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Gevrey Well-Posedness of Quasi-Linear Hyperbolic Prandtl Equations

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Abstract. We study the hyperbolic version of the Prandtl system derived from the hyperbolic Navier-Stokes system with no-slip boundary condition. Compared to the classical Prandtl system, the quasi-linear terms in the hyperbolic Prandtl equation leads to an additional instability mechanism. To overcome the loss of derivatives in all directions in the quasi-linear term, we introduce a new auxiliary function for the well-posedness of the system in an anisotropic Gevrey space which is Gevrey class 3/2 in the tangential variable and is analytic in the normal variable.

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1 Introduction

We investigate the well-podedness of the following quasi-linear hyperbolic Prandtl system in the half-space $\mathbb{R}^d_+ \overset{\text{def}}{=} \{(x,y);\ x\in\mathbb{R}^{d-1}, y>0\}$ with $d=2$ or 3:

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$$\begin{cases} \eta\partial_t^2 u+\partial_t u+(u\cdot\partial_x)u+v\partial_y u+\eta\partial_t\big((u\cdot\partial_x)u+v\partial_y u\big)-\partial_y^2 u+\partial_x p=0,\\ \partial_x u+\partial_y v=0,\\ u|_{y=0}=v|_{y=0}=0, \quad u|_{y\to+\infty}=U,\\ u|_{t=0}=u_0, \qquad\quad \partial_t u|_{t=0}=u_1, \end{cases} \tag{1.1}$$

where $0<\eta<1$ is a small parameter. The unknown u represents the tangential velocity which is scalar in the two-dimensional (2D) case and vector-valued in 3D. And the functions $p=p(t,x)$ and $U=U(t,x)$ in (1.1) are the traces of the tangential velocity field and pressure of the outer flow on the boundary satisfying that

$$\eta\partial_t^2 U+\partial_t U+U\cdot\partial_x U+\eta\partial_t(U\cdot\partial_x U)+\partial_x p=0.$$

This degenerate hyperbolic system (1.1) can be derived from the hyperbolic Navier-Stokes equations with the no-slip boundary condition. It is well-known that the classical Navier-Stokes system can be obtained from the Newtonian law. And its parabolic structure leads to the property of infinite speed of propagation which seems to be a paradox from the physical point of view. To have finite propagation, Cattaneo [4,5] proposed to replace the Fourier law by the so-called Cattaneo law, where a small time delay η is introduced in stress tensors. And this yields the following hyperbolic version of Navier-Stokes equations:

$$\eta\partial_t^2 u^{NS}+\partial_t u^{NS}+(u^{NS}\cdot\nabla)u^{NS}+\eta\partial_t\big((u^{NS}\cdot\nabla)u^{NS}\big)-\varepsilon\Delta u^{NS}+\nabla p^{NS}=0, \tag{1.2}$$

where the gradient operator ∇ is taken with respect to all spatial variables, and similarly for the Laplace operator Δ. In the whole space, the system (1.2) with fixed viscosity $\varepsilon>0$ was studied by Coulaud *et al.* [7] in almost optimal function spaces (see also [2,30]). On the other hand, it is natural to study the inviscid limit of (1.2) as $\varepsilon\to 0$, in particular in the situation when the fluid domain has a physical boundary. In fact, when we analyze the asymptotic expansion with respect to the viscosity ε of (1.2) with the no-slip boundary condition, a Prandtl type boundary layer is expected to take care of the mismatched tangential velocities. In fact, the governing equation of the boundary layer is the system (1.1) by following the Prandtl's ansatz.

When $\eta=0$, the system (1.1) is the classical Prandtl equations. The mathematical study of the classical Prandtl boundary layer has a long history with fruitful results and developed approaches in analysis. It has been well studied in various function spaces, see, e.g. [3,6,8–10,12,14–16,18,20,23,25–28,32–35] and the references therein. Due to the loss of tangential derivatives in the nonlocal term $v\partial_y u$, the Prandtl system is usually ill-posed in Sobolev spaces. It is now well understood that, for initial data without any structural assumption, the Prandtl system is well-posed in Gevrey class with optimal Gevrey index 2 by the instability analysis of Gérard-Varet and Dormy [10] and the work on well-posedness of Dietert-Gérard-Varet [8] and Li *et al.* [17]. The key observation in [8, 17] is about some kind of intrinsic structure that is similar to hyperbolic feature for one order loss of tangential derivatives. Recently, inspired by the stabilizing effect of the intrinsic hyperbolic type structure, Li *et al.* [22] showed the global well-posedness of a Prandtl Model from MHD in the Gevrey 2 setting.

The hyperbolic Prandtl system (1.1) is more complicated in terms of loss of derivatives due to the quasi-linear term $\eta\partial_t(u\cdot\partial_x u+v\partial_y u)$. In fact, as to be seen below, the loss of derivatives occurs not only for the tangential variable but also for the normal variable. Inspired by the abstract Cauchy-Kowalewski theory, one can expect the well-posedness of (1.1) in the full analytic spaces (i.e. space of functions that are analytic in all variables). However, it is hard to relax the analyticity to Gevrey class because the nonlinearity and non-locality in the term $\partial_t((u\cdot\partial_x)u+v\partial_y u)$ that prevent us to apply the techniques developed for the classical Prandtl equation directly. An attempt is to consider the following semilinear model:

$$\eta\partial_t^2 u+\partial_t u+(u\cdot\partial_x)u+v\partial_y u-\partial_y^2 u+\partial_x p=0 \tag{1.3}$$

by removing the quasi-linear term $\eta\partial_t((u\cdot\partial_x)u+v\partial_y u)$ in (1.1). For this, the local well-posedness of (1.3) in Gevrey 2 space is obtained by [21] which is the same Gevrey space for the classical Prandtl equation. However, the Gevrey index 2 may not be optimal for the well-posedness of (1.3). Hence, it is interesting to find out whether there is a larger Gevrey index for well-posedness by exploring the stabilizing effect of the hyperbolic perturbation $\eta\partial_t^2$ therein.

Similar problems occur when investigating the following hyperbolic hydrostatic Navier-Stokes equations:

$$\begin{cases} \eta\partial_t^2\tilde{u}+\partial_t\tilde{u}+(\tilde{u}\cdot\partial_x)\tilde{u}+\tilde{v}\partial_y\tilde{u} & \\ \quad +\eta\partial_t\big((\tilde{u}\cdot\partial_x)\tilde{u}+\tilde{v}\partial_y\tilde{u}\big)-\partial_y^2\tilde{u}+\partial_x\tilde{p}=0, & (x,y)\in\mathbb{R}\times\,]0,1[, \\ \partial_y\tilde{p}=0, & (x,y)\in\mathbb{R}\times\,]0,1[, \\ \partial_x\tilde{u}+\partial_y\tilde{v}=0, & (x,y)\in\mathbb{R}\times\,]0,1[, \\ \tilde{u}|_{y=0,1}=\tilde{v}|_{y=0,1}=0, & x\in\mathbb{R}, \\ \tilde{u}|_{t=0}=\tilde{u}_0, \quad \partial_t\tilde{u}|_{t=0}=\tilde{u}_1, & (x,y)\in\mathbb{R}\times\,]0,1[. \end{cases} \tag{1.4}$$

The hydrostatic Navier-Stokes system (1.4) has a similar degeneracy feature as the Prandtl equation. This system that can be used to describe the large scale motion of geophysical flow plays an important role in the atmospheric and oceanic sciences. It is a limit of the hyperbolic Navier-Stokes equations (1.2) in a thin domain where the vertical scale is significantly smaller than the horizontal one.

Compared to the classical Prandtl equation, much less is known for the hydrostatic Navier-Stokes equations (1.4). In fact, the well-posedness of the hydrostatic Navier-Stokes equations in Sobolev space is still unclear. Under the convex assumption, the Gevrey well-posedness has been established by Gérard-Varet *et al.* [13], and later improved by Wang-Wang [31] and Gérard-Varet *et al.* [11] with the optimal Gevrey index 3/2. The aforementioned works mainly focus on the hydrostatic equations of the parabolic type. To the best of our knowledge, there is no mathematical theory on the well-posedness of the hyperbolic hydrostatic Navier-Stokes equations. Here, we just mention some recent works attempting to explore the hyperbolic feature of some simplified semi-linear models (cf. [1, 24, 29]) in order to investigate the wave type property that lead to some kind of stability effect compared to the parabolic counterparts. In addition, a quasi-linear model was recently studied by [19]. However, the well-posedness property for the full quasi-linear system (1.4) remains as a challenging problem.

This paper aims to investigate the hyperbolic Prandtl system (1.1) in an anisotropic Gevrey space (see Definition 1.1 below). To simplify the argument, we assume without loss of generality that $\eta=1$ and $\partial_x p=U\equiv 0$. Then we consider

$$\begin{cases} \partial_t^2 u+\partial_t u+(u\cdot\partial_x)u+v\partial_y u+\partial_t\big((u\cdot\partial_x)u+v\partial_y u\big)-\partial_y^2 u=0, & (x,y)\in\mathbb{R}_+^d, \\ \partial_x u+\partial_y v=0, \\ u|_{y=0}=v|_{y=0}=0, \quad u|_{y\to+\infty}=0, \\ u|_{t=0}=u_0, \qquad \partial_t u|_{t=0}=u_1. \end{cases} \tag{1.5}$$

Notations. In the half-space $\mathbb{R}_+^d$ with $d=2$ or 3, we will use $\|\cdot\|_{L^2}$ and $(\cdot,\cdot)_{L^2}$ to denote the norm and inner product of $L^2=L^2(\mathbb{R}_+^d)$ and use the notation $\|\cdot\|_{L_x^2}$ and $(\cdot,\cdot)_{L_x^2}$ when the variable x is specified. Similar notations will be used for L^∞. And $L_x^pL_y^q=L^p(\mathbb{R}^{d-1};L^q(\mathbb{R}_+))$.

Definition 1.1. *The anisotropic Gevrey space $G_{\rho,\ell}^{3/2,1}$ consists of all smooth functions $h(x,y)$ that are analytic in y and of Gevrey class $3/2$ in x satisfying*

$$\|h\|_{G_{\rho,\ell}^{3/2,1}}<+\infty$$

with

$$\begin{aligned}\|h\|_{G_{\rho,\ell}^{3/2,1}}^2 \overset{def}{=} &\sum_{m=0}^{+\infty}\Big(N_{\rho,m}\big\|\langle y\rangle^{\ell-1}\partial_x^m h\big\|_{L^2}\Big)^2 \\ &+\sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty}\Big((m+1)H_{\rho,m+1,k}\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\partial_y h\big\|_{L^2}\Big)^2,\end{aligned}$$

where $\langle y\rangle \overset{def}{=} (1+|y|^2)^{1/2}$, *the number* $\ell \geq 2$ *is given, and*

$$H_{\rho,m,k} = \frac{\rho^{m+k+1}(m+k+1)^9}{(m+k)!(m!)^{\frac{1}{2}}}, \quad N_{\rho,m} = H_{\rho,m,0} = \frac{\rho^{m+1}(m+1)^9}{(m!)^{\frac{3}{2}}}. \tag{1.6}$$

The main result of the paper is stated as follows.

Theorem 1.1. *If the initial data of the hyperbolic Prandtl system* (1.5) *satisfy* $u_0 \in G^{3/2,1}_{2\rho_0,\ell}$, $u_1 \in G^{3/2,1}_{2\rho_0,\ell+1}$ *for some* $\rho_0 > 0$ *and are compatible to the boundary conditions in* (1.5). *Then problem* (1.5) *admits a unique local solution* $u \in L^\infty([0,T];\ G^{3/2,1}_{\rho,\ell})$ *for some* $T>0$ *and* $0<\rho\leq\rho_0$.

The key part in the proof of Theorem 1.1 is to derive the a priori estimate for (1.5) so that the existence and uniqueness follow from a standard argument. Hence, for brevity, we only present the proof of the a priori estimate.

The paper is organized as follows. Sections 2-5 are for proving the a priori estimate in 2D. The proof in 3D we present in Section 6.

2 An a priori estimate in 2D

In this section, we state an a priori estimate for the hyperbolic Prandtl system (1.5) when $d=2$ and its proof we give in Sections 3-5.

In the following argument we assume the initial data in (1.5) satisfy that $u_0 \in G^{3/2,1}_{2\rho_0,\ell}$ and $u_1 \in G^{3/2,1}_{2\rho_0,\ell+1}$ for some $\rho_0>0$, and suppose that $u \in L^\infty([0,T];G^{3/2,1}_{\rho,\ell})$ solves (1.5), where

$$\rho = \rho(t) \overset{\text{def}}{=} \rho_0 e^{-\mu t}, \quad 0 \leq t \leq T \tag{2.1}$$

with $\mu > 1$ begin a given large constant to be determined later. By using the notation that

$$\varphi = \partial_t u + u\partial_x u + v\partial_y u \quad \text{with} \quad v(t,x,y) = -\int_0^y \partial_x u(t,x,\tilde{y})d\tilde{y}, \tag{2.2}$$

we can reformulate system (1.5) in 2D as

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = \varphi, & (2.3a) \\ \partial_t \varphi + \varphi - \partial_y^2 u = 0, & (2.3b) \\ u|_{y=0} = \varphi|_{y=0} = 0, & (2.3c) \\ u|_{t=0} = u_0, \quad \varphi|_{t=0} = \varphi_0, & (2.3d) \end{cases}$$

where

$$\varphi_0 = u_1 + u_0\partial_x u_0 - (\partial_y u_0)\int_0^y \partial_x u_0(x,\tilde{y})d\tilde{y}. \tag{2.4}$$

As for the classical Prandtl equation, the loss of one order tangential derivatives occurs in the Eq. (2.3a). To overcome this difficulty, inspired by [17] we introduce two auxiliary functions $\mathcal{U}$ and λ. Precisely, let $\mathcal{U}$ be a solution to the Cauchy problem

$$\begin{cases} (\partial_t + u\partial_x + v\partial_y)\int_0^y \mathcal{U}d\tilde{y} = -\partial_x v, \\ \mathcal{U}|_{t=0} = 0. \end{cases} \tag{2.5}$$

The existence of $\mathcal{U}$ follows from the standard theory of transport equations. In fact, one can first apply the existence theory for linear transport equations to construct a solution f to the Cauchy problem

$$\begin{cases} (\partial_t + u\partial_x + v\partial_y)f = -\partial_x v, \\ f|_{y=0} = 0, \\ f|_{t=0} = 0, \end{cases}$$

and then set

$$f = \int_0^y \mathcal{U}(t,x,\tilde{y})d\tilde{y}.$$

By virtue of $\mathcal{U}$ and

$$\lambda \stackrel{\text{def}}{=} \partial_x u - (\partial_y u)\int_0^y \mathcal{U}(t,x,\tilde{y})d\tilde{y}, \tag{2.6}$$

we can cancel the term involving v with the highest tangential derivative as shown in the following equation (2.8). The two auxiliary functions have the relation

$$(\partial_t + u\partial_x + v\partial_y)\mathcal{U} = \partial_x\lambda + (\partial_x\partial_y u)\int_0^y \mathcal{U}(t,x,\tilde{y})d\tilde{y} + (\partial_x u)\mathcal{U}. \tag{2.7}$$

In addition, we apply ∂_x to the Eq. (2.3a) and multiply (2.5) by $\partial_y u$. Then the subtraction of these two equations yields the following equation for λ:

$$(\partial_t + u\partial_x + v\partial_y)\lambda = \partial_x\varphi - (\partial_x u)\partial_x u - (\partial_y\varphi)\int_0^y \mathcal{U}d\tilde{y}. \tag{2.8}$$

Definition 2.1. *Let $\ell \geq 2$ be the number given in Definition* 1.1, *and let $\varphi, \mathcal{U}, \lambda$ be given in* (2.2), (2.5) *and* (2.6), *respectively. By denoting*

$$\vec{a} = (u, \mathcal{U}, \lambda, \varphi),$$

we define $|\vec{a}|_{X_\rho}$ *and* $|\vec{a}|_{Y_\rho}$ *by*

$$\begin{aligned}|\vec{a}|^2_{X_\rho}=&\sum_{m=0}^{\infty}N^2_{\rho,m+1}\big\|\partial_x^m\mathcal{U}\big\|^2_{L^2}+\sum_{m=0}^{\infty}(m+1)N^2_{\rho,m+1}\big\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\big\|^2_{L^2}\\&+\sum_{m,k\geq 0}(m+1)^2H^2_{\rho,m+1,k}\Big(\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\varphi\big\|^2_{L^2}+\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\partial_y u\big\|^2_{L^2}\Big),\end{aligned} \tag{2.9}$$

$$\begin{aligned}|\vec{a}|^2_{Y_\rho}=&\sum_{m=0}^{\infty}(m+1)N^2_{\rho,m+1}\big\|\partial_x^m\mathcal{U}\big\|^2_{L^2}+\sum_{m=0}^{\infty}(m+1)^2N^2_{\rho,m+1}\big\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\big\|^2_{L^2}\\&+\sum_{m,k\geq 0}(m+k+1)(m+1)^2H^2_{\rho,m+1,k}\Big(\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\varphi\big\|^2_{L^2}+\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\partial_y u\big\|^2_{L^2}\Big).\end{aligned} \tag{2.10}$$

Recall that $N_{\rho,m}$ *and* $H_{\rho,m,k}$ *are given in* (1.6) *with* ρ *defined in* (2.1).

Remark 2.1. From (2.9) and (2.10), it follows directly that

$$|\vec{a}|_{X_\rho}\leq|\vec{a}|_{Y_\rho}. \tag{2.11}$$

Moreover, as shown in Lemma 3.1 below,

$$\|u\|_{G^{3/2,1}_{\rho,\ell}}\leq C|\vec{a}|_{X_\rho},$$

where C is a constant depending only on ρ_0,ℓ and the Sobolev embedding constants. Denote $\vec{a}(0)=\vec{a}|_{t=0}$. Then there exists a constant $C_0>0$ such that

$$|\vec{a}(0)|_{X_{\rho_0}}\leq C_0\left(\|u_0\|_{G^{3/2,1}_{2\rho_0,\ell}}+\|u_1\|_{G^{3/2,1}_{2\rho_0,\ell+1}}\right). \tag{2.12}$$

For clear presentation, the proof of (2.12) we give in Appendix A.

We can now state an a priori estimate.

Theorem 2.1 (A Priori Estimate)**.** *Under the assumption of Theorem* 1.1*, there exists a constant* $\mu\geq1$ *depending only on* ℓ,ρ_0*, the Sobolev embedding constants and the initial data such that if*

$$\sup_{0\leq t\leq T}|\vec{a}(t)|_{X_\rho}+\left(\int_0^T|\vec{a}(t)|^2_{Y_\rho}dt\right)^{\frac12}\leq 2C_0\Big(\|u_0\|_{G^{3/2,1}_{2\rho_0,\ell}}+\|u_1\|_{G^{3/2,1}_{2\rho_0,\ell+1}}\Big) \tag{2.13}$$

with ρ *defined by* (2.1) *and* $T=\mu^{-1}$*, then*

$$\sup_{0\leq t\leq T}|\vec{a}(t)|_{X_\rho}+\left(\int_0^T|\vec{a}(t)|^2_{Y_\rho}dt\right)^{\frac12}\leq C_0\Big(\|u_0\|_{G^{3/2,1}_{2\rho_0,\ell}}+\|u_1\|_{G^{3/2,1}_{2\rho_0,\ell+1}}\Big). \tag{2.14}$$

The proof of Theorem 2.1 we give in Sections 3-5. We first list some facts for later use. In view of (1.6) and (2.1), we have

$$\frac{d}{dt}N_{\rho,m} = -\mu(m+1)N_{\rho,m}, \quad \frac{d}{dt}H_{\rho,m,k} = -\mu(m+k+1)H_{\rho,m,k}, \quad \forall m,k \geq 0, \tag{2.15}$$

and

$$e^{-1}\rho_0 \leq \rho(t) \leq \rho_0, \quad \forall 0 \leq t \leq T = \mu^{-1}. \tag{2.16}$$

Similarly, if we denote

$$L_{\rho,k} \stackrel{\text{def}}{=} H_{\rho,1,k} = \frac{\rho^{k+2}(k+2)^9}{(k+1)!}, \tag{2.17}$$

then

$$\frac{d}{dt}L_{\rho,k} = -\mu(k+2)L_{\rho,k}, \quad \forall k \geq 0. \tag{2.18}$$

We will use the following Young's inequality for discrete convolution:

$$\left[\sum_{m=0}^{\infty}\left(\sum_{j=0}^{m} p_j q_{m-j}\right)^2\right]^{\frac{1}{2}} \leq \left(\sum_{m=0}^{\infty} q_m^2\right)^{\frac{1}{2}} \sum_{j=0}^{\infty} p_j, \tag{2.19}$$

where $\{p_j\}_{j\geq 0}$ and $\{q_j\}_{j\geq 0}$ are positive sequences.

3 Tangential derivatives of u

To simplify the notations, we will use C in Sections 3-5 to denote a generic constant which may vary from line to line and depend only on ℓ,ρ_0 and the Sobolev embedding constants, but is independent of μ in (2.1) and the order of derivatives.

Compared with Definition 1.1, the tangential derivatives of u are not specified in the definitions (2.9) and (2.10) of $|\vec{a}|_{X_\rho}$ and $|\vec{a}|_{Y_\rho}$. As a preliminary step to prove Theorem 2.1, we first use (2.6) to control the tangential derivatives of u in terms of $|\vec{a}|_{X_\rho}$ and $|\vec{a}|_{Y_\rho}$.

Lemma 3.1. *Under the assumptions of Theorem* 2.1, *we have*

$$\sum_{m=0}^{\infty} N_{\rho,m}^2 \left\| \langle y\rangle^{\ell-1}\partial_x^m u\right\|_{L^2}^2 \leq C\big(1+|\vec{a}|_{X_\rho}^2\big)\,|\vec{a}|_{X_\rho}^2,$$

$$\sum_{m=0}^{\infty} (m+1)N_{\rho,m}^2 \left\| \langle y\rangle^{\ell-1}\partial_x^m u\right\|_{L^2}^2 \leq C\big(1+|\vec{a}|_{X_\rho}^2\big)\,|\vec{a}|_{Y_\rho}^2.$$

Recall that $N_{\rho,m}$ *is defined by* (1.6).

Proof. We first prove the first statement. In view of (2.6), we have

$$\partial_x^{m+1}u=\partial_x^m\lambda+\sum_{j=0}^m\binom{m}{j}\left(\partial_x^j\partial_y u\right)\int_0^y\partial_x^{m-j}\mathcal{U}(t,x,\tilde{y})d\tilde{y}.$$

Thus, we first multiply both sides of the above equation by $\langle y\rangle^{\ell-1}$ and then take the L^2-product with $\langle y\rangle^{\ell-1}\partial_x^{m+1}u$. Then this with the fact that

$$\left|\int_0^y\partial_x^{m-j}\mathcal{U}(t,x,\tilde{y})d\tilde{y}\right|\leq\langle y\rangle^{\frac{1}{2}}\left\|\partial_x^{m-j}\mathcal{U}(x,\cdot)\right\|_{L_y^2},\quad\forall(x,y)\in\mathbb{R}_+^2$$

implies

$$\sum_{m=0}^{\infty}N_{\rho,m+1}^2\left\|\langle y\rangle^{\ell-1}\partial_x^{m+1}u\right\|_{L^2}^2\leq I_1+I_2, \tag{3.1}$$

where

$$I_1=\sum_{m=0}^{+\infty}N_{\rho,m+1}^2\left\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\right\|_{L^2}^2\leq|\vec{a}|_{X_\rho}^2 \tag{3.2}$$

due to (2.9), and

$$\begin{aligned}I_2=&2\sum_{m=0}^{+\infty}\left[\sum_{j=0}^{[m/2]}\binom{m}{j}N_{\rho,m+1}\left\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\right\|_{L_x^\infty L_y^2}\left\|\partial_x^{m-j}\mathcal{U}\right\|_{L^2}\right]^2\\&+2\sum_{m=0}^{+\infty}\left[\sum_{j=[m/2]+1}^{m}\binom{m}{j}N_{\rho,m+1}\left\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\right\|_{L^2}\left\|\partial_x^{m-j}\mathcal{U}\right\|_{L_x^\infty L_y^2}\right]^2.\end{aligned} \tag{3.3}$$

As usual $[m/2]$ represents the largest integer less than or equal to $m/2$. To estimate I_2, we first write

$$\begin{aligned}&\sum_{j=0}^{[m/2]}\binom{m}{j}N_{\rho,m+1}\left\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\right\|_{L_x^\infty L_y^2}\left\|\partial_x^{m-j}\mathcal{U}\right\|_{L^2}\\=&\sum_{j=0}^{[m/2]}\frac{m!}{j!(m-j)!}\frac{N_{\rho,m+1}}{N_{\rho,j+3}N_{\rho,m-j+1}}\left(N_{\rho,j+3}\left\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\right\|_{L_x^\infty L_y^2}\right)\left(N_{\rho,m-j+1}\left\|\partial_x^{m-j}\mathcal{U}\right\|_{L^2}\right).\end{aligned}$$

By using the estimate (see Appendix A)

$$\frac{m!}{j!(m-j)!}\frac{N_{\rho,m+1}}{N_{\rho,j+3}N_{\rho,m-j+1}}\leq\frac{C}{j+1},\quad j\leq[m/2] \tag{3.4}$$

and the Young's inequality (2.19), we obtain

$$\begin{aligned}&\sum_{j=0}^{m}\left[\sum_{j=0}^{[m/2]}\binom{m}{j}N_{\rho,m+1}\big\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\big\|_{L_x^\infty L_y^2}\big\|\partial_x^{m-j}\mathcal{U}\big\|_{L^2}\right]^2\\&\leq C\sum_{j=0}^{m}\left[\sum_{j=0}^{[m/2]}\frac{N_{\rho,j+3}\big\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\big\|_{L_x^\infty L_y^2}}{j+1}\Big(N_{\rho,m-j+1}\big\|\partial_x^{m-j}\mathcal{U}\big\|_{L^2}\Big)\right]^2\\&\leq C\left(\sum_{j=0}^{+\infty}\frac{N_{\rho,j+3}\big\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\big\|_{L_x^\infty L_y^2}}{j+1}\right)^2\sum_{j=0}^{+\infty}N_{\rho,j+1}^2\big\|\partial_x^j\mathcal{U}\big\|_{L^2}^2\\&\leq C\left(\sum_{j=1}^{+\infty}j^{-2}\right)\left(\sum_{j=0}^{+\infty}N_{\rho,j+3}^2\big\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\big\|_{L_x^\infty L_y^2}^2\right)|\vec{a}|_{X_\rho}^2\leq C|\vec{a}|_{X_\rho}^4,\end{aligned}\tag{3.5}$$

where in the last inequality we have used the Sobolev embedding and (2.9).

Similarly, by using the estimate (see Appendix A)

$$\frac{m!}{j!(m-j)!}\frac{N_{\rho,m+1}}{N_{\rho,j+1}N_{\rho,m-j+3}}\leq\frac{C}{m-j+1},\quad [m/2]+1\leq j\leq m,\tag{3.6}$$

we have

$$\begin{aligned}&\sum_{m=0}^{+\infty}\left[\sum_{j=[m/2]+1}^{m}\binom{m}{j}N_{\rho,m+1}\big\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\big\|_{L^2}\big\|\partial_x^{m-j}\mathcal{U}\big\|_{L_x^\infty L_y^2}\right]^2\\&\leq\sum_{m=0}^{+\infty}\left[\sum_{j>[m/2]}^{m}\Big(N_{\rho,j+1}\big\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\big\|_{L^2}\Big)\frac{N_{\rho,m-j+3}\big\|\partial_x^{m-j}\mathcal{U}\big\|_{L_x^\infty L_y^2}}{m-j+1}\right]^2\leq C|\vec{a}|_{X_\rho}^4.\end{aligned}$$

This together with (3.5) and (3.3) yields

$$I_2\leq C|\vec{a}|_{X_\rho}^4.\tag{3.7}$$

Substituting the above estimate and (3.2) into (3.1), we get

$$\begin{aligned}\sum_{m=0}^{+\infty}N_{\rho,m}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m u\big\|_{L^2}^2&=N_{\rho,0}^2\big\|\langle y\rangle^{\ell-1}u\big\|_{L^2}^2+\sum_{m=0}^{+\infty}N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^{m+1}u\big\|_{L^2}^2\\&\leq N_{\rho,0}^2\big\|\langle y\rangle^{\ell-1}u\big\|_{L^2}^2+C\big(|\vec{a}|_{X_\rho}^2+|\vec{a}|_{X_\rho}^4\big).\end{aligned}$$

On the other hand, by Hardy's inequality and (2.16), we have

$$
\begin{aligned}
N_{\rho,0}\big\|\langle y\rangle^{\ell-1}u\big\|_{L^2} &\leq C\frac{N_{\rho,0}}{N_{\rho,1}}N_{\rho,1}\big\|\langle y\rangle^{\ell}\partial_y u\big\|_{L^2}\\
&\leq \frac{C}{\rho}N_{\rho,1}\big\|\langle y\rangle^{\ell}\partial_y u\big\|_{L^2}\leq C|\vec{a}|_{X_\rho}.
\end{aligned}
$$

Thus, the first statement in the lemma follows.

The second statement can be proved similarly. In fact, firstly we have

$$
\begin{aligned}
&\sum_{m=0}^{+\infty}(m+1)N_{\rho,m}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m u\big\|_{L^2}^2\\
&\leq C|\vec{a}|_{X_\rho}^2+\sum_{m=1}^{+\infty}(m+1)N_{\rho,m}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m u\big\|_{L^2}^2.
\end{aligned}
$$

Similar to the estimates (3.1) and (3.2), by using the fact that $m+1\leq C(m-j)$ for $j\leq[m/2]$ and $m+1\leq Cj$ for $[m/2]+1\leq j\leq m$, we have by following the argument for (3.7) that

$$
\begin{aligned}
&\sum_{m=1}^{+\infty}(m+1)N_{\rho,m}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m u\big\|_{L^2}^2\\
&\leq C\sum_{m=0}^{+\infty}(m+1)N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^{m+1} u\big\|_{L^2}^2\\
&\leq C\sum_{m=0}^{+\infty}\left[\sum_{j=0}^{[m/2]}\binom{m}{j}N_{\rho,m+1}\big\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\big\|_{L_x^\infty L_y^2}(m-j+1)^{\frac12}\big\|\partial_x^{m-j}\mathcal{U}\big\|_{L^2}\right]^2\\
&\quad+C\sum_{m=0}^{+\infty}\left[\sum_{j=[m/2]+1}^{m}\binom{m}{j}N_{\rho,m+1}(j+1)^{\frac12}\big\|\langle y\rangle^{\ell}\partial_x^j\partial_y u\big\|_{L^2}\big\|\partial_x^{m-j}\mathcal{U}\big\|_{L_x^\infty L_y^2}\right]^2\\
&\leq C|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2.
\end{aligned}
$$

Combining the above estimates gives

$$
\sum_{m=0}^{+\infty}(m+1)N_{\rho,m}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m u\big\|_{L^2}^2\leq C\big(1+|\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2.
$$

Then the proof of the lemma is complete. □

As a direct consequence of the above lemma, we have the following corollary by using

$$\left\|\partial_x^m v\right\|_{L_x^2 L_y^\infty} \leq C \left\|\langle y\rangle^{\ell-1} \partial_x^{m+1} u\right\|_{L^2}.$$

Corollary 3.1. *Under the assumptions of Theorem* 2.1*, we have*

$$\sum_{m=0}^{\infty} N_{\rho,m+1}^2 \left\|\partial_x^m v\right\|_{L_x^2 L_y^\infty}^2 \leq C\left(1+|\vec{a}|_{X_\rho}^2\right)|\vec{a}|_{X_\rho}^2,$$
$$\sum_{m=0}^{\infty} (m+1) N_{\rho,m+1}^2 \left\|\partial_x^m v\right\|_{L_x^2 L_y^\infty}^2 \leq C\left(1+|\vec{a}|_{X_\rho}^2\right)|\vec{a}|_{Y_\rho}^2.$$

4 Gevrey norm of φ and $\partial_y u$

In this section, we will study the last term in definition (2.9) of $|\vec{a}|_{X_\rho}$ involving the mixed derivatives of φ and $\partial_y u$. The estimate is stated in the following proposition.

Proposition 4.1. *Under the assumptions of Theorem* 2.1*, we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty}(m+1)^2 H_{\rho,m+1,k}^2\left(\left\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\varphi\right\|_{L^2}^2+\left\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\partial_y u\right\|_{L^2}^2\right)\\
&\leq C\left(1+|\vec{a}|_{X_\rho}^4\right)|\vec{a}|_{Y_\rho}^2-\mu\sum_{m,k\geq 0}(m+k+1)(m+1)^2 H_{\rho,m+1,k}^2\\
&\quad\times\left(\left\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\varphi\right\|_{L^2}^2+\left\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\partial_y u\right\|_{L^2}^2\right).
\end{aligned}$$

To have a clear presentation, we first deal with the normal and tangential derivatives in Sections 4.1 and 4.2, respectively. The estimate on the mixed derivatives will then be presented in the last subsection.

4.1 Normal derivatives

We first prove the following estimate on the normal derivatives of φ and $\partial_y u$.

Lemma 4.1. *Under the assumptions of Theorem* 2.1*, we have*

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{k=0}^{+\infty}L_{\rho,k}^2\left(\left\|\langle y\rangle^{\ell}\partial_y^k\varphi\right\|_{L^2}^2+\left\|\langle y\rangle^{\ell}\partial_y^k\partial_y u\right\|_{L^2}^2\right)\\
&\leq C\left(1+|\vec{a}|_{X_\rho}^2\right)|\vec{a}|_{Y_\rho}^2-\mu\sum_{k=0}^{+\infty}(k+1)L_{\rho,k}^2\left(\left\|\langle y\rangle^{\ell}\partial_y^k\varphi\right\|_{L^2}^2+\left\|\langle y\rangle^{\ell}\partial_y^k\partial_y u\right\|_{L^2}^2\right),
\end{aligned}$$

where $L_{\rho,k}$ is defined by (2.17).

Proof. We apply $\langle y\rangle^\ell\partial_y^{k+1}$ and $\langle y\rangle^\ell\partial_y^k$ to the Eqs. (2.3a) and (2.3b), respectively to have that

$$\left(\partial_t+u\partial_x+v\partial_y\right)\langle y\rangle^\ell\partial_y^{k+1}u=\langle y\rangle^\ell\partial_y^{k+1}\varphi+\mathcal{R}_k, \tag{4.1a}$$

$$\left(\partial_t+1\right)\langle y\rangle^\ell\partial_y^k\varphi=\langle y\rangle^\ell\partial_y^{k+2}u, \tag{4.1b}$$

where

$$\mathcal{R}_k=v\left(\partial_y\langle y\rangle^\ell\right)\partial_y^{k+1}u-\sum_{i=1}^{k+1}\binom{k+1}{i}\langle y\rangle^\ell\left[\left(\partial_y^iu\right)\partial_x\partial_y^{k+1-i}u+\left(\partial_y^iv\right)\partial_y^{k+2-i}u\right]. \tag{4.2}$$

Then we take the L^2-product with $\langle y\rangle^\ell\partial_y^{k+1}u$ for the Eq. (4.1a), and with $\langle y\rangle^\ell\partial_y^k\varphi$ for the Eq. (4.1b). By using

$$\begin{aligned}\left(\langle y\rangle^\ell\partial_y^{k+2}u,\langle y\rangle^\ell\partial_y^k\varphi\right)_{L^2}=&-\left(\langle y\rangle^\ell\partial_y^{k+1}u,\langle y\rangle^\ell\partial_y^{k+1}\varphi\right)_{L^2}-\left(\left(\partial_y\langle y\rangle^{2\ell}\right)\partial_y^{k+1}u,\partial_y^k\varphi\right)_{L^2}\\&-\int_{\mathbb{R}}\left[\left(\partial_y^{k+1}u\right)\partial_y^k\varphi\right]\big|_{y=0}dx,\end{aligned}$$

we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|\langle y\rangle^\ell\partial_y^k\varphi\|_{L^2}^2+\|\langle y\rangle^\ell\partial_y^{k+1}u\|_{L^2}^2\right)+\|\langle y\rangle^\ell\partial_y^k\varphi\|_{L^2}^2\\&=\left(\mathcal{R}_k,\langle y\rangle^\ell\partial_y^{k+1}u\right)_{L^2}-\left(\left(\partial_y\langle y\rangle^{2\ell}\right)\partial_y^{k+1}u,\partial_y^k\varphi\right)_{L^2}-\int_{\mathbb{R}}\left[\left(\partial_y^{k+1}u\right)\partial_y^k\varphi\right]\big|_{y=0}dx.\end{aligned} \tag{4.3}$$

In view of definitions (2.9)-(2.10) of $|\vec{a}|_{X_\rho}$ and $|\vec{a}|_{Y_\rho}$ and (2.11), we have

$$-\sum_{k=0}^{+\infty}L_{\rho,k}^2\left(\left(\partial_y\langle y\rangle^{2\ell}\right)\partial_y^{k+1}u,\partial_y^k\varphi\right)_{L^2}\le C|\vec{a}|_{X_\rho}^2\le C|\vec{a}|_{Y_\rho}^2.$$

This together with the Sobolev inequality

$$\|f\|_{L_y^\infty}^2\le 2\|f\|_{L_y^2}\|\partial_yf\|_{L_y^2}$$

implies

$$\begin{aligned}&-\sum_{k=0}^{+\infty}L_{\rho,k}^2\int_{\mathbb{R}}\left[\left(\partial_y^{k+1}u\right)\partial_y^k\varphi\right]\Big|_{y=0}dx\\&\le C\sum_{k=0}^{+\infty}(k+1)L_{\rho,k}L_{\rho,k+1}\left(\|\partial_y^k\varphi\|_{L^2}\|\partial_y^{k+1}\varphi\|_{L^2}+\|\partial_y^{k+1}u\|_{L^2}\|\partial_y^{k+2}u\|_{L^2}\right)\\&\le C\sum_{k=0}^{+\infty}(k+1)L_{\rho,k}^2\left(\|\partial_y^k\varphi\|_{L^2}^2+\|\partial_y^k\partial_yu\|_{L^2}^2\right)\le C|\vec{a}|_{Y_\rho}^2,\end{aligned}$$

where in the first inequality we have used the fact that

$$\frac{L_{\rho,k}}{L_{\rho,k+1}}=\frac{k+2}{\rho}\leq C(k+1)$$

due to (2.16). As a result, by the above estimates and (2.18), we multiply Eq. (4.3) by $L^2_{\rho,k}$ and then take the summation over $k\geq 0$ to obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{k=0}^{+\infty}L^2_{\rho,k}\Big(\|\langle y\rangle^{\ell}\partial_y^k\varphi\|^2_{L^2}+\|\langle y\rangle^{\ell}\partial_y^k\partial_y u\|^2_{L^2}\Big)\\&\leq-\mu\sum_{k=0}^{+\infty}(k+1)L^2_{\rho,k}\Big(\|\langle y\rangle^{\ell}\partial_y^k\varphi\|^2_{L^2}+\|\langle y\rangle^{\ell}\partial_y^k\partial_y u\|^2_{L^2}\Big)\\&\quad+\sum_{k=0}^{+\infty}L^2_{\rho,k}\Big(\mathcal{R}_k,\langle y\rangle^{\ell}\partial_y^{k+1}u\Big)_{L^2}+C|\vec{a}|^2_{Y_\rho},\end{aligned}\tag{4.4}$$

where $\mathcal{R}_m$ is given by (4.2). For the term in the last inequality, we claim that

$$\sum_{k=0}^{+\infty}L^2_{\rho,k}\Big(\mathcal{R}_k,\ \langle y\rangle^{\ell}\partial_y^{k+1}u\Big)_{L^2}\leq C\big(1+|\vec{a}|^2_{X_\rho}\big)|\vec{a}|^2_{Y_\rho}.\tag{4.5}$$

The proof of (4.5) is postponed after the proof of the lemma. Now with the claim and the above two estimates, we complete the proof of lemma. □

Proof of assertion (4.5). We use the estimate

$$\Big(\mathcal{R}_k,\langle y\rangle^{\ell}\partial_y^{k+1}u\Big)_{L^2}\leq(k+1)\|\langle y\rangle^{\ell}\partial_y^{k+1}u\|^2_{L^2}+\Big[(k+1)^{-\frac{1}{2}}\|\mathcal{R}_k\|_{L^2}\Big]^2$$

and definition (2.10) of $|\vec{a}|_{Y_\rho}$ to obtain

$$\sum_{k=0}^{+\infty}L^2_{\rho,k}\Big(\mathcal{R}_k,\langle y\rangle^{\ell}\partial_y^{k+1}u\Big)_{L^2}\leq|\vec{a}|^2_{Y_\rho}+\sum_{k=0}^{+\infty}\Big[(k+1)^{-\frac{1}{2}}L_{\rho,k}\|\mathcal{R}_k\|_{L^2}\Big]^2.\tag{4.6}$$

Moreover, in view of (4.2), it follows that

$$\begin{aligned}&\sum_{k=0}^{+\infty}\Big[(k+1)^{-\frac{1}{2}}L_{\rho,k}\|\mathcal{R}_k\|_{L^2}\Big]^2\\&\leq C|\vec{a}|^3_{X_\rho}+\sum_{k=0}^{+\infty}\left[(k+1)^{-\frac{1}{2}}L_{\rho,k}\sum_{i=1}^{k+1}\binom{k+1}{i}\big\|\langle y\rangle^{\ell}(\partial_y^i u)\partial_x\partial_y^{k+1-i}u\big\|_{L^2}\right]^2\\&\quad+\sum_{k=0}^{+\infty}\left[(k+1)^{-\frac{1}{2}}L_{\rho,k}\sum_{i=1}^{k+1}\binom{k+1}{i}\big\|\langle y\rangle^{\ell}(\partial_y^i v)\partial_y^{k+2-i}u\big\|_{L^2}\right]^2.\end{aligned}\tag{4.7}$$

For the last term the right-hand side, we use the decomposition

$$\sum_{i=1}^{k+1}=\sum_{i=1}^{[(k+1)/2]}+\sum_{i=[(k+1)/2]+1}^{k+1},$$

to write

$$(k+1)^{-\frac{1}{2}}L_{\rho,k}\sum_{i=1}^{k+1}\binom{k+1}{i}\big\|\langle y\rangle^{\ell}(\partial_y^i v)\partial_y^{k+2-i}u\big\|_{L^2}\leq p_k+q_k \tag{4.8}$$

with

$$\begin{aligned}p_k=&\sum_{i=1}^{[(k+1)/2]}\frac{(k+1)!}{i!(k+1-i)!}\frac{(k+1)^{-\frac{1}{2}}L_{\rho,k}}{H_{\rho,4,i-1}L_{\rho,k+1-i}}\\&\qquad\times\Big(H_{\rho,4,i-1}\big\|\partial_y^i v\big\|_{L^\infty}\Big)\Big(L_{\rho,k+1-i}\big\|\langle y\rangle^{\ell}\partial_y^{k+2-i}u\big\|_{L^2}\Big),\\q_k=&\sum_{i=[(k+1)/2]+1}^{k+1}\frac{(k+1)!}{i!(k+1-i)!}\frac{(k+1)^{-\frac{1}{2}}L_{\rho,k}}{H_{\rho,2,i-2}H_{\rho,3,k+2-i}}\\&\qquad\times\Big(H_{\rho,2,i-2}\big\|\partial_y^i v\big\|_{L^2}\Big)\Big(H_{\rho,3,k+2-i}\big\|\langle y\rangle^{\ell}\partial_y^{k+2-i}u\big\|_{L^\infty}\Big).\end{aligned}$$

For the term p_k, we first note the following estimate (see Appendix A for its proof) that

$$\frac{(k+1)!}{i!(k+1-i)!}\frac{(k+1)^{-\frac{1}{2}}L_{\rho,k}}{H_{\rho,4,i-1}L_{\rho,k+1-i}}\leq C\frac{(k+2-i)^{\frac{1}{2}}}{i+1},\quad \forall 1\leq i\leq[(k+1)/2]. \tag{4.9}$$

Following an argument similar to (3.5), we have

$$\begin{aligned}&\sum_{k=0}^{+\infty}p_k^2\leq C\sum_{k=0}^{+\infty}\left[\sum_{i=1}^{k+1}\frac{H_{\rho,4,i-1}\big\|\partial_y^i v\big\|_{L^\infty}}{i+1}(k+2-i)^{\frac{1}{2}}L_{\rho,k+1-i}\big\|\langle y\rangle^{\ell}\partial_y^{k+2-i}u\big\|_{L^2}\right]^2\\&\leq C\left(\sum_{i=1}^{+\infty}\frac{H_{\rho,4,i-1}\big\|\partial_y^i v\big\|_{L^\infty}}{i+1}\right)^2\sum_{i=0}^{+\infty}(i+1)L_{\rho,i}^2\big\|\langle y\rangle^{\ell}\partial_y^{i+1}u\big\|_{L^2}^2\\&\leq C|\vec{a}|_{Y_\rho}^2\sum_{i=1}^{+\infty}H_{\rho,4,i-1}^2\big\|\partial_y^i v\big\|_{L^\infty}^2,\end{aligned}$$

where we have used (2.19) in the second inequality. Moreover, using the Sobolev embedding inequality and Hardy's inequality, we obtain

$$\sum_{i=1}^{+\infty}H_{\rho,4,i-1}^2\big\|\partial_y^i v\big\|_{L^\infty}^2\leq H_{\rho,4,0}^2\|\partial_x u\|_{L^\infty}^2+\sum_{i=2}^{+\infty}H_{\rho,4,i-1}^2\big\|\partial_x\partial_y^{i-1}u\big\|_{L^\infty}^2$$

$$\leq C\|\langle y\rangle^{\ell}\partial_x\partial_y u\|^2_{L^\infty_x L^2_y}+C|\vec{a}|^2_{X_\rho}\leq C|\vec{a}|^2_{X_\rho}.$$

Combining the above estimates gives

$$\sum_{k=0}^{+\infty} p_k^2 \leq C|\vec{a}|^2_{X_\rho}|\vec{a}|^2_{Y_\rho}. \tag{4.10}$$

Similarly, by using the estimate (see Appendix A)

$$\frac{(k+1)!}{i!(k+1-i)!}\frac{(k+1)^{-\frac{1}{2}}L_{\rho,k}}{H_{\rho,2,i-2}H_{\rho,3,k+2-i}}\leq\frac{C}{k+3-i},\quad [(k+1)/2]<i\leq k+1, \tag{4.11}$$

and observing $|\vec{a}|_{X_\rho}\leq|\vec{a}|_{Y_\rho}$, we have

$$\sum_{k=0}^{+\infty} q_k^2 \leq C\left(\sum_{i=0}^{+\infty}\frac{H_{\rho,3,i}\|\partial_y^i u\|_{L^\infty}}{i+1}\right)^2\sum_{i=0}^{+\infty}H^2_{\rho,2,i}\|\partial_x\partial_y^{i+1}u\|^2_{L^2}\leq C|\vec{a}|^2_{X_\rho}|\vec{a}|^2_{Y_\rho}.$$

This together with (4.10) and (4.8) yields

$$\sum_{k=0}^{+\infty}\left[(k+1)^{-\frac{1}{2}}L_{\rho,k}\sum_{i=1}^{k+1}\binom{k+1}{i}\|\langle y\rangle^{\ell}(\partial_y^i v)\partial_y^{k+2-i}u\|_{L^2}\right]^2\leq C|\vec{a}|^2_{X_\rho}|\vec{a}|^2_{Y_\rho}.$$

Similarly,

$$\sum_{k=0}^{+\infty}\left[(k+1)^{-\frac{1}{2}}L_{\rho,k}\sum_{i=1}^{k+1}\binom{k+1}{i}\|\langle y\rangle^{\ell}(\partial_y^i u)\partial_x\partial_y^{k+1-i}u\|_{L^2}\right]^2\leq C|\vec{a}|^2_{X_\rho}|\vec{a}|^2_{Y_\rho}.$$

Finally, by substituting the above two estimates into (4.7) and using (2.11), we obtain

$$\sum_{k=0}^{+\infty}\left[(k+1)^{-\frac{1}{2}}L_{\rho,k}\|\mathcal{R}_k\|_{L^2}\right]^2\leq C\big(1+|\vec{a}|^2_{X_\rho}\big)|\vec{a}|^2_{Y_\rho}.$$

This with (4.6) yields the desired assertion (4.5). The proof of the claim (4.5) is complete. □

4.2 Tangential derivatives

In this subsection, we consider the tangential derivatives of φ and $\partial_y u$.

Lemma 4.2. *Under the assumptions of Theorem* 2.1, *we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}(m+1)^2N_{\rho,m+1}^2\Big(\|\langle y\rangle^{\ell}\partial_x^m\varphi\|_{L^2}^2+\|\langle y\rangle^{\ell}\partial_x^m\partial_y u\|_{L^2}^2\Big)\\&\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2-\mu\sum_{m=0}^{+\infty}(m+1)^3N_{\rho,m+1}^2\Big(\|\langle y\rangle^{\ell}\partial_x^m\varphi\|_{L^2}^2+\|\langle y\rangle^{\ell}\partial_x^m\partial_y u\|_{L^2}^2\Big).\end{aligned}$$

Recall $N_{\rho,m}$ *is defined by* (1.6).

Proof. We apply $\langle y\rangle^{\ell}\partial_x^m\partial_y$ and $\langle y\rangle^{\ell}\partial_x^m$ to the Eqs. (2.3a) and (2.3b), respectively, to have

$$\begin{cases}(\partial_t+u\partial_x+v\partial_y)\langle y\rangle^{\ell}\partial_x^m\partial_y u=\langle y\rangle^{\ell}\partial_x^m\partial_y\varphi+\mathcal{P}_m,\\(\partial_t+1)\langle y\rangle^{\ell}\partial_x^m\varphi=\langle y\rangle^{\ell}\partial_x^m\partial_y^2u,\end{cases}$$

where

$$\mathcal{P}_m=v(\partial_y\langle y\rangle^{\ell})\partial_x^m\partial_y u-\sum_{j=1}^{m}\binom{m}{j}\langle y\rangle^{\ell}\Big[(\partial_x^j u)\partial_x^{m-j+1}\partial_y u+(\partial_x^j v)\partial_x^{m-j}\partial_y^2u\Big]. \tag{4.12}$$

Following a similar argument as the proof of Lemma 4.1, by observing that

$$\begin{aligned}\Big(\langle y\rangle^{\ell}\partial_x^m\partial_y^2u,\langle y\rangle^{\ell}\partial_x^m\varphi\Big)_{L^2}=&-\Big(\langle y\rangle^{\ell}\partial_x^m\partial_y u,\langle y\rangle^{\ell}\partial_x^m\partial_y\varphi\Big)_{L^2}\\&-\Big((\partial_y\langle y\rangle^{2\ell})\partial_x^m\partial_y u,\ \partial_x^m\varphi\Big)_{L^2},\end{aligned}$$

we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|\langle y\rangle^{\ell}\partial_x^m\varphi\|_{L^2}^2+\|\langle y\rangle^{\ell}\partial_x^m\partial_y u\|_{L^2}^2\Big)+\|\langle y\rangle^{\ell}\partial_x^m\varphi\|_{L^2}^2\\&=-\Big((\partial_y\langle y\rangle^{2\ell})\partial_x^m\partial_y u,\partial_x^m\varphi\Big)_{L^2}+\Big(\mathcal{P}_m,\ \langle y\rangle^{\ell}\partial_x^m\partial_y u\Big)_{L^2}.\end{aligned}$$

Thus, by using (2.15) and the fact that $-\mu(m+2)\leq-\mu(m+1)$, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}(m+1)^2N_{\rho,m+1}^2\Big(\|\langle y\rangle^{\ell}\partial_x^m\varphi\|_{L^2}^2+\|\langle y\rangle^{\ell}\partial_x^m\partial_y u\|_{L^2}^2\Big)\\&\leq-\mu\sum_{m=0}^{+\infty}(m+1)^3N_{\rho,m+1}^2\Big(\|\langle y\rangle^{\ell}\partial_x^m\varphi\|_{L^2}^2+\|\langle y\rangle^{\ell}\partial_x^m\partial_y u\|_{L^2}^2\Big)\\&\quad+C|\vec{a}|_{X_\rho}^2+\sum_{m=0}^{+\infty}(m+1)^2N_{\rho,m+1}^2\big(\mathcal{P}_m,\ \langle y\rangle^{\ell}\partial_x^m\partial_y u\big)_{L^2}.\end{aligned}$$

It remains to estimate the last term on the right-hand side. Similar to (4.6), it holds that

$$\begin{aligned}&\sum_{m=0}^{+\infty}(m+1)^2N_{\rho,m+1}^2\big(\mathcal{P}_m,\ \langle y\rangle^{\ell}\partial_x^m\partial_y u\big)_{L^2}\\&\leq|\vec{a}|_{Y_\rho}^2+\sum_{m=0}^{+\infty}\Big[(m+1)^{\frac12}N_{\rho,m+1}\|\mathcal{P}_m\|_{L^2}\Big]^2.\end{aligned}\tag{4.13}$$

Then Lemma 4.2 holds by the above inequalities if

$$\sum_{m=0}^{+\infty}\Big[(m+1)^{\frac12}N_{\rho,m+1}\|\mathcal{P}_m\|_{L^2}\Big]^2\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.\tag{4.14}$$

We now turn to prove (4.14). In view of (4.12), we use the fact that $\|v\|_{L_y^\infty}\leq C\|\langle y\rangle^{\ell}\partial_x\partial_y u\|_{L_y^2}$ by Hardy's inequality to obtain

$$\begin{aligned}&\sum_{m=0}^{+\infty}\Big[(m+1)^{\frac12}N_{\rho,m+1}\|\mathcal{P}_m\|_{L^2}\Big]^2\\&\leq C|\vec{a}|_{X_\rho}^3+\sum_{m=0}^{+\infty}\left[(m+1)^{\frac12}N_{\rho,m+1}\sum_{j=1}^{m}\binom{m}{j}\|\langle y\rangle^{\ell}(\partial_x^j u)\partial_x^{m-j+1}\partial_y u\|_{L^2}\right]^2\\&\quad+\sum_{m=0}^{+\infty}\left[(m+1)^{\frac12}N_{\rho,m+1}\sum_{j=1}^{m}\binom{m}{j}\|\langle y\rangle^{\ell}(\partial_x^j v)\partial_x^{m-j}\partial_y^2 u\|_{L^2}\right]^2.\end{aligned}\tag{4.15}$$

For the last term above inequality as (4.8), we write

$$(m+1)^{\frac12}N_{\rho,m+1}\sum_{j=1}^{m}\binom{m}{j}\|\langle y\rangle^{\ell}(\partial_x^j v)\partial_x^{m-j}\partial_y^2 u\|_{L^2}=A_m+B_m\tag{4.16}$$

with

$$\begin{aligned}A_m=&\sum_{j=1}^{[m/2]}\frac{m!}{j!(m-j)!}\frac{(m+1)^{\frac12}N_{\rho,m+1}}{N_{\rho,j+3}H_{\rho,m-j+1,1}}\\&\times\Big(N_{\rho,j+3}\|\partial_x^j v\|_{L^\infty}\Big)\Big(H_{\rho,m-j+1,1}\|\langle y\rangle^{\ell}\partial_x^{m-j}\partial_y^2 u\|_{L^2}\Big),\\B_m=&\sum_{j=[m/2]+1}^{m}\frac{m!}{j!(m-j)!}\frac{(m+1)^{\frac12}N_{\rho,m+1}}{N_{\rho,j+1}H_{\rho,m-j+3,1}}\\&\times\Big(N_{\rho,j+1}\|\partial_x^j v\|_{L_x^2L_y^\infty}\Big)\Big(H_{\rho,m-j+3,1}\|\langle y\rangle^{\ell}\partial_x^{m-j}\partial_y^2 u\|_{L_x^\infty L_y^2}\Big).\end{aligned}$$

Moreover, by using the estimate (see Appendix A)

$$\frac{m!}{j!(m-j)!}\frac{(m+1)^{\frac{1}{2}}N_{\rho,m+1}}{N_{\rho,j+3}H_{\rho,m-j+1,1}}\leq\frac{C(m-j+1)^{\frac{3}{2}}}{j+1},\quad 1\leq j\leq[m/2], \tag{4.17}$$

we follow a similar argument as that after (4.8) to conclude that

$$\begin{aligned}\sum_{m=0}^{+\infty}A_m^2&\leq C\left(\sum_{j=0}^{+\infty}\frac{N_{\rho,j+3}\|\partial_x^j v\|_{L^\infty}}{j+1}\right)^2\sum_{j=0}^{+\infty}(j+1)^3H_{\rho,j+1,1}^2\|\langle y\rangle^\ell\partial_x^j\partial_y^2u\|_{L^2}^2\\&\leq C\big(1+|\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{X_\rho}^2|\vec{a}|_{Y_\rho}^2\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2,\end{aligned}$$

where we have used Corollary 3.1 in the second inequality. On the other hand, we note that

$$\frac{m!}{j!(m-j)!}\frac{(m+1)^{\frac{1}{2}}N_{\rho,m+1}}{N_{\rho,j+1}H_{\rho,m-j+3,1}}\leq C\frac{(j+1)^{\frac{1}{2}}}{m-j+1},\quad\forall[m/2]+1\leq j\leq m, \tag{4.18}$$

where its proof is given in Appendix A. Thus, following a similar argument as that after (4.8) and using Corollary 3.1, we conclude

$$\begin{aligned}\sum_{m=0}^{+\infty}B_m^2&\leq C\left(\sum_{j=0}^{+\infty}\frac{H_{\rho,j+3,1}\|\langle y\rangle^\ell\partial_x^j\partial_y^2u\|_{L_x^\infty L_y^2}}{j+1}\right)^2\sum_{j=0}^{+\infty}(j+1)N_{\rho,j+1}^2\|\partial_x^j v\|_{L_x^2L_y^\infty}^2\\&\leq C|\vec{a}|_{X_\rho}^2\big(1+|\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.\end{aligned}$$

As a result, we combine the above estimates with (4.16) to obtain

$$\sum_{m=0}^{+\infty}\left[(m+1)^{\frac{1}{2}}N_{\rho,m+1}\sum_{j=1}^{m}\binom{m}{j}\|\langle y\rangle^\ell(\partial_x^jv)\partial_x^{m-j}\partial_y^2u\|_{L^2}\right]^2\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2. \tag{4.19}$$

Similar argument also yields

$$\sum_{m=0}^{+\infty}\left[(m+1)^{\frac{1}{2}}N_{\rho,m+1}\sum_{j=1}^{m}\binom{m}{j}\|\langle y\rangle^\ell(\partial_x^ju)\partial_x^{m-j+1}\partial_yu\|_{L^2}\right]^2\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.$$

Then substituting the above two inequalities into (4.15), the estimate (4.14) follows. The proof of the lemma is complete. □

4.3 Proof of Proposition 4.1

Proof. We will follow the argument used in the Sections 4.1 and 4.2 to derive the estimate on the mixed derivatives. For this, we apply $\langle y\rangle^\ell\partial_x^m\partial_y^{k+1}$ and $\langle y\rangle^\ell\partial_x^m\partial_y^k$ to the Eqs. (2.3a) and (2.3b) to obtain

$$\begin{cases} (\partial_t+u\partial_x+v\partial_y)\langle y\rangle^\ell\partial_x^m\partial_y^{k+1}u=\langle y\rangle^\ell\partial_x^m\partial_y^{k+1}\varphi+\mathcal{Q}_{m,k},\\ (\partial_t+1)\langle y\rangle^\ell\partial_x^m\partial_y^k\varphi=\langle y\rangle^\ell\partial_x^m\partial_y^k\partial_y^2u, \end{cases}$$

where

$$\begin{aligned} \mathcal{Q}_{m,k}=&v\big(\partial_y\langle y\rangle^\ell\big)\partial_x^m\partial_y^{k+1}u-\sum_{i+j\geq1}\binom{m}{j}\binom{k+1}{i}\langle y\rangle^\ell\big(\partial_x^j\partial_y^iu\big)\partial_x^{m-j+1}\partial_y^{k+1-i}u\\ &-\sum_{\substack{i+j\geq1\\ j\leq m,i\leq k+1}}\binom{m}{j}\binom{k+1}{i}\langle y\rangle^\ell\big(\partial_x^j\partial_y^iv\big)\partial_x^{m-j}\partial_y^{k+2-i}u. \end{aligned} \tag{4.20}$$

Thus, we follow a similar argument as in the proof of (4.4) and observe the fact that

$$\frac12\frac{d}{dt}\big(H^2_{\rho,m+1,k}\big)=-\mu(m+k+2)H^2_{\rho,m+1,k}\leq-\mu(m+k+1)H^2_{\rho,m+1,k},$$

to obtain

$$\begin{aligned} &\frac12\frac{d}{dt}\sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty}(m+1)^2H^2_{\rho,m+1,k}\Big(\big\|\langle y\rangle^\ell\partial_x^m\partial_y^k\varphi\big\|^2_{L^2}+\big\|\langle y\rangle^\ell\partial_x^m\partial_y^{k+1}u\big\|^2_{L^2}\Big)\\ \leq&C|\vec a|^2_{Y_\rho}+\sum_{m=0}^{+\infty}\sum_{k=0}^{+\infty}(m+1)^2H^2_{\rho,m+1,k}\big(\mathcal{Q}_{m,k},\ \langle y\rangle^\ell\partial_x^m\partial_y^{k+1}u\big)_{L^2}\\ &-\mu\sum_{m,k}(m+k+1)(m+1)^2H^2_{\rho,m+1,k}\Big(\big\|\langle y\rangle^\ell\partial_x^m\partial_y^k\varphi\big\|^2_{L^2}+\big\|\langle y\rangle^\ell\partial_x^m\partial_y^{k+1}u\big\|^2_{L^2}\Big). \end{aligned} \tag{4.21}$$

Similar to (4.6) and (4.13), by recalling $\mathcal{Q}_{m,k}$ is given by (4.20), we have

$$\sum_{m,k\geq0}(m+1)^2H^2_{\rho,m+1,k}\big(\mathcal{Q}_{m,k},\langle y\rangle^\ell\partial_x^m\partial_y^{k+1}u\big)_{L^2}\leq C|\vec a|^3_{X_\rho}+|\vec a|^2_{Y_\rho}+S_1+S_2 \tag{4.22}$$

with

$$\begin{aligned} S_1=\sum_{m,k\geq0}\Bigg[&(m+k+1)^{-\frac12}(m+1)H_{\rho,m+1,k}\\ &\times\sum_{\substack{i+j\geq1\\ j\leq m,i\leq k+1}}\binom{m}{j}\binom{k+1}{i}\big\|\langle y\rangle^\ell\big(\partial_x^j\partial_y^iu\big)\partial_x^{m-j+1}\partial_y^{k+1-i}u\big\|_{L^2}\Bigg]^2, \end{aligned}$$

$$S_2=\sum_{m,k\geq 0}\Big[(m+k+1)^{-\frac{1}{2}}(m+1)H_{\rho,m+1,k}\\ \times\sum_{\substack{i+j\geq 1\\ j\leq m,i\leq k+1}}\binom{m}{j}\binom{k+1}{i}\big\|\langle y\rangle^{\ell}(\partial_x^j\partial_y^i v)\partial_x^{m-j}\partial_y^{k+2-i}u\big\|_{L^2}\Big]^2.$$

To estimate S_2, we write

$$\begin{aligned}&(m+k+1)^{-\frac{1}{2}}(m+1)H_{\rho,m+1,k}\\&\quad\times\sum_{\substack{i+j\geq 1\\ j\leq m,i\leq k+1}}\binom{m}{j}\binom{k+1}{i}\big\|\langle y\rangle^{\ell}(\partial_x^j\partial_y^i v)\partial_x^{m-j}\partial_y^{k+2-i}u\big\|_{L^2}\\&\leq r_{m,k}+p_{m,k}+q_{m,k}\end{aligned}$$

with

$$\begin{aligned}r_{m,k}=&(m+k+1)^{-\frac{1}{2}}(m+1)H_{\rho,m+1,k}\sum_{j=1}^{m}\binom{m}{j}\big\|\langle y\rangle^{\ell}(\partial_x^j v)\partial_x^{m-j}\partial_y^{k+2}u\big\|_{L^2}\\&+(m+k+1)^{-\frac{1}{2}}(m+1)(k+1)H_{\rho,m+1,k}\sum_{j=0}^{m}\binom{m}{j}\big\|\langle y\rangle^{\ell}(\partial_x^j\partial_y v)\partial_x^{m-j}\partial_y^{k+1}u\big\|_{L^2},\end{aligned}$$

$$\begin{aligned}p_{m,k}=&\sum_{\substack{i+j\leq[(m+k+1)/2]\\ j\leq m,2\leq i\leq k+1}}\binom{m}{j}\binom{k+1}{i}\frac{(m+k+1)^{-\frac{1}{2}}(m+1)H_{\rho,m+1,k}}{H_{\rho,j+4,i-1}H_{\rho,m-j+1,k+1-i}}\\&\times\Big(H_{\rho,j+4,i-1}\big\|\partial_x^j\partial_y^i v\big\|_{L^\infty}\Big)\Big(H_{\rho,m-j+1,k+1-i}\big\|\langle y\rangle^{\ell}\partial_x^{m-j}\partial_y^{k+2-i}u\big\|_{L^2}\Big),\end{aligned}$$

$$\begin{aligned}q_{m,k}=&\sum_{\substack{i+j\geq[(m+k+1)/2]+1\\ j\leq m,2\leq i\leq k+1}}\binom{m}{j}\binom{k+1}{i}\frac{(m+k+1)^{-\frac{1}{2}}(m+1)H_{\rho,m+1,k}}{H_{\rho,j+2,i-2}H_{\rho,m-j+3,k+2-i}}\\&\times\Big(H_{\rho,j+2,i-2}\big\|\partial_x^j\partial_y^i v\big\|_{L^2}\Big)\Big(H_{\rho,m-j+3,k+2-i}\big\|\langle y\rangle^{\ell}\partial_x^{m-j}\partial_y^{k+2-i}u\big\|_{L^\infty}\Big).\end{aligned}$$

Like the proof of (4.19), we can obtain

$$\sum_{m,k\geq 0}r_{m,k}^2\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.$$

Moreover, if $1\leq i+j\leq[(m+k+1)/2]$, then

$$\begin{aligned}
&\binom{m}{j}\binom{k+1}{i}\frac{(m+k+1)^{-\frac{1}{2}}(m+1)H_{\rho,m+1,k}}{H_{\rho,j+4,i-1}H_{\rho,m-j+1,k+1-i}}\\
&\leq(j+4)(m-j+1)\frac{1}{(i+j+1)^2}(m+k-i-j+2)^{\frac{1}{2}},
\end{aligned}\tag{4.23}$$

where its proof is given in Appendix A. Then following the argument after (4.8) and (4.16), we obtain

$$\begin{aligned}
\sum_{m,k\geq0}p_{m,k}^2\leq&\left(\sum_{j\geq0,i\geq2}\frac{(j+4)H_{\rho,j+4,i-1}\left\|\partial_x^j\partial_y^iv\right\|_{L^\infty}}{(i+j+1)^2}\right)^2\\
&\times\sum_{i,j\geq0}(i+j+1)(j+1)^2H_{\rho,j+1,i}\left\|\langle y\rangle^{\ell}\partial_x^j\partial_y^i\partial_yu\right\|_{L^2}^2\\
\leq&\, C\big(1+|\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2.
\end{aligned}$$

Similarly, by using

$$\binom{m}{j}\binom{k+1}{i}\frac{(m+k+1)^{-\frac{1}{2}}(m+1)H_{\rho,m+1,k}}{H_{\rho,j+2,i-2}H_{\rho,m-j+3,k+2-i}}\leq C\frac{j+1}{(m+k-i-j+2)^2}\tag{4.24}$$

for any pair (i,j) with $[(m+k+1)/2]\leq i+j\leq m+k+1$ (see Appendix A), we have

$$\sum_{m,k\geq0}q_{m,k}^2\leq C\big(1+|\vec{a}|_{X_\rho}^2\big)|\vec{a}|_{Y_\rho}^2.$$

Thus, combining the above estimates gives

$$S_2\leq C\sum_{m,k\geq0}\big(r_{m,k}^2+p_{m,k}^2+q_{m,k}^2\big)\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.$$

Similarly,

$$S_1\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.$$

This with (4.22) and (4.21) yields the statement in Proposition 4.1. The proof is complete. □

5 Proof of the a priori estimate

We now study the Gevrey estimate on the auxiliary functions $\mathcal{U}$ and λ defined in (2.5) and (2.6) to complete the proof of Theorem 2.1.

Proposition 5.1. *Under the assumptions of Theorem* 2.1, *we have*

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}\Big(N_{\rho,m+1}^2\big\|\partial_x^m\mathcal{U}\big\|_{L^2}^2+(m+1)N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\big\|_{L^2}^2\Big)\\&\leq-\mu\sum_{m=0}^{+\infty}\Big[(m+1)N_{\rho,m+1}^2\big\|\partial_x^m\mathcal{U}\big\|_{L^2}^2+(m+1)^2N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\big\|_{L^2}^2\Big]\\&\quad+C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.\end{aligned}$$

Proof. It follows from (2.7) that

$$\big(\partial_t+u\partial_x+v\partial_y\big)\partial_x^m\mathcal{U}=\partial_x^{m+1}\lambda+\partial_x^m\left[(\partial_x\partial_y u)\int_0^y\mathcal{U}(t,x,\tilde{y})d\tilde{y}+(\partial_x u)\mathcal{U}\right].$$

This, with the fact that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}N_{\rho,m+1}^2\leq-\mu(m+1)N_{\rho,m+1}^2,\\&\sum_{m=0}^{+\infty}(m+1)N_{\rho,m+1}^2\|\partial_x^m\mathcal{U}\|_{L^2}^2\leq|\vec{a}|_{Y_\rho}^2,\end{aligned}$$

yields that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}N_{\rho,m+1}^2\big\|\partial_x^m\mathcal{U}\big\|_{L^2}^2\\&\leq-\mu\sum_{m=0}^{+\infty}(m+1)N_{\rho,m+1}^2\big\|\partial_x^m\mathcal{U}\big\|_{L^2}^2+|\vec{a}|_{Y_\rho}^2+\sum_{m=0}^{+\infty}(m+1)^{-1}N_{\rho,m+1}^2\big\|\partial_x^{m+1}\lambda\big\|_{L^2}^2\\&\quad+\sum_{m=0}^{+\infty}(m+1)^{-1}N_{\rho,m+1}^2\left\|\partial_x^m\left[(\partial_x\partial_y u)\int_0^y\mathcal{U}(t,x,\tilde{y})d\tilde{y}+(\partial_x u)\mathcal{U}\right]\right\|_{L^2}^2.\end{aligned}$$

By definition (2.10) of $|\vec{a}|_{Y_\rho}$ and the fact that $N_{\rho,m+1}/N_{\rho,m+2}\leq C(m+1)^{3/2}$, we have

$$\sum_{m=0}^{+\infty}(m+1)^{-1}N_{\rho,m+1}^2\big\|\partial_x^{m+1}\lambda\big\|_{L^2}^2$$

$$
\begin{aligned}
&\le C\sum_{m=0}^{+\infty}(m+1)^2 N_{\rho,m+2}^2\big\|\partial_x^{m+1}\lambda\big\|_{L^2}^2\\
&\le C\sum_{m=0}^{+\infty}(m+1)^2 N_{\rho,m+1}^2\big\|\partial_x^{m}\lambda\big\|_{L^2}^2\le C|\vec{a}|_{Y_\rho}^2.
\end{aligned}
$$

Following a similar argument as in the proof of Lemmas 3.1 and 4.2, we conclude

$$
\begin{aligned}
&\sum_{m=0}^{+\infty}(m+1)^{-1}N_{\rho,m+1}^2\left\|\partial_x^m\left[(\partial_x\partial_y u)\int_0^y\mathcal{U}(t,x,\tilde{y})d\tilde{y}+(\partial_x u)\mathcal{U}\right]\right\|_{L^2}^2\\
&\le C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.
\end{aligned}
$$

As a result, combining the above estimates yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}N_{\rho,m+1}^2\big\|\partial_x^m\mathcal{U}\big\|_{L^2}^2\\
&\le C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2-\mu\sum_{m=0}^{+\infty}(m+1)N_{\rho,m+1}^2\big\|\partial_x^m\mathcal{U}\big\|_{L^2}^2.
\end{aligned}\tag{5.1}
$$

It remains to estimate λ, and it follows from (2.8) that

$$
\begin{aligned}
&\big(\partial_t+u\partial_x+v\partial_y\big)\langle y\rangle^{\ell-1}\partial_x^m\lambda\\
&=v\big(\partial_y\langle y\rangle^{\ell-1}\big)\partial_x^m\lambda+\langle y\rangle^{\ell-1}\partial_x^{m+1}\varphi\\
&\quad-\langle y\rangle^{\ell-1}\partial_x^m\left[(\partial_x u)\partial_x u+(\partial_y\varphi)\int_0^y\mathcal{U}d\tilde{y}\right].
\end{aligned}
$$

This with the fact that

$$
\sum_{m=0}^{+\infty}(m+1)^2N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\big\|_{L^2}^2\le|\vec{a}|_{Y_\rho}^2,
$$

yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}(m+1)N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\big\|_{L^2}^2\\
&\le-\mu\sum_{m=0}^{+\infty}(m+1)^2N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\big\|_{L^2}^2+C|\vec{a}|_{X_\rho}^3+|\vec{a}|_{Y_\rho}^2\\
&\quad+\sum_{m=0}^{+\infty}N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^{m+1}\varphi\big\|_{L^2}^2\\
&\quad+\sum_{m=0}^{+\infty}N_{\rho,m+1}^2\left\|\langle y\rangle^{\ell-1}\partial_x^m\left[(\partial_x u)\partial_x u+(\partial_y\varphi)\int_0^y\mathcal{U}d\tilde{y}\right]\right\|_{L^2}^2.
\end{aligned}
$$

Note that

$$\begin{aligned}&\sum_{m=0}^{+\infty} N_{\rho,m+1}^2 \big\| \langle y\rangle^{\ell-1}\partial_x^{m+1}\varphi\big\|_{L^2}^2\\ &\leq C\sum_{m=0}^{+\infty}(m+2)^3 N_{\rho,m+2}^2\big\|\langle y\rangle^{\ell}\partial_x^{m+1}\varphi\big\|_{L^2}^2\leq C|\vec{a}|_{Y_\rho}^2.\end{aligned}$$

By a similar argument as the proof of Lemma 3.1, we have

$$\sum_{m=0}^{+\infty} N_{\rho,m+1}^2\left\|\langle y\rangle^{\ell-1}\partial_x^m\left[(\partial_x u)\partial_x u+(\partial_y\varphi)\int_0^y \mathcal{U}d\tilde{y}\right]\right\|_{L^2}^2\leq C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.$$

Consequently, combining the above inequalities yields

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{m=0}^{+\infty}(m+1)N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\big\|_{L^2}^2\\ &\leq-\mu\sum_{m=0}^{+\infty}(m+1)^2N_{\rho,m+1}^2\big\|\langle y\rangle^{\ell-1}\partial_x^m\lambda\big\|_{L^2}^2+C\big(1+|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2.\end{aligned}$$

This and (5.1) yields the statement in Proposition 5.1, so that the proof of Proposition 5.1 is complete. □

Proof of Theorem 2.1. With the two estimates in Propositions 4.1 and 5.1 and by noting the definitions (2.9) and (2.10) of $|\vec{a}|_{X_\rho}$ and $|\vec{a}|_{Y_\rho}$, we have

$$\frac{1}{2}\frac{d}{dt}|\vec{a}|_{X_\rho}^2\leq\big(-\mu+C+C|\vec{a}|_{X_\rho}^4\big)|\vec{a}|_{Y_\rho}^2. \tag{5.2}$$

Now we choose μ large enough such that

$$\mu\geq\frac{1}{2}+C+C(2C_0)^4\Big(\|u_0\|_{G^{3/2,1}_{2\rho_0,\ell}}+\|u_1\|_{G^{3/2,1}_{2\rho_0,\ell+1}}\Big)^4, \tag{5.3}$$

where C_0 is the constant in (2.12). Then, under assumption (2.13), (5.2) and (5.3), yield

$$\frac{1}{2}\frac{d}{dt}|\vec{a}(t)|_{X_\rho}^2\leq-\frac{1}{2}|\vec{a}(t)|_{Y_\rho}^2,\quad\forall t\in[0,T].$$

By (2.12), we have

$$\sup_{0\leq t\leq T}|\vec{a}(t)|_{X_\rho}+\left(\int_0^T|\vec{a}(t)|_{Y_\rho}^2dt\right)^{\frac{1}{2}}\leq|\vec{a}(0)|_{X_\rho}\leq C_0\left(\|u_0\|_{G^{3/2,1}_{2\rho_0,\ell}}+\|u_1\|_{G^{3/2,1}_{2\rho_0,\ell+1}}\right).$$

This is (2.14), so that the proof of Theorem 2.1 is complete. □

6 The 3D hyperbolic Prandtl equation

The discussion on the 3D hyperbolic Prandtl model is similar to that of the 2D case with slight modifications on auxiliary functions. Precisely, we will use vector-valued auxiliary functions instead of the scalar ones used in the previous sections. We denote by $\vec{u}=(u_1,u_2)$ and v the tangential and normal velocities respectively, and by (x,y) the spatial variables in $\mathbb{R}^2\times\mathbb{R}_+$ with $x=(x_1,x_2)$. As the counterparts of the auxiliary functions defined in Section 2, we set

$$\vec{\varphi}=\partial_t\vec{u}+(\vec{u}\cdot\partial_x)\vec{u}+v\partial_y\vec{u} \quad \text{with} \quad v(t,x,y)=-\int_0^y \partial_x\cdot\vec{u}(t,x,\tilde{y})d\tilde{y}.$$

Moreover, we define $\vec{\mathcal{U}}=(\mathcal{U}_1,\mathcal{U}_2)$ and $\vec{\lambda}=(\lambda_1,\lambda_2,\lambda_3,\lambda_4)$ as follows. Let $\mathcal{U}_j, j=1,2$, solve the Cauchy problem

$$\begin{cases} (\partial_t+\vec{u}\cdot\partial_x+v\partial_y)\int_0^y \mathcal{U}_j(t,x,\tilde{y})d\tilde{y}=-\partial_{x_j}v, \\ \mathcal{U}_j|_{t=0}=0. \end{cases}$$

Accordingly, set

$$\begin{cases} \lambda_1=\partial_{x_1}u_1-(\partial_y u_1)\int_0^y \mathcal{U}_1(t,x,\tilde{y})d\tilde{y}, \\ \lambda_2=\partial_{x_2}u_1-(\partial_y u_1)\int_0^y \mathcal{U}_2(t,x,\tilde{y})d\tilde{y}, \\ \lambda_3=\partial_{x_1}u_2-(\partial_y u_2)\int_0^y \mathcal{U}_1(t,x,\tilde{y})d\tilde{y}, \\ \lambda_4=\partial_{x_2}u_2-(\partial_y u_2)\int_0^y \mathcal{U}_2(t,x,\tilde{y})d\tilde{y}. \end{cases}$$

We also denote that

$$\vec{b}=(\vec{u},\vec{\mathcal{U}},\vec{\lambda},\vec{\varphi}),$$

and define $|\vec{b}|_{X_\rho}$ and $|\vec{b}|_{Y_\rho}$ as in Definition 2.1. Then the a priori estimate in Theorem 2.1 also holds with $\vec{a}$ replaced by $\vec{b}$. This can be derived in the same way for the 2D case. For brevity, we omit the details.

Appendix A. Proof of some estimates

Finally, we present the proof of some estimates used in the previous sections.

Proof of (3.4). For any $0\le j\le [m/2]$, we have that $m-j\approx m$ so that

$$\begin{aligned}&\frac{m!}{j!(m-j)!}\times\frac{N_{\rho,m+1}}{N_{\rho,j+3}N_{m-j+1,\rho}}\\&=\frac{m!}{j!(m-j)!}\times\frac{\rho^{m+2}(m+2)^9}{[(m+1)!]^{\frac{3}{2}}}\times\frac{[(j+3)!]^{\frac{3}{2}}}{\rho^{j+4}(j+4)^9}\times\frac{[(m-j+1)!]^{\frac{3}{2}}}{\rho^{m-j+2}(m-j+2)^9}\\&\lesssim (j!)^{\frac{1}{2}}[(m-j)!]^{\frac{1}{2}}\times\frac{j^{\frac{9}{2}}(m-j+1)^{\frac{3}{2}}}{\rho^4(m!)^{\frac{1}{2}}(m+1)^{\frac{3}{2}}(j+4)^9}\lesssim\frac{1}{j+1},\end{aligned}$$

where we have used the fact that $p!q!\le(p+q)!$ and (2.16) in the last inequality. This gives estimate (3.4). □

Proof of (3.6). For any j with $[m/2]+1\le j\le m$, we have

$$\begin{aligned}&\frac{m!}{j!(m-j)!}\times\frac{N_{\rho,m+1}}{N_{\rho,j+1}N_{\rho,m-j+3}}\\&=\frac{m!}{j!(m-j)!}\times\frac{\rho^{m+2}(m+2)^9}{[(m+1)!]^{\frac{3}{2}}}\times\frac{[(j+1)!]^{\frac{3}{2}}}{\rho^{j+2}(j+2)^9}\times\frac{[(m-j+3)!]^{\frac{3}{2}}}{\rho^{m-j+4}(m-j+4)^9}\\&\lesssim (j!)^{\frac{1}{2}}[(m-j)!]^{\frac{1}{2}}\times\frac{j^{\frac{3}{2}}(m-j+1)^{\frac{9}{2}}}{\rho^4(m!)^{\frac{1}{2}}(m+1)^{\frac{3}{2}}(m-j+4)^9}\lesssim\frac{1}{m-j+1}.\end{aligned}$$

This gives (3.6). □

Proof of (4.9). For $1\le i\le[(k+1)/2]$ we have $k-i\approx k$ so that

$$\begin{aligned}&\frac{(k+1)!}{i!(k+1-i)!}\times\frac{(k+1)^{-\frac{1}{2}}L_{\rho,k}}{H_{\rho,4,i-1}L_{\rho,k+1-i}}\\&\lesssim\frac{(k+1)!}{i!(k+1-i)!}(k+1)^{-\frac{1}{2}}\times\frac{\rho^{k+2}(k+2)^9}{(k+1)!}\times\frac{(i+3)!}{\rho^{i+4}(i+4)^9}\times\frac{(k+2-i)!}{\rho^{k+3-i}(k+3-i)^9}\\&\lesssim\frac{(k+1)^{-\frac{1}{2}}(k+2-i)}{(i+1)^6}\lesssim\frac{(k+2-i)^{\frac{1}{2}}}{i+1}.\end{aligned}$$

Hence, (4.9) holds. □

Proof of (4.11). We use the fact that $[(k+1)/2]+1\le i\le k+1$ to have

$$\begin{aligned}&\frac{(k+1)!}{i!(k+1-i)!}\times\frac{(k+1)^{-\frac{1}{2}}L_{\rho,k}}{H_{\rho,2,i-2}H_{\rho,3,k+2-i}}\\&\lesssim\frac{(k+1)!}{i!(k+1-i)!}(k+1)^{-\frac{1}{2}}\times\frac{\rho^{k+2}(k+2)^9}{(k+1)!}\times\frac{i!}{\rho^{i+1}(i+1)^9}\times\frac{(k+5-i)!}{\rho^{k+6-i}(k+6-i)^9}\end{aligned}$$

$$\lesssim \frac{1}{(k+5-i)^5} \lesssim \frac{1}{k+3-i},$$

which is (4.11). □

Proof of (4.17). For any $1 \le j \le [m/2]$ so that $m-j \approx m$, and thus,

$$\begin{aligned}
&\frac{m!}{j!(m-j)!} \times \frac{(m+1)^{\frac{1}{2}} N_{\rho,m+1}}{N_{\rho,j+3} H_{\rho,m-j+1,1}} \\
&= \frac{m!(m+1)^{\frac{1}{2}}}{j!(m-j)!} \times \frac{\rho^{m+2}(m+2)^9}{[(m+1)!]^{\frac{3}{2}}} \times \frac{[(j+3)!]^{\frac{3}{2}}}{\rho^{j+4}(j+4)^9} \times \frac{(m-j+2)![(m-j+1)!]^{\frac{1}{2}}}{\rho^{m-j+3}(m-j+3)^9} \\
&\lesssim (j!)^{\frac{1}{2}} [(m-j)!]^{\frac{1}{2}} (m+1)^{\frac{1}{2}} \times \frac{j^{\frac{9}{2}} (m-j+1)^{\frac{5}{2}}}{(m!)^{\frac{1}{2}} (m+1)^{\frac{3}{2}} (j+4)^9} \lesssim \frac{(m-j+1)^{\frac{3}{2}}}{j+1}.
\end{aligned}$$

The proof is thus complete. □

Proof of (4.18). For $[m/2]+1 \le j \le m$, we have

$$\begin{aligned}
&\frac{m!}{j!(m-j)!} \times \frac{(m+1)^{\frac{1}{2}} N_{\rho,m+1}}{N_{\rho,j+1} H_{\rho,m-j+3,1}} \\
&= \frac{m!(m+1)^{\frac{1}{2}}}{j!(m-j)!} \times \frac{\rho^{m+2}(m+2)^9}{[(m+1)!]^{\frac{3}{2}}} \times \frac{[(j+1)!]^{\frac{3}{2}}}{\rho^{j+2}(j+2)^9} \times \frac{(m-j+4)![(m-j+3)!]^{\frac{1}{2}}}{\rho^{m-j+5}(m-j+5)^9} \\
&\lesssim (j!)^{\frac{1}{2}} [(m-j)!]^{\frac{1}{2}} (m+1)^{\frac{1}{2}} \times \frac{j^{\frac{3}{2}} (m-j+1)^{\frac{11}{2}}}{(m!)^{\frac{1}{2}} (m+1)^{\frac{3}{2}} (m-j+5)^9} \lesssim \frac{(j+1)^{\frac{1}{2}}}{m-j+1}.
\end{aligned}$$

This gives (4.18). □

Proof of (4.23). For $1 \le i+j \le [(m+k+1)/2]$, we have $m+k-i-j \approx m+k$ so that

$$\begin{aligned}
&\binom{m}{j}\binom{k+1}{i}(m+k+1)^{-\frac{1}{2}}(m+1) \times \frac{H_{\rho m+1,k}}{H_{\rho,j+4,i-1} H_{\rho,m-j+1,k+1-i}} \\
&= \frac{m!}{j!(m-j)!} \frac{(k+1)!}{i!(k+1-i)!} (m+k+1)^{-\frac{1}{2}} (m+1) \times \frac{\rho^{m+k+2}(m+k+2)^9}{(m+k+1)![(m+1)!]^{\frac{1}{2}}} \\
&\quad \times \frac{(i+j+3)![(j+4)!]^{\frac{1}{2}}}{\rho^{i+j+4}(i+j+4)^9} \times \frac{(m+k-i-j+2)![(m-j+1)!]^{\frac{1}{2}}}{\rho^{m+k-i-j+3}(m+k-i-j+3)^9} \\
&\lesssim \frac{(m!)^{\frac{1}{2}}}{(j!)^{\frac{1}{2}} [(m-j)!]^{\frac{1}{2}}} \frac{(k+1)!}{i!(k+1-i)!} \times \frac{(i+j+3)!(m+k-i-j+2)!}{(m+k+1)!}
\end{aligned}$$

$$\times(m+k+1)^{-\frac{1}{2}}(m+1)^{\frac{1}{2}}\times\frac{(j+1)^2(m-j+1)^{\frac{1}{2}}}{(i+j+4)^9}.$$

Moreover, by using

$$\binom{\alpha}{\beta}\leq\binom{|\alpha|}{|\beta|},\quad \forall\alpha,\beta\in\mathbb{Z}_+^2\quad\text{with}\quad\beta\leq\alpha,$$

we have

$$\begin{aligned}&\frac{(m!)^{\frac{1}{2}}}{(j!)^{\frac{1}{2}}[(m-j)!]^{\frac{1}{2}}}\times\frac{(k+1)!}{i!(k+1-i)!}\\&\leq\frac{m!}{j!(m-j)!}\times\frac{(k+1)!}{i!(k+1-i)!}\leq\frac{(m+k+1)!}{(i+j)!(m+k+1-i-j)!}.\end{aligned}$$

We then combine the above estimates and observe that $m+k-i-j\approx m+k$ for $1\leq i+j\leq[(m+k+1)/2]$ to have

$$\begin{aligned}&\binom{m}{j}\binom{k+1}{i}(m+k+1)^{-\frac{1}{2}}(m+1)\times\frac{H_{\rho,m+1,k}}{H_{\rho,j+4,i-1}H_{\rho,m-j+1,k+1-i}}\\&\lesssim(m+k+1)^{-\frac{1}{2}}(m+1)^{\frac{1}{2}}\times\frac{(j+1)^2(m-j+1)^{\frac{1}{2}}(i+j+1)^3(m+k-i-j+2)}{(i+j+4)^9}\\&\lesssim(m+1)^{\frac{1}{2}}(m-j+1)^{\frac{1}{2}}\frac{1}{(i+j+3)^4}(m+k-i-j+2)^{\frac{1}{2}}\\&\lesssim(j+4)(m-j+1)\frac{1}{(i+j+1)^2}(m+k-i-j+2)^{\frac{1}{2}}.\end{aligned}$$

The proof of (4.23) is complete. □

Proof of (4.24). For any $[(m+k+1)/2]\leq i+j\leq m+k+1$, following a similar argument as above, we have

$$\begin{aligned}&\binom{m}{j}\binom{k+1}{i}\frac{(m+k+1)^{-\frac{1}{2}}(m+1)H_{\rho,m+1,k}}{H_{\rho,j+2,i-2}H_{\rho,m-j+3,k+2-i}}\\&=\frac{m!}{j!(m-j)!}\frac{(k+1)!}{i!(k+1-i)!}(m+k+1)^{-\frac{1}{2}}(m+1)\times\frac{\rho^{m+k+2}(m+k+2)^9}{(m+k+1)![(m+1)!]^{\frac{1}{2}}}\\&\quad\times\frac{(i+j)![(j+2)!]^{\frac{1}{2}}}{\rho^{i+j+1}(i+j+1)^9}\times\frac{(m+k-i-j+5)![(m-j+3)!]^{\frac{1}{2}}}{\rho^{m+k-i-j+6}(m+k-i-j+6)^9}\end{aligned}$$

$$
\begin{aligned}
&\lesssim \frac{(m!)^{\frac{1}{2}}}{(j!)^{\frac{1}{2}}[(m-j)!]^{\frac{1}{2}}}\times\frac{(k+1)!}{i!(k+1-i)!}\times\frac{(i+j)!(m+k-i-j+5)!}{(m+k+1)!}\\
&\qquad\times(m+k+1)^{-\frac{1}{2}}(m+1)^{\frac{1}{2}}\times\frac{(j+1)(m-j+1)^{\frac{3}{2}}}{(m+k-i-j+6)^{9}}\\
&\lesssim (m+k+1)^{-\frac{1}{2}}(m+1)^{\frac{1}{2}}\times\frac{(j+1)(m-j+1)^{\frac{3}{2}}}{(m+k-i-j+6)^{5}}\lesssim\frac{(j+1)}{(m+k-i-j+2)^{2}}.
\end{aligned}
$$

Hence, (4.24) follows. □

Proof of (2.12). By (2.9), (2.5) and (2.6), we have

$$
\begin{aligned}
|\vec{a}(0)|^2_{X_{\rho_0}}=&\sum_{m=0}^{\infty}(m+1)N^2_{\rho_0,m+1}\big\|\langle y\rangle^{\ell-1}\partial_x^{m+1}u_0\big\|^2_{L^2}\\
&+\sum_{m,k\geq 0}(m+1)^2H^2_{\rho_0,m+1,k}\Big(\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\varphi_0\big\|^2_{L^2}+\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\partial_y u_0\big\|^2_{L^2}\Big).
\end{aligned}
$$

This together with

$$
\sum_{m,k\geq 0}(m+1)^2H^2_{\rho_0,m+1,k}\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\partial_y u_0\big\|^2_{L^2}\leq\|u_0\|^2_{G^{3/2,1}_{2\rho_0,\ell}},
$$

and

$$
\begin{aligned}
&\sum_{m=0}^{\infty}(m+1)N^2_{\rho_0,m+1}\big\|\langle y\rangle^{\ell-1}\partial_x^{m+1}u_0\big\|^2_{L^2}\\
=&\sum_{m=0}^{\infty}(m+1)\left(\frac{1}{2}\right)^{m+2}N^2_{2\rho_0,m+1}\big\|\langle y\rangle^{\ell-1}\partial_x^{m+1}u_0\big\|^2_{L^2}\leq 2\|u_0\|^2_{G^{3/2,1}_{2\rho_0,\ell}},
\end{aligned}
$$

yields

$$
|\vec{a}(0)|^2_{X_{\rho_0}}\leq 3\|u_0\|^2_{G^{3/2,1}_{2\rho_0,\ell}}+\sum_{m,k\geq 0}(m+1)^2H^2_{\rho_0,m+1,k}\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\varphi_0\big\|^2_{L^2}. \tag{A.1}
$$

It follows from (2.4) and $\ell\geq 2$ that

$$
\begin{aligned}
\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\varphi_0\big\|_{L^2}\leq&\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k u_1\big\|_{L^2}+C\sum_{j\leq m,i\leq k}\binom{m}{j}\binom{k}{i}\big\|\langle y\rangle^{\ell-1}\partial_x^{m+1-j}\partial_y^{k-i}u_0\big\|_{L^2}\\
&\times\Big(\big\|\langle y\rangle^{\ell-1}\partial_x^j\partial_y^i u_0\big\|_{L^\infty}+\big\|\langle y\rangle^{\ell}\partial_x^j\partial_y^i\partial_y u_0\big\|_{L^\infty_x L^2_y}\Big).
\end{aligned}
$$

Hence, by using Young's inequality (2.19) for discrete convolution and the fact that $kr^k \leq (1-r)^{-1}$ for any pair $(r,k)\in[0,1/2]\times\mathbb{Z}_+$, we conclude that

$$\sum_{m,k\geq 0}(m+1)^2 H^2_{\rho_0,m+1,k}\big\|\langle y\rangle^{\ell}\partial_x^m\partial_y^k\varphi_0\big\|^2_{L^2}\leq C\Big(\|u_1\|^2_{G^{3/2,1}_{2\rho_0,\ell+1}}+\|u_0\|^2_{G^{3/2,1}_{2\rho_0,\ell}}\Big).$$

This and (A.1) imply (2.12). □

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