{aligned}&\left(\frac{R\theta}{2\rho^2}\phi_{xx}^2\right)_t + \left(\frac{Ru\theta}{2\rho^2}\phi_{xx}^2\right)_x + \frac{R\theta}{\rho}\phi_{xx}\psi_{xxx}\\&=\left\{\left(\frac{R\theta}{2\rho^2}\right)_t + \left(\frac{Ru\theta}{2\rho^2}\right)_x\right\}\phi_{xx}^2 + \frac{R\theta}{\rho^2}\phi_{xx}(H_{1x} - u_x\phi_{xx} - \rho_x\psi_{xx}).\end{aligned}\tag{4.55}$$

Multiplying (4.6b) by ψ_{xx}/ρ, we have

$$\begin{aligned}&\left(\frac{\psi_{xx}^2}{2}\right)_t + \left(\frac{u}{2}\psi_{xx}^2\right)_x + \frac{R\theta}{\rho}\psi_{xx}\phi_{xxx} + R\psi_{xx}\xi_{xxx}\\&=\frac{u_x}{2}\psi_{xx}^2 + \psi_{xx}\left\{\left(\frac{H_2}{\rho}\right)_x - \left(\frac{R\theta}{\rho}\right)_x\phi_{xx}\right\}.\end{aligned}\tag{4.56}$$

Multiplying (4.6c) by $\xi_{xx}/(\rho\theta)$, we obtain

$$\begin{aligned}&\left(\frac{C_v}{2\theta}\xi_{xx}^2\right)_t + \left(\frac{C_v u}{2\theta}\xi_{xx}^2\right)_x + R\psi_{xxx}\xi_{xx} + \frac{\xi_{xx}}{\rho\theta}w_{xxx}\\&=\left\{\left(\frac{C_v}{2\theta}\right)_t + \left(\frac{C_v u}{2\theta}\right)_x\right\}\xi_{xx}^2 + \frac{\xi_{xx}}{\rho\theta}\{H_{3x} - C_v\rho_x\xi_{tx} - C_v(\rho u)_x\xi_{xx} - p_x\psi_{xx}\}.\end{aligned}\tag{4.57}$$

Multiplying (4.6d) by $w_{xx}/(4b\rho\theta^4)$, we get

$$
\begin{aligned}
&-\left(\frac{w_{xx}}{4b\rho\theta^4}w_{xxx}\right)_x+\left(\frac{w_{xx}}{4b\rho\theta^4}\right)_x w_{xxx}+\frac{aw_{xx}^2}{4b\rho\theta^4}+\frac{w_{xx}}{\rho\theta}\xi_{xxx}\\
=&\frac{w_{xx}}{4b\rho\theta^4}H_{4x}-\theta_x\xi_{xx}\frac{w_{xx}}{\rho\theta^2}.
\end{aligned}\tag{4.58}
$$

Combining (4.55)-(4.58), one has

$$
\begin{aligned}
&\left\{\frac{R\theta}{2\rho^2}\phi_{xx}^2+\frac{\psi_{xx}^2}{2}+\frac{C_v}{2\theta}\xi_{xx}^2\right\}_t+\left(\frac{w_{xx}}{4b\rho\theta^4}\right)_x w_{xxx}+\frac{aw_{xx}^2}{4b\rho\theta^4}+I_{4x}\\
=&\mathcal{O}(1)\big(\epsilon+N(t)\big)\big|\big(\phi_x,\psi_x,\xi_x,w_x,\phi_{xx},\psi_{xx},\xi_{xx},w_{xx}\big)\big|^2\\
&+\mathcal{O}(1)\big|\big(\widetilde{\theta}_x,\widetilde{\theta}_{xx},\widetilde{\theta}_{xxx}\big)\big|^2|(\phi,\psi,\xi)|^2,
\end{aligned}\tag{4.59}
$$

where

$$
\begin{aligned}
I_4:=&\frac{Ru\theta}{2\rho^2}\phi_{xx}^2+\frac{u}{2}\psi_{xx}^2+\frac{C_v u}{2\theta}\xi_{xx}^2+R\psi_{xx}\xi_{xx}\\
&+\frac{R\theta}{\rho}\phi_{xx}\psi_{xx}+\frac{w_{xx}}{\rho\theta}\xi_{xx}-\frac{w_{xx}}{4b\rho\theta^4}w_{xxx}.
\end{aligned}\tag{4.60}
$$

Integrating (4.59) over $\mathbb{R}^+\times[0,t]$, choosing ϵ and $N(t)$ suitable small, we have

$$
\begin{aligned}
&\|(\phi_{xx},\psi_{xx},\xi_{xx})(t)\|^2+\int_0^t\big(\|w_{xx}(\tau)\|_1^2+\|\xi_{xx}(\tau)\|^2\big)d\tau\\
\lesssim&\|(\phi_0,\psi_0,\xi_0)\|_2^2+\epsilon^{\frac{1}{8}}+\big(\epsilon+N(t)\big)\int_0^t\|(\phi_x,\psi_x,\xi_x)(\tau)\|_1^2 d\tau\\
&+\int_0^t I_4(0,\tau)d\tau+\int_0^t\int_{\mathbb{R}^+}\big|\big(\widetilde{\theta}_x,\widetilde{\theta}_{xx},\widetilde{\theta}_{xx}\big)\big|^2|(\phi,\psi,\xi)|^2 dxd\tau,
\end{aligned}\tag{4.61}
$$

where we used

$$
\int_0^t\|\xi_{xx}(\tau)\|^2 d\tau\lesssim\int_0^t\int_{\mathbb{R}^+}\big(w_{xxx}^2+w_x^2+H_{4x}^2\big)dxd\tau.\tag{4.62}
$$

Step 2. Deal with the boundary integral $\int_0^t I_4(0,\tau)d\tau$.

Due to $u(0,t)=0$, we get

$$
I_4(0,t)=\frac{1}{\rho}(R\rho\xi_{xx}+R\theta\phi_{xx})\psi_{xx}(0,t)+\frac{w_{xx}}{4b\rho\theta^4}\big(4b\theta^3\xi_{xx}-w_{xxx}\big)(0,t).\tag{4.63}
$$

At first, one has from (4.35)

$$\begin{aligned} &\phi_{tx}(0,t)+\rho\psi_{xx}(0,t)=H_1(0,t), &(4.64a)\\ &\rho\psi_{tx}(0,t)+R\rho\xi_{xx}(0,t)+R\theta\phi_{xx}(0,t)=H_2(0,t), &(4.64b)\\ &p\psi_{xx}(0,t)+w_{xx}(0,t)=H_3(0,t)-C_v\rho\xi_{tx}(0,t), &(4.64c)\\ &-w_{xxx}(0,t)+4b\theta^3\xi_{xx}(0,t)=H_4(0,t)-aw_x(0,t). &(4.64d)\end{aligned}$$

It holds by (4.64c) and $\xi_x(0,t)=-\widetilde{\theta}_x(0,t)$ that

$$\begin{aligned}\int_0^t\psi_{xx}^2(0,\tau)d\tau&\lesssim\int_0^t\left(w_{xx}^2+\xi_{tx}^2+H_3^2\right)(0,\tau)d\tau\\ &\lesssim\epsilon^{\frac18}+\left(\epsilon+N(t)\right)\int_0^t\left(\psi_x^2+\xi_x^2\right)(0,\tau)d\tau\\ &\quad+\int_0^t\left(\left|(\widetilde{\theta}_x,\widetilde{\theta}_{xx})(\phi,\xi)\right|^2+\widetilde{q}_{xx}^2\right)(0,\tau)d\tau\\ &\lesssim\left(\epsilon+N(t)\right)\int_0^t\|(\psi_x,\widetilde{\theta}_x)(\tau)\|_\infty^2d\tau+\epsilon^{\frac18}\\ &\lesssim\left(\epsilon+N(t)\right)\int_0^t\|(\psi_x,\psi_{xx})(\tau)\|^2d\tau+\epsilon^{\frac18}. \end{aligned}\tag{4.65}$$

In addition, we see from (4.35d) that

$$-w_{txx}+aw_t+4b\theta^3\xi_{tx}=\widetilde{q}_{txx}-12b\theta^2\theta_t\xi_x-4b\left[\widetilde{\theta}_x\xi\left(\theta^2+\theta\widetilde{\theta}+\widetilde{\theta}^2\right)\right]_t,\tag{4.66}$$

and $\xi_x(0,t)=-\widetilde{\theta}_x(0,t)$ and $w(0,t)=-\widetilde{q}(0,t)$ yield

$$\int_0^t w_{txx}^2(0,\tau)d\tau\lesssim\int_0^t\left(w_t^2+\xi_{tx}^2+\widetilde{q}_{txx}^2+\theta_t^2\xi_x^2+\widetilde{\theta}_{tx}^2\xi^2\right)(0,\tau)d\tau\lesssim\epsilon^{\frac18}.\tag{4.67}$$

Recall that $C_v\rho(\xi_{tx}+u\xi_{xx})+p\psi_{xx}+w_{xx}=H_3$, which implies

$$\begin{aligned}&C_v\rho(\xi_{ttx}+u\xi_{txx})+p\psi_{txx}+w_{txx}=H_{3t}-C_v\rho_t\xi_{tx}-C_v(\rho u)_t\xi_{xx}-p_t\psi_{xx},\\ &[C_v\rho\xi_{ttx}+p\psi_{txx}+w_{txx}](0,t)=(H_{3t}-C_v\rho_t\xi_{tx}-p_t\psi_{xx})(0,t),\end{aligned}\tag{4.68}$$

where

$$\begin{aligned}H_{3t}:=&h_{3tx}-\left[C_v\rho_x(\xi_t+u\xi_x)+C_v\rho u_x\xi_x+p_x\psi_x\right]_t\\ =&\left(-R\rho\xi\widetilde{u}_x-C_v\widetilde{\theta}_x\rho\psi-\widetilde{q}_x\right)_{tx}-\left[C_v\frac{\rho_x}{\rho}(h_3-p\psi_x-w_x)+C_v\rho u_x\xi_x+p_x\psi_x\right]_t\end{aligned}$$

$$
\begin{aligned}
&=\big(-R\rho\xi\widetilde{u}_x-C_v\widetilde{\theta}_x\rho\psi-\widetilde{q}_x\big)_{tx}-\left[\frac{\rho_x}{\rho}(h_3-w_x)+C_v\rho u_x\xi_x+R\theta_x\rho\psi_x\right]_t\\
&=\big(-R\rho\xi\widetilde{u}_x-\widetilde{q}_x\big)_{tx}-C_v\widetilde{\theta}_x\rho\psi_{tx}-C_v\widetilde{\theta}_x\rho_{tx}\psi-C_v\widetilde{\theta}_x(\rho_x\psi_t+\rho_t\psi_x)-C_v\widetilde{\theta}_{txx}\rho\psi\\
&\quad-\left[\frac{\rho_x}{\rho}(h_3-w_x)\right]_t-C_v\rho\xi_x\widetilde{u}_{tx}-C_v\rho\xi_x\psi_{tx}-R\theta_x\rho\psi_{tx}-C_v(\rho\xi_x)_tu_x-R(\theta_x\rho)_t\psi_x\\
&=\big(-R\rho\xi\widetilde{u}_x-\widetilde{q}_x\big)_{tx}-C_v\widetilde{\theta}_x\rho_{tx}\psi-C_v\widetilde{\theta}_x(\rho_x\psi_t+\rho_t\psi_x)-C_v\widetilde{\theta}_{txx}\rho\psi\\
&\quad-\left[\frac{\rho_x}{\rho}(h_3-w_x)\right]_t-C_v\rho\xi_x\widetilde{u}_{tx}-\gamma C_v\rho\theta_x\psi_{tx}-C_v(\rho\xi_x)_tu_x-R(\rho\theta_x)_t\psi_x\\
&=\mathcal{O}(1)\big(|(\phi_{tx},\xi_{tx},w_{tx})|+|(\widetilde{u}_x,\widetilde{\theta}_x)(\phi_x,\psi_x,\xi_x)|+|(\widetilde{u}_{txx},\widetilde{\theta}_{txx})(\phi,\xi)|\\
&\qquad+\widetilde{q}_{txx}+\gamma C_v\rho\theta_x\psi_{tx}\big).
\end{aligned}\tag{4.69}
$$

From (4.68) and (4.69), $\theta_x(0,t)=0$ and $\xi_x(0,t)=-\widetilde{\theta}_x(0,t)$ yield

$$
\begin{aligned}
\int_0^t\psi_{txx}^2(0,\tau)d\tau&\lesssim\int_0^t|(\xi_{ttx},w_{txx},H_{3t},\xi_{tx},\psi_{xx})|^2(0,\tau)d\tau\\
&\lesssim\int_0^t|(w_{txx},\psi_{xx},w_{tx},\phi_{tx})|^2(0,\tau)d\tau+\int_0^t\big|\big(\widetilde{\theta}_{ttx},\widetilde{\theta}_{tx},\widetilde{q}_{txx}\big)\big|^2(0,\tau)d\tau\\
&\quad+\int_0^t\Big(\big|\big(\widetilde{u}_x,\widetilde{\theta}_x\big)(\phi_x,\psi_x,\xi_x)\big|^2+\big|\big(\widetilde{u}_{txx},\widetilde{\theta}_{txx}\big)(\phi,\xi)\big|^2\Big)(0,\tau)d\tau\\
&\lesssim\big(\epsilon+N(t)\big)\int_0^t\|(\phi_x,\psi_x)(\tau)\|_1^2d\tau+\int_0^t w_{tx}^2(0,\tau)d\tau+\epsilon^{\frac{1}{8}}.
\end{aligned}\tag{4.70}
$$

Hence, it holds

$$
\begin{aligned}
\int_0^t\psi_{tx}\psi_{xx}(0,\tau)d\tau&=\int_0^t[\psi_{tx}\psi_{xx}(0,\tau)]_\tau d\tau-\int_0^t\psi_x\psi_{txx}(0,\tau)d\tau\\
&\lesssim\psi_{tx}\psi_{xx}(0,t)-\psi_{tx}\psi_{xx}(0,0)+\int_0^t\big(\psi_{txx}^2+\psi_x^2\big)(0,\tau)d\tau\\
&\lesssim\|(\psi_{tx},\psi_{xx})(t)\|^2+\|(\psi_{tx},\psi_{xx})(0)\|^2+\epsilon^{\frac{1}{8}}\\
&\quad+\big(\epsilon+N(t)\big)\int_0^t\|(\phi_x,\psi_x,\xi_x)(\tau)\|_1^2d\tau+\int_0^t w_{tx}^2(0,\tau)d\tau.
\end{aligned}\tag{4.71}
$$

Therefore, one has

$$
\begin{aligned}
\int_0^t I_4(0,\tau)d\tau\lesssim&\int_0^t\psi_{xx}\psi_{tx}(0,\tau)d\tau-\int_0^t\frac{1}{\rho}|\psi_{xx}H_2|(0,\tau)d\tau\\
&+\int_0^t|w_{xx}(H_4-aw_x)|(0,\tau)d\tau.
\end{aligned}\tag{4.72}
$$

The second and third terms on the right-hand side of (4.72) are estimated as follows:

$$\int_0^t \psi_{xx} H_2(0,\tau)d\tau \lesssim \int_0^t \psi_{xx}^2(0,\tau)d\tau + \big(\epsilon+N(t)\big)\int_0^t \big(\phi_x^2+\psi_x^2\big)(0,\tau)d\tau + \int_0^t \big|\big(\widetilde{\rho}_{xx},\widetilde{u}_{xx},\widetilde{\theta}_{xx}\big)(\phi,\xi)\big|^2(0,\tau)d\tau, \tag{4.73}$$

$$\int_0^t |w_{xx}(H_4-aw_x)|(0,\tau)d\tau \lesssim \int_0^t \big(w_x^2+w_{xx}^2+\xi_x^2+\widetilde{\theta}_x^2\xi^2+\widetilde{q}_{xxx}^2\big)(0,\tau)d\tau. \tag{4.74}$$

Finally, by Lemma 4.4, we get

$$\int_0^t I_4(0,\tau)d\tau \lesssim \|(\phi_0,\psi_0,\xi_0)\|_2^2+\epsilon^{\frac{1}{8}}+\big(\epsilon+N(t)\big)\int_0^t \|(\phi_x,\psi_x)(\tau)\|_1^2 d\tau. \tag{4.75}$$

Substituting (4.75) into (4.61), we get (4.54). This proves the Lemma 4.5. □

Combining the results in Lemmas 4.1-4.5, we get

$$\begin{aligned}&\|(\phi,\psi,\xi)(t)\|_2^2+\int_0^t |(\psi_x,w_x,w_{xx},\psi_{xx},\phi_{tx},w_{tx},w_{txx})|^2(0,\tau)d\tau\\&+\int_0^t \big(\|(\phi_x,\psi_x,\xi_x,\xi_{xx},\xi_{tx})(\tau)\|^2+\|w(\tau)\|_3^2+\|w_t(\tau)\|_1^2\big)d\tau\\&\lesssim \|(\phi_0,\psi_0,\xi_0)\|_2^2+\epsilon^{\frac{1}{8}}+\big(\epsilon+N(t)\big)\int_0^t \|(\phi_{xx},\psi_{xx})(\tau)\|^2 d\tau.\end{aligned} \tag{4.76}$$

Finally, we will deal with $\int_0^t \|(\phi_{xx},\psi_{xx})(\tau)\|^2 d\tau$.

Lemma 4.6. *Under the assumptions of Proposition* 4.1*, if ϵ and $N(t)$ are suitably small, it holds*

$$\int_0^t \|(\phi_{xx},\psi_{xx})(\tau)\|^2 d\tau \lesssim \|(\phi_0,\psi_0,\xi_0)\|_2^2+\epsilon^{\frac{1}{8}}. \tag{4.77}$$

Proof. Multiplying (4.6b) by $R\theta\phi_{xx}/(2\rho)$ and (4.6c) by ψ_{xx}, we can get

$$\begin{aligned}&\left(\frac{R\theta}{2}\phi_{xx}\psi_x\right)_t-\left(\frac{R\theta}{2}\phi_{tx}\psi_x\right)_x+\frac{p}{2}\psi_{xx}^2+\frac{R^2\theta^2}{2\rho}\phi_{xx}^2\\&=\frac{R\theta}{2\rho}\phi_{xx}H_2-\frac{R^2\theta}{2}\phi_{xx}\xi_{xx}+\left(\frac{R\theta}{2}\right)_t\phi_{xx}\psi_x-\left(\frac{R\theta}{2}\right)_x\phi_{tx}\psi_x\\&\quad-\frac{R\theta}{2}\phi_{xx}H_1+H_3\psi_{xx}-C_v\rho(\xi_{tx}+u\xi_{xx})\psi_{xx}-w_{xx}\psi_{xx}.\end{aligned} \tag{4.78}$$

Integrating (4.78) over $\mathbb{R}^+\times[0,t]$, choosing ϵ and $N(t)$ suitable small, we have

$$\begin{aligned}\int_0^t \|(\phi_{xx},\psi_{xx})(\tau)\|^2 d\tau \lesssim & \|(\phi_{xx},\psi_x)(t)\|^2+\|(\phi_{0xx},\psi_{0x})\|^2+\int_0^t |\phi_{tx}\psi_x|(0,\tau)d\tau \\ &+\int_0^t \|(\xi_{tx},\xi_{xx},w_{xx},\phi_x,\psi_x,\xi_x)(\tau)\|^2 d\tau \\ &+\int_0^t\int_{\mathbb{R}^+} \left|\left(\widetilde{\theta}_x,\widetilde{\theta}_{xx},\widetilde{\theta}_{tx}\right)\right|^2 |(\phi,\psi,\xi)|^2 dxd\tau, \end{aligned} \tag{4.79}$$

$$\begin{aligned}\int_0^t |\phi_{tx}\psi_x|(0,\tau)d\tau \lesssim & \int_0^t \phi_{tx}^2(0,\tau)d\tau+\int_0^t \psi_x^2(0,\tau)d\tau \\ \lesssim & \int_0^t \phi_{tx}^2(0,\tau)d\tau+\int_0^t \|\psi_x(\tau)\|_\infty^2 d\tau \\ \lesssim & \int_0^t \phi_{tx}^2(0,\tau)d\tau+\frac{1}{8}\int_0^t \|\psi_{xx}(\tau)\|^2 d\tau+\int_0^t \|\psi_x(\tau)\|^2 d\tau. \end{aligned} \tag{4.80}$$

Using (4.76), we obtain (4.77). This completes the proof. □

At last, (4.6d) yields

$$\xi_{tx}=\mathcal{O}(1)|(H_3,\xi_{xx},\psi_{xx},w_{xx})|, \tag{4.81}$$

$$w_t=\mathcal{O}(1)|(\widetilde{h}_4,\xi_{tx},w_{txx})|. \tag{4.82}$$

Combining the results in Lemmas 4.1-4.6, we get

$$\begin{aligned}&\|(\phi,\psi,\xi)(t)\|_2^2+\int_0^t |(\phi_x,\psi_x,w_x,w_{xx},\psi_{xx},\phi_{tx},w_{tx},w_{txx})|^2(0,\tau)d\tau \\ &\quad+\int_0^t \left(\|(\phi_x,\psi_x,\xi_x)(\tau)\|_1^2+\|\xi_{tx}(\tau)\|^2+\|w(\tau)\|_3^2+\|w_t(\tau)\|_2^2\right)d\tau \\ &\lesssim \|(\phi_0,\psi_0,\xi_0)\|_2^2+\epsilon^{\frac{1}{8}}. \end{aligned} \tag{4.83}$$

5 Energy estimates on radiative perturbation w

In this section, we will establish the estimates on the radiative perturbation $\|w(t)\|_3$ and $\|w_t(t)\|_2$ as follows.

Lemma 5.1. *Under the same assumptions listed in Proposition* 4.1, *if* $\epsilon, N(t)$ *are suitably small, it holds*

$$\sup_{\tau\in[0,t]}\left\{\|w(t)\|_3^2+\|w_t(t)\|_2^2\right\}\lesssim \|(\phi_0,\psi_0,\xi_0)\|_2^2+\epsilon^{\frac{1}{8}}. \tag{5.1}$$

Proof. The proof is divided into three steps as follows.

Step 1. Multiplying (2.12d) by w, we get

$$-(w_xw)_x+aw^2+w_x^2+4b\theta^3\xi_xw+4b\widetilde{\theta}_xw\xi\left(\theta^2+\theta\widetilde{\theta}+\widetilde{\theta}^2\right)=\widetilde{q}_{xx}w. \tag{5.2}$$

Integrating (5.2) over $\mathbb{R}^+$, choosing $\epsilon,N(t)$ suitable small, we have

$$\int_{\mathbb{R}^+}\left(aw^2+w_x^2\right)(x,t)dx\lesssim\int_{\mathbb{R}^+}\left(\xi_x^2+\widetilde{\theta}_x^2\xi^2+\widetilde{q}_{xx}^2\right)(x,t)dx+|w_xw|(0,t), \tag{5.3}$$

and

$$\begin{aligned}|w_xw|(0,t)&\lesssim\left|\widetilde{\theta}_xw_x\right|(0,t)\lesssim\left\|\widetilde{\theta}_x(t)\right\|_\infty\|w_x(t)\|_\infty\\&\lesssim\epsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}\|w_x(t)\|^{\frac{1}{2}}\|w_{xx}(t)\|^{\frac{1}{2}}\\&\lesssim\epsilon^{\frac{1}{8}}\left(\|w_{xx}(t)\|^2+\|w_x(t)\|^2\right)+\epsilon^{\frac{1}{8}}.\end{aligned}$$

Thus, we obtain

$$\|w(t)\|_1^2\lesssim\epsilon^{\frac{1}{8}}\|w_{xx}(t)\|^2+\|\xi(t)\|_1^2+\epsilon^{\frac{1}{8}}. \tag{5.4}$$

Step 2. Multiplying (4.6d) by $-w_{xxx}$, we get

$$w_{xxx}^2+aw_{xx}^2-(aw_xw_{xx})_x-4b\theta^3\xi_{xx}w_{xxx}=H_{4x}w_{xx}. \tag{5.5}$$

Integrating (5.5) over $\mathbb{R}^+$, choosing $\epsilon,N(t)$ suitable small, we obtain

$$\int_{\mathbb{R}^+}\left(aw_{xx}^2+w_{xxx}^2\right)(x,t)dx\lesssim\int_{\mathbb{R}^+}\left(\xi_{xx}^2+H_4^2\right)(x,t)dx+|w_xw_{xx}|(0,t). \tag{5.6}$$

Furthermore, by Hölder inequality, Agmon inequality and Young inequality, one has

$$\begin{aligned}|w_xw_{xx}|(0,t)&\lesssim\|w_x(\tau)\|_\infty\|w_{xx}(\tau)\|_\infty\\&\lesssim\|w_x(\tau)\|^{\frac{1}{2}}\|w_{xx}(\tau)\|\|w_{xxx}(\tau)\|^{\frac{1}{2}}\\&\lesssim\frac{1}{8}\|w_{xxx}(\tau)\|^2+8\|w_x(\tau)\|^{\frac{2}{3}}\|w_{xx}(\tau)\|^{\frac{4}{3}}\\&\lesssim\frac{1}{8}\|w_{xxx}(\tau)\|^2+\frac{1}{8}\|w_{xx}(\tau)\|^2+\|w_x(\tau)\|^2.\end{aligned} \tag{5.7}$$

Therefore, combining (5.4), (5.6) with (4.83), we obtain

$$\|w(t)\|_3^2\lesssim\|(\phi_0,\psi_0,\xi_0)\|_2^2+\epsilon^{\frac{1}{8}}. \tag{5.8}$$

Step 3. Multiplying (4.35d) by w_t and $-w_{txx}$ respectively, we get

$$-(w_t w_{tx})_x + w_{tx}^2 + a w_t^2 + 4b\theta^3 w_t \xi_{tx} = \widetilde{h}_4 w_t, \tag{5.9}$$

$$w_{txx}^2 + a w_{tx}^2 - (a w_t w_{tx})_x - 4b\theta^3 w_{txx}\xi_{tx} = -w_{txx}\widetilde{h}_4. \tag{5.10}$$

Integrating (5.9) and (5.10) over $\mathbb{R}^+$ respectively, choosing ϵ and $N(t)$ suitable small, one has by (4.83)

$$\|w_t(t)\|_2^2 \lesssim \int_{\mathbb{R}^+} (\xi_{tx}^2 + \widetilde{h}_4^2)\,dx + |w_t w_{tx}|(0,t). \tag{5.11}$$

Similar to (5.7), it holds

$$\begin{aligned} |w_t w_{tx}(0,t)| &\lesssim |w_t(0,t)|\,\|w_{tx}(t)\|_\infty \\ &\lesssim \epsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{8}}\|w_{tx}(t)\|^{\frac{1}{2}}\|w_{txx}(t)\|^{\frac{1}{2}} \\ &\lesssim \epsilon^{\frac{1}{8}}\|w_{txx}(t)\|^2 + \epsilon^{\frac{1}{8}}(1+t)^{-\frac{7}{6}}\|w_{tx}(t)\|^{\frac{4}{3}} \\ &\lesssim \epsilon^{\frac{1}{8}}\|w_{txx}(t)\|^2 + \epsilon^{\frac{1}{8}}\|w_{tx}(t)\|^2 + \epsilon^{\frac{1}{8}}. \end{aligned} \tag{5.12}$$

Thus, we get

$$\|w_t(t)\|_2^2 \lesssim \|(\phi_0,\psi_0,\xi_0)\|_2^2 + \epsilon^{\frac{1}{8}}. \tag{5.13}$$

Combining (5.6) with (5.13), one has (5.1). This completes the proof of Lemma 5.1. □

Combining (5.1) and (4.83), we can get (4.2). This completes the proof of Proposition 4.1.

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