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Global Regularity of the Vlasov-Poisson-Boltzmann System Near Maxwellian Without Angular Cutoff for Soft Potential

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Abstract. We consider the non-cutoff Vlasov-Poisson-Boltzmann (VPB) system of two species with soft potential in the whole space $\mathbb{R}^3$ when an initial data is near Maxwellian. Continuing the work Deng [Comm. Math. Phys. 387 (2021)] for hard potential case, we prove the global regularity of the Cauchy problem to VPB system for the case of soft potential in the whole space for the whole range $0<s<1$. This completes the smoothing effect of the Vlasov-Poisson-Boltzmann system, which shows that any classical solutions are smooth with respect to (t,x,v) for any positive time $t>0$. The proof is based on the time-weighted energy method building upon the pseudo-differential calculus.

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Key words: Vlasov-Poisson-Boltzmann system, regularity, without angular cutoff, regularizing effect, soft potentials.

1 Introduction

The Vlasov-Poisson-Boltzmann system is an important physical model to describe the time evolution of plasma particles of two species (e.g. ions and electrons). In this work, we study the smoothing effect of solutions to non-cutoff

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Vlasov-Poisson-Boltzmann system with $-3/2-2s<\gamma\le -2s$ and $0<s<1$. We find that the solutions enjoy the same smoothing phenomenon as the Boltzmann equation, which gives the regularity of the Vlasov-Poisson-Boltzmann system. Since Duan and Liu [17] found the global solution for non-cutoff soft potential with $1/2\le s<1$, the smoothing effect for the VPB system is an open interesting problem. In [14], the author finds out the smoothing effect for hard potential. In this work, we finally recover the smoothing effect for non-cutoff soft potential with the whole range $0<s<1$.

1.1 Equations

We consider the Vlasov-Poisson-Boltzmann system of two species in the whole space $\mathbb{R}^3$, cf. [22,26]

$$\begin{aligned}
&\partial_t F_+ + v\cdot\nabla_x F_+ + E\cdot\nabla_v F_+ = Q(F_+,F_+)+Q(F_-,F_+),\\
&\partial_t F_- + v\cdot\nabla_x F_- - E\cdot\nabla_v F_- = Q(F_-,F_-)+Q(F_+,F_-).
\end{aligned}\tag{1.1}$$

The self-consistent electrostatic field is taken as $E(t,x)=-\nabla_x\phi$, with the electric potential ϕ given by

$$-\Delta_x\phi=\int_{\mathbb{R}^3}(F_+-F_-)\,dv,\quad \phi\ \to\ 0\quad \text{as}\quad |x|\to\infty.\tag{1.2}$$

The initial data of the system is

$$F_\pm(0,x,v)=F_{\pm,0}(x,v).\tag{1.3}$$

The unknown function $F_\pm(t,x,v)\ge 0$ represents the velocity distribution for the particle with position $x\in\mathbb{R}^3$ and velocity $v\in\mathbb{R}^3$ at time $t\ge 0$. The bilinear collision term $Q(F,G)$ on the right-hand side of (1.1) is given by

$$Q(F,G)(v)=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-v_*,\sigma)\big(F'_*G'-F_*G\big)\,d\sigma dv_*,$$

where

$$F'=F(x,v',t),\quad G'_*=G(x,v'_*,t),\quad F=F(x,v,t),\quad G_*=G(x,v_*,t).$$

Velocity pairs (v,v_*) and (v',v'_*) are velocities before and after binary elastic collision respectively. They are defined by

$$v'=\frac{v+v_*}{2}+\frac{|v-v_*|}{2}\sigma,\quad v'_*=\frac{v+v_*}{2}-\frac{|v-v_*|}{2}\sigma.$$

These two pairs of velocities satisfy the conservation law of momentum and energy

$$v+v_*=v'+v'_*,\quad |v|^2+|v_*|^2=|v'|^2+|v'_*|^2.$$

1.2 Collision kernel

The Boltzmann collision kernel B is defined as

$$B(v-v_*,\sigma)=|v-v_*|^\gamma b(\cos\theta)$$

for some function b and γ determined by the intermolecular interactive mechanism with $\cos\theta=((v-v_*)/|v-v_*|)\cdot\sigma$. Without loss of generality, we can assume $B(v-v_*,\sigma)$ is supported on $(v-v_*)\cdot\sigma\geq 0$, which corresponds to $\theta\in(0,\pi/2]$, since B can be replaced by its symmetrized form

$$\overline{B}(v-v_*,\sigma)=B(v-v_*,\sigma)+B(v-v_*,-\sigma)$$

in $Q(f,f)$. The angular function $\sigma\mapsto b(\cos\theta)$ is not integrable on $\mathbb{S}^2$. Moreover, there exists $0<s<1$ such that

$$\frac{1}{C}\theta^{-1-2s}\leq \sin\theta b(\cos\theta)\leq C\theta^{-1-2s} \quad \text{on } \theta\in(0,\pi/2]$$

for some $C>0$. It is convenient to call soft potential when $\gamma+2s<0$, and hard potential when $\gamma+2s\geq 0$. In this work, we always assume

$$0<s<1, \quad -3/2<\gamma\leq -2s.$$

In this paper, we are going to establish the smoothing effect of the solutions to the Cauchy problem (1.1)-(1.3) of the Vlasov-Poisson-Boltzmann system near the global Maxwellian equilibrium. For global existence, Guo [22] firstly investigates the hard-sphere model of the Vlasov-Poisson-Boltzmann system in a periodic box. Since then, the energy method was largely developed for the Boltzmann equation with the self-consistent electric and magnetic fields. Duan and Strain [19] analyzes the optimal time decay rate for the Vlasov-Maxwell-Boltzmann system with cutoff hard potential. Guo [24] gives the global existence of the Vlasov-Poisson-Landau system by using an elegant weight $e^{\pm\phi}$. Duan and Liu [17] investigated the Vlasov-Poisson-Boltzmann system without angular cutoff for the case of soft potential when $1/2\leq s<1$. For the smoothing effect of the Boltzmann equation, since the work [1] discovered the entropy dissipation property for the non-cutoff linearized Boltzmann operator, there have been many discussions in different contexts. See [2,5,7,21,29] for the dissipation estimate of collision operator, and [3,4,8,10,11,13] for C^∞ smoothing effect for the solution to Boltzmann equation in a different aspect. We refer to [9,16] for the Gevrey smoothing effect for the spatially inhomogeneous Boltzmann equation. Recently, the author [14,15] established the smoothing effect of the Cauchy problem for VPB system with hard potential and VPL system for Coulomb interactions. These works show that the Boltzmann operator behaves locally like a fractional operator

$$Q(f,g) \sim (-\Delta_v)^s g + \text{lower order terms}.$$

More precisely, according to the symbolic calculus developed by [2], the linearized Boltzmann operator behaves essentially as

$$L \sim \langle v \rangle^\gamma \big(-\Delta_v + |v \wedge \partial_v|^2 + |v|^2\big)^s + \text{lower order terms}.$$

We also mention [25] for global regularity of the Boltzmann equation without angular cutoff.

1.3 Reformulation

We will reformulate the problem near Maxwellian as in [22]. For this, we denote a normalized global Maxwellian μ by

$$\mu(v) = (2\pi)^{-\frac{3}{2}} e^{-\frac{|v|^2}{2}}.$$

Set $F_\pm(t,x,v) = \mu(v) + \mu^{1/2} f_\pm(t,x,v)$. Denote $f = (f_+, f_-)$ and $f_0 = (f_{+,0}, f_{-,0})$. Then the Cauchy problem (1.1)-(1.3) can be reformulated as

$$\partial_t f_\pm + v \cdot \nabla_x f_\pm \pm \frac{1}{2} \nabla_x \phi \cdot v f_\pm \mp \nabla_x \phi \cdot \nabla_v f_\pm \pm \nabla_x \phi \cdot v \mu^{\frac{1}{2}} - L_\pm f = \Gamma_\pm(f,f), \tag{1.4}$$

$$-\Delta_x \phi = \int_{\mathbb{R}^3} (f_+ - f_-) \mu^{\frac{1}{2}} dv, \quad \phi \to 0 \quad \text{as} \quad |x| \to \infty \tag{1.5}$$

with initial data

$$f_\pm(0,x,v) = f_{\pm,0}(x,v). \tag{1.6}$$

The linearized operator $L = (L_+, L_-)$ and bilinear collision operator $\Gamma = (\Gamma_+, \Gamma_-)$ are given by

$$\begin{aligned} L_\pm f &= \mu^{-\frac{1}{2}} \Big(2Q\big(\mu, \mu^{\frac{1}{2}} f_\pm\big) + Q\big(\mu^{\frac{1}{2}}(f_\pm + f_\mp), \mu\big) \Big), \\ \Gamma_\pm(f,g) &= \mu^{-\frac{1}{2}} \Big(Q\big(\mu^{\frac{1}{2}} f_\pm, \mu^{\frac{1}{2}} g_\pm\big) + Q\big(\mu^{\frac{1}{2}} f_\mp, \mu^{\frac{1}{2}} g_\pm\big) \Big). \end{aligned}$$

For later use, we introduce the bilinear operator $\mathcal{T}$ by

$$\mathcal{T}_\beta(h_1,h_2) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_*, \sigma) \partial_\beta\big(\mu_*^{\frac{1}{2}}\big) \big(h_1(v'_*) h_2(v') - h_1(v_*) h_2(v)\big) d\sigma dv_*$$

for two scalar functions h_1, h_2, and in particular, we set $\mathcal{T} = \mathcal{T}_0$. Thus,

$$\begin{aligned} L_\pm f &= 2\mathcal{T}\big(\mu^{\frac{1}{2}}, f_\pm\big) + \mathcal{T}\big(f_\pm + f_\mp, \mu^{\frac{1}{2}}\big), \\ \Gamma_\pm(f,g) &= \mathcal{T}(f_\pm, g_\pm) + \mathcal{T}(f_\mp, g_\pm). \end{aligned}$$

1.4 Notations

Through the paper, C denotes some positive constant (generally large) and λ denotes some positive constant (generally small), where both C and λ may take different values in different lines. For any $v\in\mathbb{R}^3$, we denote $\langle v\rangle=(1+|v|^2)^{1/2}$. For multi-indices $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,\beta_2,\beta_3)$, write

$$\partial^\alpha_\beta=\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}\partial^{\beta_1}_{v_1}\partial^{\beta_2}_{v_2}\partial^{\beta_3}_{v_3}.$$

The length of α is $|\alpha|=\alpha_1+\alpha_2+\alpha_3$. The notation $a\approx b$ (respectively $a\gtrsim b, a\lesssim b$) for positive real function a,b means there exists $C>0$ not depending on possible free parameters such that $C^{-1}a\le b\le Ca$ (respectively $a\ge C^{-1}b, a\le Cb$) on their domain. $\mathscr{S}$ denotes the Schwartz space. $\mathrm{Re}(a)$ means the real part of complex number a. $[a,b]=ab-ba$ is the commutator between operators. $\{a(v,\eta),b(v,\eta)\}=\partial_\eta a_1\partial_v a_2-\partial_v a_1\partial_\eta a_2$ is the Poisson bracket. $\Gamma=|dv|^2+|d\eta|^2$ is the admissible metric and $S(m)=S(m,\Gamma)$ is the symbol class. For pseudo-differential calculus, we write $(x,v)\in\mathbb{R}^3\times\mathbb{R}^3$ to be the space-velocity variable and $(y,\eta)\in\mathbb{R}^3\times\mathbb{R}^3$ to be the corresponding variable in frequency space (the variable after Fourier transform). The $L^2_{v,x}$ space is defined as $L^2_{v,x}=L^2(\mathbb{R}^3_v\times\mathbb{R}^3_x)$. $L^2(B_C)$ is the L^2_v space on Euclidean ball B_C of radius C at the origin. For the usual Sobolev space, we will use the notation

$$\|f\|_{H^k_vH^m_x}=\sum_{|\beta|\le k,|\alpha|\le m}\big\|\partial^\alpha_\beta f\big\|_{L^2_{v,x}}$$

for $k,m\ge 0$. We also define the standard velocity-space mixed Lebesgue space $Z_1=L^2(\mathbb{R}^3_v;L^1(\mathbb{R}^3_x))$ with the norm

$$\|f\|_{Z_1}=\big\|\|f\|_{L^1_x}\big\|_{L^2_v}.$$

In this paper, we write Fourier transform and inverse Fourier transform on x as

$$\widehat{f}(y)=\int_{\mathbb{R}^3}f(x)e^{-ix\cdot y}dx,\quad f^\vee(x)=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3}f(y)e^{iy\cdot x}dx.$$

(i) As in [23], the null space of L is given by

$$\ker L=\mathrm{span}\Big\{[1,0]\mu^{\frac12},[0,1]\mu^{\frac12},[1,1]v_i\mu^{\frac12}\,(1\le i\le 3),[1,1]|v|^2\mu^{\frac12}\Big\}.$$

We denote $\mathbf{P}_\pm$ to be the orthogonal projection from $L^2_v\times L^2_v$ onto $\ker L$, which is defined by

$$\mathbf{P}f=\big(a_+(t,x)[1,0]+a_-(t,x)[0,1]+v\cdot b(t,x)[1,1]+(|v|^2-3)c(t,x)[1,1]\big)\mu^{\frac12},\quad (1.7)$$

or equivalently by

$$\mathbf{P}_{\pm}f=\big(a_{\pm}(t,x)+v\cdot b(t,x)+(|v|^2-3)c(t,x)\big)\mu^{\frac{1}{2}}.$$

Then for given f, one can decompose f uniquely as

$$f=\mathbf{P}f+(\mathbf{I}-\mathbf{P})f.$$

The function $a_{\pm},b,c$ are given by

$$\begin{aligned}
&a_{\pm}=\big(\mu^{\frac{1}{2}},f_{\pm}\big)_{L^2_v}=\big(\mu^{\frac{1}{2}},\mathbf{P}_{\pm}f\big)_{L^2_v},\\
&b_j=\frac{1}{2}\big(v_j\mu^{\frac{1}{2}},f_{+}+f_{-}\big)_{L^2_v}=\big(v_j\mu^{\frac{1}{2}},\mathbf{P}_{\pm}f\big)_{L^2_v},\\
&c=\frac{1}{12}\big((|v|^2-3)\mu^{\frac{1}{2}},f_{+}+f_{-}\big)_{L^2_v}=\frac{1}{6}\big((|v|^2-3)\mu^{\frac{1}{2}},\mathbf{P}_{\pm}f\big)_{L^2_v}.
\end{aligned}$$

(ii) To describe the behavior of linearized Boltzmann collision operator, [6] introduce the norm $|||f|||$ while [20] introduce the norm $N_l^{s,\gamma}$. The work [2] give the pseudo-differential-type norm $\|(\tilde{a}^{1/2})^w f\|_{L^2_v}$. They are all equivalent and we list their results as follows.

Let $\mathscr{S}'$ be the space of tempered distribution functions. $N^{s,\gamma}$ denotes the weighted geometric fractional Sobolev space

$$N^{s,\gamma}=\{f\in\mathscr{S}':|f|_{N^{s,\gamma}}<\infty\}$$

with the anisotropic norm

$$|f|^2_{N^{s,\gamma}}:=\big\|\langle v\rangle^{\frac{\gamma}{2}+s}f\big\|^2_{L^2}+\int\big(\langle v\rangle\langle v'\rangle\big)^{\frac{\gamma+2s+1}{2}}\frac{(f'-f)^2}{d(v,v')^{d+2s}}\mathbf{1}_{d(v,v')\le 1}$$

with

$$d(v,v'):=\sqrt{|v-v'|^2+\frac{1}{4}(|v|^2-|v'|^2)^2}.$$

In order to describe the velocity weight $\langle v\rangle$, as in [20], we define

$$|f|^2_{N^{s,\gamma}_l}=\big|w_l\langle v\rangle^{\frac{\gamma}{2}+s}f\big|^2_{L^2_v}+\int_{\mathbb{R}^3}dv\,w_l\langle v\rangle^{\gamma+2s+1}\int_{\mathbb{R}^3}dv'\,\frac{(f'-f)^2}{d(v,v')^{d+2s}}\mathbf{1}_{d(v,v')\le 1},$$

which turns out to be equivalent with $|w_lf|_{N^{s,\gamma}}$. This follows from the proof of [20, Proposition 5.1] since the ψ therein has a nice support.

On the other hand, as in [6], we define

$$|||f|||^2 := \int B(v-v_*,\sigma)\Big(\mu_*(f'-f)^2 + f_*^2\big((\mu')^{\frac{1}{2}} - \mu^{\frac{1}{2}}\big)^2\Big)\, d\sigma dv_* dv.$$

For pseudo-differential calculus as in [2], one may refer to the [14, Appendix] as well as [28] for more information. Let $\Gamma = |dv|^2 + |d\eta|^2$ be an admissible metric. We say that $a \in S(M) = S(M,\Gamma)$, if for $\alpha,\beta \in \mathfrak{N}^d, v,\eta \in \mathbb{R}^3$,

$$\big|\partial_v^\alpha \partial_\eta^\beta a(v,\eta,\xi)\big| \le C_{\alpha,\beta} M$$

with $C_{\alpha,\beta}$ a constant depending only on α and β. The space $S(M,\Gamma)$ endowed with the seminorms

$$\|a\|_{k;S(M,\Gamma)} = \max_{0\le|\alpha|+|\beta|\le k} \sup_{(v,\eta)\in\mathbb{R}^{2d}} \big|M(v,\eta)^{-1}\partial_v^\alpha \partial_\eta^\beta a(v,\eta,\xi)\big|$$

becomes a Fréchet space. Define

$$\tilde{a}(v,\eta) := \langle v\rangle^\gamma (1+|\eta|^2 + |\eta\wedge v|^2 + |v|^2)^s + K_0 \langle v\rangle^{\gamma+2s} \tag{1.8}$$

to be a Γ-admissible weight, where $K_0>0$ is chosen as the following. Applying [2, Theorem 4.2] and [12, Lemmas 2.1, 2.2], there exists $K_0>0$ such that the Weyl quantizations $\tilde{a}^w : H(\tilde{a}c) \to H(c)$ and $(\tilde{a}^{1/2})^w : H(\tilde{a}^{1/2}c) \to H(c)$ are invertible, with c being any Γ-admissible metric. The weighted Sobolev space $H(c)$ is defined by

$$H(M,\Gamma) := \big\{u \in \mathscr{S}' : \|u\|_{H(M,\Gamma)} < \infty\big\},$$

where

$$\|u\|_{H(M,\Gamma)} := \int M(Y)^2 \|\varphi_Y^w u\|_{L^2}^2 |\Gamma_Y|^{\frac{1}{2}} dY < \infty,$$

and $(\varphi_Y)_{Y\in\mathbb{R}^{2d}}$ is any uniformly confined family of symbols which is a partition of unity. If $a \in S(M)$ is a isomorphism from $H(M')$ to $H(M'M^{-1})$, then $(a^w u, a^w v)$ is an equivalent Hilbertian structure on $H(M)$. The symbol $\tilde{a}$ is real and gives the formal self-adjointness of Weyl quantization $\tilde{a}^w$. By the invertibility of $(\tilde{a}^{1/2})^w$, we have equivalence

$$\Big\|(\tilde{a}^{\frac{1}{2}})^w(\cdot)\Big\|_{L_v^2} \approx \|\cdot\|_{H(\tilde{a}^{\frac{1}{2}})_v},$$

and hence, we will equip $H(\tilde{a}^{1/2})_v$ with norm $\|(\tilde{a}^{1/2})^w(\cdot)\|_{L_v^2}$, see [14, Appendix]. Also,

$$\big\|w_l(\tilde{a}^{\frac{1}{2}})^w(\cdot)\big\|_{L_v^2} \approx \big\|(\tilde{a}^{\frac{1}{2}})^w w_l(\cdot)\big\|_{L_v^2}$$

due to Lemma 2.1.

The three norms defined above are equivalent since for $l\in\mathbb{R}$,

$$\|(\tilde{a}^{\frac{1}{2}})^w f\|_{L^2_v}^2\approx |||f|||^2\approx |f|^2_{N^{s,\gamma}}\approx(-Lf,f)_{L^2_v}+\|\langle v\rangle^l f\|_{L^2_v},$$

which follows from [20, Eqs. (2.13), (2.15)], [6, Proposition 2.1] and [2, Theorem 1.2]. An important result from [12, Section 3] is that

$$L\in S(\tilde{a}),$$

where $S(\tilde{a})=S(\tilde{a},\Gamma)$ is the pseudo-differential symbol class, see [28, Chapter 2]. This implies that

$$|(Lf,f)_{L^2_v}|\lesssim\|(\tilde{a}^{\frac{1}{2}})^w f\|^2_{L^2}.$$

For brevity, we denote dissipation norms

$$\|f\|_{L^2_D}=\|(\tilde{a}^{\frac{1}{2}})^w f\|_{L^2_v},\quad \|f\|_{L^2_xL^2_D}=\|(\tilde{a}^{\frac{1}{2}})^w f\|_{L^2_xL^2_v}.$$

In order to extract the smoothing effect on x, we define a symbol $\tilde{b}$ by

$$\tilde{b}(v,y)=\langle v\rangle^{l_0}|y|^{\delta_1}, \tag{1.9}$$

where l_0,δ_1 are defined by (3.16). This symbol will help us find out the smoothing rate on spatial variables.

1.5 Main results

To state the result of the paper, we let $K\geq 0$ to be the total order of derivatives on v,x and define the velocity weight function w_l for any $l\in\mathbb{R}$ by

$$w_l(\alpha,\beta)=\langle v\rangle^{l-p|\alpha|-q|\beta|+Kp}, \tag{1.10}$$

where $p,q>0$ are given by

$$p=-\gamma-\frac{2\gamma(1-s)}{s}+1,\quad q=-\frac{2\gamma}{s}+1.$$

For brevity, we write $w_l=w_l(0,0)$ and $w(|\alpha|,|\beta|)=w(\alpha,\beta)$. To extract the smoothing effect, as in [14], we define a useful coefficient

$$\psi_k=\begin{cases}1, & \text{if}\quad k\leq 0,\\ \psi^k, & \text{if}\quad k>0,\end{cases} \tag{1.11}$$

where $\psi=1$ in Theorem 2.1 (for existence) and $\psi=t^N$ with $N=N(\alpha,\beta)>0$ large in Theorem 1.1 and Section 3 (for regularity). When considering $\psi=t^N$ in proving regularity, we always assume $0\le t\le 1$, since regularity is a local property. In any case, we have $\psi\le 1$. The motivation of this weight is that when $\psi=t^N$, the initial high-order energy functional defined in (1.12) would vanish at the initial time $t=0$. This shows that high-order energy for any $t>0$ is controlled by low-order initial energy and we obtain the regularizing effect.

Corresponding to given $f=f(t,x,v)$, we introduce the instant energy functional $\mathcal{E}_{K,l}(t)$ satisfying the equivalent relation

$$\begin{aligned}\mathcal{E}_{K,l}(t)\approx&\sum_{|\alpha|\le K}\left\|\psi_{|\alpha|-4}\partial^\alpha E\right\|^2_{L^2_x}+\sum_{|\alpha|\le K}\left\|\psi_{|\alpha|-4}\partial^\alpha \mathbf{P}f\right\|^2_{L^2_{v,x}}\\&+\sum_{|\alpha|+|\beta|\le K}\left\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{L^2_{v,x}}.\end{aligned}\tag{1.12}$$

The precise definition will be given in (3.14). Also, we define the dissipation rate functional $\mathcal{D}_{K,l}$ by

$$\begin{aligned}\mathcal{D}_{K,l}(t)=&\sum_{|\alpha|\le K-1}\left\|\psi_{|\alpha|-4}\partial^\alpha E\right\|^2_{L^2_x}+\sum_{1\le|\alpha|\le K}\left\|\psi_{|\alpha|-4}\partial^\alpha \mathbf{P}f\right\|^2_{L^2_{v,x}}\\&+\sum_{|\alpha|+|\beta|\le K}\left\|\psi_{|\alpha|+|\beta|-4}\big(\tilde{a}^{\frac12}\big)^w w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{L^2_{v,x}}.\end{aligned}\tag{1.13}$$

Here $E=E(t,x)$ is determined by $f(t,x,v)$ in terms of $E=-\nabla_x\phi$ and (1.5). Notice that one can change the order of $(\tilde{a}^{1/2})^w$ and $w_l(\alpha,\beta)$ due to Lemma 2.1. The main result of this paper is stated as follows.

Theorem 1.1. *Let $-3/2-2s<\gamma\le-2s, 0<s<1, 0<\tau<T\le\infty$ and $l\ge 0$. For any $K\ge 4$ and multi-indices $|\alpha|+|\beta|\le K$, assume $\psi=t^N$ with $N>1$ large if $|\alpha|=0$ or $|\alpha|+|\beta|\le 4$, and $N=N(\alpha,\beta)>1$ defined by* (3.19) *or* (3.22) *or* (3.23) *otherwise. Let (f,E) be the solution to* (1.4)-(1.6) *satisfying that for $n>0$, there exists $C_n>0$ such that*

$$\sup_{0\le t\le T}\|\langle v\rangle^n f(t)\|_{L^2_{v,x}}\le C_n<\infty.\tag{1.14}$$

Then the following holds true:

(1) If

$$\epsilon_1=\big(\mathcal{E}_{4,l}(0)\big)^{\frac12}\tag{1.15}$$

is sufficiently small, then for $|\alpha|+|\beta|\le K, T<\infty$,

$$\sup_{\tau\le t\le T}\Big(\|w_l(\alpha,\beta)\partial^\alpha_\beta f\|^2_{L^2_{v,x}}+\|\partial^\alpha\nabla_x\phi\|^2_{L^2_x}\Big)\le\epsilon_1^2 C_{\tau,T,K,l}, \tag{1.16}$$

where $C_{\tau,T,K,l}>0$ *depends on* τ,T,K,l.

(2) There exists $C_{K,l}>0$ *such that if* $\mathcal{E}_{4,C_{K,l}}(0)$ *is sufficiently small, then for* $|\alpha|+|\beta|\le K$, $k\ge 0$, $T<\infty$, *we have*

$$\sup_{\tau\le t\le T}\Big(\|w_l(\alpha,\beta)\partial^\alpha_\beta\partial^k_t f\|^2_{L^2_{v,x}}+\|\partial^\alpha\partial^k_t\nabla_x\phi\|^2_{L^2_x}\Big)\le C_{\tau,T,k,K,l}<\infty, \tag{1.17}$$

where $C_{\tau,T,k,K,l}$ *is a constant depending on* τ,T,k,K,l. *Consequently, we obtain that* $f\in C^\infty(\mathbb{R}^+_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v)$.

(3) If additionally, the initial data satisfies that

$$\epsilon_0=\big(\mathcal{E}_{4,l+l_1}(0)\big)^{\frac12}+\|w_{l_2}f_0\|_{Z_1}+\|E_0\|_{L^1_x} \tag{1.18}$$

is sufficiently small, where

$$l>\max\Big\{-\frac{3(\gamma+2s)}{4},K\Big\},\quad l_1=-\frac{5(\gamma+2s)}{4(1-p)},\quad l_2>-\frac{5(\gamma+2s)}{4}$$

are constants. Then the constants in (1.16) *and* (1.17) *can be chosen independent of* T *and* T *can take the value* ∞.

(4) Suppose that there exists sufficiently large $C_{K,l}>0$ *such that if the solution* (f,E) *satisfies that,*

$$\epsilon_{0,K,l}=\big(\mathcal{E}_{4,C_{K,l}+l_1}(0)\big)^{\frac12}+\|w_{l_2}f_0\|_{Z_1}+\|E_0\|_{L^1_x} \tag{1.19}$$

is sufficiently small. Then the condition (1.14) *can be removed and we have* (1.16) *and* (1.17). *Also, the constants in* (1.16) *and* (1.17) *can be chosen independent of* T *and* T *can take the value* ∞.

One way to obtain the existence of solutions to the VPB system with soft potentials satisfying (1.14) could be provided for example by using the weighted arguments in [18], which is for the Vlasov-Maxwell-Boltzmann system with angular non-cutoff and soft potentials, if we replace the weight function $\langle v\rangle^{\gamma\tau}\times\exp\{\lambda/(1+t)^\vartheta\langle v\rangle\}$ by $\langle v\rangle^{\gamma\tau}\exp\{q\langle v\rangle+\lambda/(1+t)^\vartheta\langle v\rangle\}$ (with $0<q\ll 1$) in which

case we can obtain the sub-exponential decay in v. However, this is not the main topic of the current work, so we do not give detailed proof.

Notice that (1.16) gives the smoothing effect on velocity and spatial variables. If we assume the initial data has more velocity decay, then we have the smoothing effect on the time variable as (1.17). If we assume the initial data as in the existence theory (cf. Theorem 2.1), then the constants can be independent of time T. Moreover, if we assume higher velocity decay, then we can derive (1.14) from existence theory instead of assuming it at the beginning. These results show that the solutions to the Vlasov-Poisson-Boltzmann system enjoy a similar smoothing effect to the Boltzmann equation, see [3,11]. That is, whenever the initial data has algebraic decay in any order, the solution f is smooth in (t,x,v) for any positive time t.

In what follows let us point out several technical points in the proof of Theorem 1.1. We use $K\geq 4$ because $H^2_x(\mathbb{R}^3)$ is a Banach algebra when controlling (2.4), where there are already second derivatives on v, and H^2_x is useful to control the spatial variable when dealing with the trilinear estimate. The next technical point concerns the choice of $\psi=t^N$ in Theorem 1.1 and the usage of $\tilde{b}$, $\psi_{|\alpha|+|\beta|-4}$ is Section 3. Recall (1.11) for the definition of ψ_k. Whenever $|\alpha|+|\beta|>4$, $\psi_{|\alpha|+|\beta|-4}=t^{N(|\alpha|+|\beta|-4)}$ is equal to 0 at $t=0$. Plugging this into energy estimate, the higher order derivatives are canceled at $t=0$ and one can control the higher order instant energy by lower order initial data. Then one can easily deduce the smoothing effect locally in time. By using the global energy control obtained in Theorem 2.1, the local-in-time regularity becomes global-in-time regularity. Notice that we use -4 to eliminate the index arising from Sobolev embedding $\|\cdot\|_{H^2_vL^\infty_x}\lesssim\|\cdot\|_{H^2_vH^2_x}$, where the latter has derivatives of forth order. However, after adding $\psi_{|\alpha|+|\beta|-4}$, one need to deal with the term

$$\Big(\partial_t(\psi_{|\alpha|+|\beta|-4})\partial^\alpha_\beta f,e^{\pm\phi}w^2_l(\alpha,\beta)\partial^\alpha_\beta f\Big)_{L^2_{v,x}}. \tag{1.20}$$

This is where we need $\tilde{b}$ given in (1.9). By choosing $N=N(\alpha,\beta)$ properly, one has interpolation

$$\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}\lesssim\delta\tilde{b}^{\frac12}\psi_{|\alpha|+|\beta|-4}+C_{0,\delta}\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}|y|^{-|\alpha|}\psi_{|\beta|-4-\frac{1}{2N}}.$$

The first term can be absorbed while the second term eliminates α derivatives on x. Applying a similar interpolation on v with $\tilde{a}$, we can control (1.20) by a high-order term and an algebraic decay term

$$\delta^2\big\|\psi_{|\alpha|+|\beta|-4}\tilde{b}^{\frac12}w_l(\alpha,\beta)\big(\partial^\alpha_\beta f\big)^\wedge(v,y)\big\|^2_{L^2_{v,y}}+\delta^2\mathcal{D}_{K,l}+C_\delta\|\langle v\rangle^{C_{K,l}}f\|^2_{L^2_{v,x}}.$$

Defining θ by (3.18), using the Eqs. (1.4)-(1.6) and Poisson bracket $\{v\cdot y,\theta\}$, one can control the high-order term by using functional $\mathcal{E}_{K,l}$ and $\mathcal{D}_{K,l}$, where δ_1 in $\tilde{b}$ should be chosen properly. Hence, we can obtain a closed energy estimate locally. Here, when dealing with soft potential, there occurs an algebraic decay term in v: $\|\langle v\rangle^{C_{K,l}} f\|_{L^2_{v,x}}$ and we need to assume such norms are bounded initially, as observed in the Boltzmann equation, cf. [11]. After obtaining a local regularity, we can combine it with the global energy control from existence theory, cf. [17]. Then one can deduce the regularity globally in time.

The rest of the paper is arranged as follows. In Section 2, we present some basic lemmas for existence theory, estimate on L,Γ, and some tricks in energy estimates. In Section 3, we present the proof for regularity.

2 Preliminaries

In this section, we list several basic lemmas corresponding to the existence theory of the Vlasov-Poisson-Boltzmann system, linearized Boltzmann collision term $L_\pm$ and the bilinear Boltzmann collision operator $\Gamma_\pm$. The following theorem comes from [17, Theorem 1.1], except that we improve the index $K\geq 8$ to $K\geq 4$ and $1/2\leq s<1$ to $0<s<1$.

Theorem 2.1 ([17, Theorem 1.1]). *Let* $-3/2-2s<\gamma\leq -2s, 0<s<1, K\geq 4, p\in(1/2,1)$. *Assume* $l\geq 0, l>-3(\gamma+2s)/4, l_1=-5(\gamma+2s)/(4(1-p))$ *and* $f_0(x,v)=(f_{0,+}(x,v), f_{0,-}(x,v))$ *satisfying*

$$F_\pm(0,x,v)=\mu(v)+\sqrt{\mu(v)}f_{0,\pm}(x,v)\geq 0.$$

Assume $\psi=1$. *If*

$$\epsilon_0=\big(\mathcal{E}_{K,l+l_1}(0)\big)^{\frac{1}{2}}+\|w_{l_2}f_0\|_{Z_1}+\|E_0\|_{L^1_x} \tag{2.1}$$

is sufficiently small, where $E_0(x)=E(0,x), l_2>-5(\gamma+2s)/4$ *is a constant. Then there exists a unique global solution* $f(t,x,v)$ *to the Cauchy problem* (1.4)-(1.6) *of the Vlasov-Poisson-Boltzmann system such that*

$$F_\pm(t,x,v)=\mu(v)+\big(\mu(v)\big)^{\frac{1}{2}}f_\pm(t,x,v)\geq 0,$$

and

$$\begin{aligned}&\mathcal{E}_{K,l+l_1}(t)\lesssim\epsilon_0^2,\\&\mathcal{E}_{K,l}(t)\lesssim\epsilon_0^2(1+t)^{-\frac{3}{2}},\\&\mathcal{E}^h_{K,l}(t)\lesssim\epsilon_0^2(1+t)^{-\frac{3}{2}-p},\end{aligned} \tag{2.2}$$

for any $t\geq 0$. *Here,* $\mathcal{E}_{K,l}(t)$ *is defined by* (1.12) *with* $\psi=1$.

Here the instant energy functional $\mathcal{E}^h_{K,l}$ is given by

$$\begin{aligned}\mathcal{E}^h_{K,l}(t) \approx &\sum_{|\alpha|\le K} \|\partial^\alpha E(t)\|^2_{L^2_x} + \sum_{1\le|\alpha|\le K} \|\partial^\alpha \mathbf{P} f\|^2_{L^2_{v,x}} \\ &+ \sum_{|\alpha|+|\beta|\le K} \left\|w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f\right\|^2_{L^2_{v,x}},\end{aligned}$$

and we assume $\psi=1$ in this theorem.

Proof. The proof is similar to [17, Theorem 1.1] and we only illustrate the difference. The first one is that we use $\|E_0\|_{L^1_x}$ in (2.1) instead of $\|(1+|x|)\rho_0\|_{L^1}$, where

$$\rho_0 = \int_{\mathbb{R}^3} \big(f_+(0) - f_-(0)\big)\mu^{\frac{1}{2}} dv.$$

The only place involving this term is estimate (4.25) in [17, Theorem 1.1]. One can use

$$\|\widehat{E_0}(y)\|_{L^\infty_y} \le \|E_0\|_{L^1_x}$$

instead and hence, in (2.1), we can use $\|E_0\|_{L^1_x}$ instead.

The second difference is that we use $K\ge 4$ instead of $K\ge 8$. This is because, in Corollary 2.1 below, we only require $K\ge 4$. Replacing estimate in [17, Theorem 7.1, Eqs. (7.11)-(7.12)] by Corollary 2.1 below, we can use such index on K instead.

The third difference is to improve index from $1/2\le s<1$ to $0<s<1$. The work [17] is restricted to $1/2\le s<1$ because of [17, Lemmas 3.6, 3.7], where the authors used Fourier transform on $v\in\mathbb{R}^3$ to control the gradient ∇_v. Using Lemma 2.10 below instead, we are able to obtain the result for $0<s<1$. Then following the same proof of [17, Lemma 7.1 and Theorem 1.1], we complete the proof of Theorem 2.1. □

Here we introduce the following lemmas from [12] on pseudo-differential calculus, which will be frequently used in our analysis. Notice that the condition $l\le m$ in [12] is unnecessary.

Lemma 2.1 ([12, Lemma 2.3]). *Let m,c be Γ-admissible weight and $a\in S(m)$. Assume $a^w: H(mc)\to H(c)$ is invertible. If $b\in S(m)$, then there exists $C>0$, depending only on the seminorms of symbols to $(a^w)^{-1}$ and b^w, such that for $f\in H(mc)$*

$$\|b(v,D_v)f\|_{H(c)} + \|b^w(v,D_v)f\|_{H(c)} \le C\|a^w(v,D_v)f\|_{H(c)}.$$

Consequently, if $a^w: H(m_1)\to L^2\in Op(m_1), b^w: H(m_2)\to L^2\in Op(m_2)$ are invertible, then for $f\in\mathscr{S}$

$$\|b^w a^w f\|_{L^2} \lesssim \|a^w b^w f\|_{L^2},$$

where the constant depends only on seminorms of symbols to $a^w, b^w, (a^w)^{-1}, (b^w)^{-1}$.

Lemma 2.2 ([12, Lemma 2.4]). *Denote* $a_{K,l} := a + Kl, m_{K,l} := m + Kl$ *for* $K > 1$, *where* m, l *are* Γ*-admissible weights. Assume* $a \in S(m), \partial_\eta(a_{K,l}) \in S(K^{-\kappa} m_{K,l})$ *uniformly in* K *and* $a_{K,l} \gtrsim m_{K,l}$. *Let* $\rho > 0$ *and* $b \in S(\varepsilon m_{K,l} + \varepsilon^{-\rho} l)$, *uniformly in* $\varepsilon \in (0,1)$. *Then there exists* $K_0 > 0$ *such that for* $f \in H(mc), \varepsilon \in (0,1)$,

$$\begin{aligned} &\|b(v,D_v)f\|_{H(c)} + \|b^w(v,D_v)f\|_{H(c)} \\ &\le C_{K,l}\big(\varepsilon \|a^w(v,D_v)f\|_{H(c)} + \varepsilon^{-\rho}\|l^w f\|_{H(c)}\big). \end{aligned}$$

For composition of pseudodifferential operator we have $a^w b^w = (a\#b)^w$ with

$$a\#b = ab + \frac{1}{4\pi i}\{a,b\} + \sum_{2\le k\le \nu} 2^{-k} \sum_{|\alpha|+|\beta|=k} \frac{(-1)^{|\beta|}}{\alpha!\beta!} D_\eta^\alpha \partial_x^\beta a D_\eta^\beta \partial_x^\alpha b + r_\nu(a,b), \tag{2.3}$$

where $X = (v,\eta)$,

$$\begin{aligned} &r_\nu(a,b)(X) = R_\nu\big(a(X)\otimes b(Y)\big)|_{X=Y}, \\ &R_\nu = \int_0^1 \frac{(1-\theta)^{\nu-1}}{(\nu-1)!} \exp\left(\frac{\theta}{4\pi i}\langle \sigma\partial_X, \partial_Y\rangle\right) d\theta \left(\frac{1}{4\pi i}\langle\sigma\partial_X,\partial_Y\rangle\right)^\nu. \end{aligned}$$

Let $a_1(v,\eta) \in S(M_1,\Gamma), a_2(v,\eta) \in S(M_2,\Gamma)$, then $a_1^w a_2^w = (a_1\#a_2)^w, a_1\#a_2 \in S(M_1M_2,\Gamma)$ with

$$\begin{aligned} &a_1\#a_2(v,\eta) = a_1(v,\eta)a_2(v,\eta) + \int_0^1 (\partial_\eta a_1 \#_\theta \partial_v a_2 - \partial_v a_1 \#_\theta \partial_\eta a_2)\, d\theta, \\ &g\#_\theta h(Y) := \frac{2^{2d}}{\theta^{-2n}} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\frac{4\pi i}{\theta}\sigma(X-Y_1)\cdot(X-Y_2)} (4\pi i)^{-1} \langle \sigma\partial_{Y_1}, \partial_{Y_2}\rangle g(Y_1)h(Y_2)\, dY_1 dY_2 \end{aligned}$$

with $Y = (v,\eta), \sigma = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. For any non-negative integer k, there exists l, C independent of $\theta \in [0,1]$ such that

$$\|g\#_\theta h\|_{k;S(M_1M_2,\Gamma)} \le C\|g\|_{l,S(M_1,\Gamma)}\|h\|_{l,S(M_2,\Gamma)}.$$

Thus, if $\partial_\eta a_1, \partial_\eta a_2 \in S(M_1',\Gamma)$ and $\partial_v a_1, \partial_v a_2 \in S(M_2',\Gamma)$, then $[a_1,a_2] \in S(M_1'M_2',\Gamma)$, where $[\cdot,\cdot]$ is the commutator defined by $[A,B] := AB - BA$. As a consequence of composition and Lemma 2.1, we have the following.

Lemma 2.3. *Let m,c be Γ-admissible weight and $a^{1/2}\in S(m^{1/2})$. Assume $(a^{1/2})^w: H(mc)\to H(c)$ is invertible and $L\in S(m)$. Then*

$$(Lf,f)_{L^2}=\Big(\underbrace{((a^{\frac12})^w)^{-1}L}_{\in S(m^{1/2})}f,(a^{\frac12})^w f\Big)_{L^2}\lesssim \|(a^{\frac12})^w f\|^2_{L^2}.$$

The following lemma concerns with dissipation of $L_\pm$, whose proof can be found in [20, Lemma 2.6, Theorem 8.1].

Lemma 2.4. *For any $l\in\mathbb{R}$, multi-indices α,β, we have the followings:*

(i) It holds that

$$(-Lg,g)_{L^2_v}\gtrsim \|(\mathbf{I}-\mathbf{P})g\|^2_{L^2_D}.$$

(ii) There exists $C>0$ such that

$$-\big(w_l^2Lg,g\big)_{L^2_v}\gtrsim \|w_lg\|^2_{L^2_D}-C\|g\|^2_{L^2_v(B_C)}.$$

(iii) For any $\eta>0$

$$-\big(w_l^2(\alpha,\beta)\partial^\alpha_\beta Lg,\partial^\alpha_\beta g\big)_{L^2_v}\gtrsim \big\|w_l(\alpha,\beta)\partial^\alpha_\beta g\big\|^2_{L^2_D}-\eta\sum_{|\beta_1|\le|\beta|}\big\|w_l(\alpha,\beta_1)\partial^\alpha_{\beta_1}g\big\|^2_{L^2_D} -C_\eta\|\partial^\alpha g\|^2_{L^2(B_{C_\eta})}.$$

Notice that in Carleman representation (cf. [2, Appendix]), the derivative on v will apply to f,g and $\mu^{1/2}$ respectively. Then

$$\begin{aligned}&\psi_{|\alpha|+|\beta|-4}\partial^\alpha_\beta\mathcal{T}(f,g)\\ =&\sum_{\alpha_1+\alpha_2=\alpha}\ \sum_{\beta_1+\beta_2+\beta_3=\beta}C_\alpha^{\alpha_1,\alpha_2}C_\beta^{\beta_1,\beta_2,\beta_3}\psi_{|\alpha|+|\beta|-4}\mathcal{T}_{\beta_3}(\partial^{\alpha_1}_{\beta_1}f,\partial^{\alpha_2}_{\beta_2}g)\psi_{|\beta_3|}.\end{aligned}$$

The next lemma concerns the estimates on the nonlinear collision operator $\Gamma_\pm$, which comes from [17, Lemma 2.2] and [30, Proposition 3.1].

Lemma 2.5. *Assume $\gamma+2s\le 0$. For any $l\ge 0,m\ge 0$ and multi-index β, we have the upper bound*

$$\begin{aligned}&\Big|\big(w_l^2(\alpha,\beta)\partial^\alpha_\beta\Gamma_\pm(f,g),\partial^\alpha_\beta h\big)_{L^2_{v,x}}\Big|\\ \lesssim&\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2\le\beta}}\int_{\mathbb{R}^3}\big\|\partial^{\alpha_1}_{\beta_1}f\big\|_{L^2_v}\big\|w_l(\alpha,\beta)\partial^{\alpha_2}_{\beta_2}g\big\|_{L^2_D}\big\|w_l(\alpha,\beta)\partial^\alpha_\beta h\big\|_{L^2_D}dx\end{aligned}$$

$$
\begin{aligned}
&+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2\le\beta}}\int_{\mathbb{R}^3}\|w_l(\alpha,\beta)\partial^{\alpha_1}_{\beta_1}f\|_{L^2_v}\|\partial^{\alpha_2}_{\beta_2}g\|_{L^2_D}\|w_l(\alpha,\beta)\partial^{\alpha}_{\beta}h\|_{L^2_D}dx\\
&+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2\le\beta}}\int_{\mathbb{R}^3}\min\{A_1,A_2\}\|w_l(\alpha,\beta)\partial^{\alpha}_{\beta}h\|_{L^2_D}dx,
\end{aligned}
\tag{2.4}
$$

where

$$
\begin{aligned}
A_1&=\sum_{|\beta'|\le 2}\|w^{-m}\partial^{\alpha_1}_{\beta_1+\beta'}f\|_{L^2_v}\|w_l(\alpha,\beta)\partial^{\alpha_2}_{\beta_2}g\|_{L^2_D},\\
A_2&=\|w^{-m}\partial^{\alpha_1}_{\beta_1}f\|_{L^2_v}\sum_{|\beta'|\le 2}\|w_l(\alpha,\beta)\partial^{\alpha_2}_{\beta_2+\beta'}g\|_{L^2_D}.
\end{aligned}
$$

Let $i=1$ *if* $0<s<1/2$ *and* $i=2$ *if* $1/2\le s<1$, *then*

$$
\begin{aligned}
&\|\langle v\rangle^l\Gamma(f,g)\|_{L^2_v}\\
\lesssim&\min\Big\{\|\langle v\rangle^{l+\frac{\gamma+2s}{2}}f\|_{H^2_v}\|\langle v\rangle^{l+\frac{\gamma+2s}{2}}g\|_{H^i_v},\|\langle v\rangle^{l+\frac{\gamma+2s}{2}}f\|_{L^2_v}\|\langle v\rangle^{l+\frac{\gamma+2s}{2}}g\|_{H^{i+2}_v}\Big\}.
\end{aligned}
\tag{2.5}
$$

In order to obtain a suitable norm estimate of $\mathcal{T}$ on x. We write a fundamental estimate, which is very useful throughout our analysis.

Lemma 2.6. *For any* $u,v\in H^2_x$, *we have*

$$
\|uv\|_{L^2_x}\lesssim\min\Big\{\|\nabla_x u\|_{H^1_x}\|v\|_{L^2_x},\|\nabla_x u\|_{L^2_x}\|v\|_{H^1_x}\Big\}.
\tag{2.6}
$$

Proof. The proof is straightforward. Notice that this lemma gives that H^2_x is a Banach algebra. By Gagliardo-Nirenberg interpolation inequality and Sobolev embedding, cf. [27, Theorem 12.83] and [31, Proposition 2.2, Lemma 5.1], we have

$$
\begin{aligned}
&\|u\|_{L^\infty}\lesssim\|\nabla_x u\|^{\frac12}\|\nabla_x^2 u\|^{\frac12}\lesssim\|\nabla_x u\|_{H^1},\\
&\|uv\|_{L^2}\lesssim\|u\|_{L^6}\|v\|_{L^3}\lesssim\|\nabla_x u\|_{L^2}\|v\|_{H^1}.
\end{aligned}
$$

Then (2.6) follows from Hölder's inequality. □

The following corollary describes the behavior of nonlinear terms in the Vlasov-Poisson-Boltzmann system.

Corollary 2.1. *Let* $l\geq 0$ *and* $K\geq 4$. *Define* $i=1$ *if* $0<s<1/2$ *and* $i=2$ *if* $1/2\leq s<1$. *Assume* $l>\max\{-3(\gamma+2s)/4+i+1,-5(\gamma+2s)/4+2\}$. *Then, there exists* $l_*>-5(\gamma+2s)/4$ *such that*

$$\big\|\langle v\rangle^{l_*}g_\pm\big\|_{Z_1}+\big\|\langle v\rangle^{l_*}\nabla_x g_\pm\big\|_{L^2_{v,x}}\lesssim\mathcal{E}_{K,l},$$

where

$$g_\pm=\pm\nabla_x\phi\cdot\nabla_v f_\pm\mp\frac{1}{2}\nabla_x\phi\cdot v f_\pm+\Gamma_\pm(f,f).$$

Proof. By using (2.5) and Young's inequality, we have

$$\begin{aligned}\|\langle v\rangle^{l_*}\Gamma(f,f)\|_{Z_1}&\lesssim\int dx\big\|\langle v\rangle^{l_*+\frac{\gamma}{2}+s}f\big\|_{H^2_v}\big\|\langle v\rangle^{l_*+\frac{\gamma}{2}+s}f\big\|_{H^i_v}\\&\lesssim\big\|\langle v\rangle^{l_*+\frac{\gamma+2s}{2}}f\big\|^2_{H^2_vL^2_x}\lesssim\mathcal{E}_{K,l},\end{aligned}$$

whenever $l\geq l_*+(\gamma+2s)/2+2$. On the other hand,

$$\begin{aligned}&\big\|\langle v\rangle^{l_*}\nabla_x\phi\cdot\nabla_v f_\pm\big\|_{Z_1}\lesssim\|\nabla_x\phi\|_{L^2_x}\big\|\langle v\rangle^{l_*}\nabla_v f\big\|_{L^2_{v,x}}\lesssim\mathcal{E}_{K,l},\\&\big\|\langle v\rangle^{l_*}v\cdot\nabla_x\phi f_\pm\big\|_{Z_1}\lesssim\|\nabla_x\phi\|_{L^2_x}\big\|\langle v\rangle^{l_*}vf_\pm\big\|_{L^2_{v,x}}\lesssim\mathcal{E}_{K,l},\end{aligned}$$

whenever $l\geq l_*+1$. Similarly, by using (2.6),

$$\begin{aligned}\big\|\langle v\rangle^{l_*}\nabla_x\Gamma(f,f)\big\|_{L^2_{v,x}}\lesssim&\Big\|\big\|\langle v\rangle^{l_*+\frac{\gamma+2s}{2}}\nabla_x f\big\|_{L^2_v}\big\|\langle v\rangle^{l_*+\frac{\gamma+2s}{2}}f\big\|_{H^i_v}\Big\|_{L^2_x}\\&+\Big\|\|\big\|\langle v\rangle^{l_*+\frac{\gamma+2s}{2}}f\big\|_{L^2_v}\big\|\langle v\rangle^{l_*+\frac{\gamma+2s}{2}}\nabla_x f\big\|_{H^i_v}\Big\|_{L^2_x}\\\lesssim&\big\|\langle v\rangle^{l_*+\frac{\gamma+2s}{2}}f\big\|_{L^2_vH^2_x}\big\|\langle v\rangle^{l_*+\frac{\gamma+2s}{2}}f\big\|_{H^i_vH^1_x}\lesssim\mathcal{E}_{K,l},\end{aligned}$$

whenever $l\geq l_*+(\gamma+2s)/2+i+1$. By (2.6),

$$\begin{aligned}&\big\|\langle v\rangle^{l_*}\nabla_x(\nabla_x\phi\cdot\nabla_v f_\pm)\big\|_{L^2_{v,x}}\lesssim\|\nabla_x\phi\|_{H^2_x}\big\|\langle v\rangle^{l_*}f_\pm\big\|_{H^1_vH^1_x}\lesssim\mathcal{E}_{K,l},\\&\big\|\langle v\rangle^{l_*}\nabla_x(v\cdot\nabla_x\phi f_\pm)\big\|_{L^2_{v,x}}\lesssim\|\nabla_x\phi\|_{H^1_x}\big\|\langle v\rangle^{l_*}vf_\pm\big\|_{L^2_vH^1_x}\lesssim\mathcal{E}_{K,l},\end{aligned}$$

whenever $l\geq l_*+2$. Now we verify that such l_* exists. From the restriction above, we need to choose l_* such that

$$-\frac{5(\gamma+2s)}{4}<l_*\leq l-\frac{\gamma+2s}{2}-i-1,\quad l_*\leq l-2.$$

Such choice exists, since $l>\max\{-3(\gamma+2s)/4+i+1,-5(\gamma+2s)/4+2\}$. □

With the help of Lemmas 2.5 and 2.6, we can control the trilinear term as the following.

Lemma 2.7. *Let $K\geq 4$. For any multi-indices $|\alpha|+|\beta|\leq K$ and real number $l\geq 0$, we have*

$$
\begin{aligned}
&\big|\big(\psi_{2|\alpha|+2|\beta|-8}w_l^2(\alpha,\beta)\partial_\beta^\alpha\Gamma_\pm(f,g),\partial_\beta^\alpha h\big)_{L^2_{v,x}}\big|\\
\lesssim&\Big(\sum_{|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4}\partial_\beta^\alpha f\big\|_{L^2_{v,x}}\sum_{|\alpha|\geq 1,|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha g\big\|_{L^2_xL^2_D}\\
&\quad+\sum_{|\alpha|\geq 1,|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4}\partial_\beta^\alpha f\big\|_{L^2_{v,x}}\sum_{|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha g\big\|_{L^2_xL^2_D}\\
&\quad+\sum_{|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha f\big\|_{L^2_{v,x}}\sum_{|\alpha|\geq 1,|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4}\partial_\beta^\alpha g\big\|_{L^2_xL^2_D}\\
&\quad+\sum_{|\alpha|\geq 1,|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha f\big\|_{L^2_{v,x}}\sum_{|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4}\partial_\beta^\alpha g\big\|_{L^2_xL^2_D}\Big)\\
&\quad\times\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha h\big\|_{L^2_xL^2_D},
\end{aligned}
$$

where we restrict $t\in[0,1]$ when considering $\psi=t^N$ as in Theorem 1.1.

Proof. Using the estimate (2.4), we have

$$
\begin{aligned}
&\big|\big(\psi_{2|\alpha|+2|\beta|-8}w_l^2(\alpha,\beta)\partial_\beta^\alpha\Gamma_\pm(f,g),\partial_\beta^\alpha h\big)_{L^2_{v,x}}\big|\\
\lesssim&\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2\leq\beta}}\big\|\psi_{|\alpha|+|\beta|-4}\big\|\partial_{\beta_1}^{\alpha_1}f\big\|_{L^2_v}\big\|w_l(\alpha,\beta)\partial_{\beta_2}^{\alpha_2}g\big\|_{L^2_D}\big\|_{L^2_x}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha h\big\|_{L^2_xL^2_D}\\
&+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2\leq\beta}}\big\|\psi_{|\alpha|+|\beta|-4}\big\|w_l(\alpha,\beta)\partial_{\beta_1}^{\alpha_1}f\big\|_{L^2_v}\big\|\partial_{\beta_2}^{\alpha_2}g\big\|_{L^2_D}\big\|_{L^2_x}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha h\big\|_{L^2_xL^2_D}\\
&+\sum_{\substack{\alpha_1+\alpha_2=\alpha\\ \beta_1+\beta_2\leq\beta}}\big\|\psi_{|\alpha|+|\beta|-4}\min\{B_1,B_2\}\big\|_{L^2_x}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha h\big\|_{L^2_xL^2_D},
\end{aligned}\tag{2.7}
$$

where

$$
\begin{aligned}
B_1&=\sum_{|\beta'|\leq 2}\big\|w^{-m}\partial_{\beta_1+\beta'}^{\alpha_1}f\big\|_{L^2_v}\big\|w_l(\alpha,\beta)\partial_{\beta_2}^{\alpha_2}g\big\|_{L^2_D},\\
B_2&=\big\|w^{-m}\partial_{\beta_1}^{\alpha_1}f\big\|_{L^2_v}\sum_{|\beta'|\leq 2}\big\|w_l(\alpha,\beta)\partial_{\beta_2+\beta'}^{\alpha_2}g\big\|_{L^2_D}.
\end{aligned}
$$

Here we divide the summation into several parts. For brevity, we denote the first terms in the norm $\|\cdot\|_{L^2_x}$ inside the summation $\sum_{\alpha_1+\alpha_2=\alpha,\beta_1+\beta_2\le\beta}$ on the right-hand side of (2.7) to be I,J,K and discuss their value in several cases. If $2\le|\alpha_1|+|\beta_1|\le K$, then $|\alpha_2|+|\beta_2|\le|\alpha|+|\beta|-2$ and $|\alpha_2+\alpha'|+|\beta_2|\le|\alpha|+|\beta|$ for any $1\le|\alpha'|\le 2$. Notice that in this case

$$\psi_{|\alpha|+|\beta|-4}\le\psi_{|\alpha_1|+|\beta_1|-4}\psi_{|\alpha_2+\alpha'|+|\beta_2|-4}.$$

By using (2.6), we have

$$\begin{aligned}
I&\lesssim\psi_{|\alpha|+|\beta|-4}\big\|\partial^{\alpha_1}_{\beta_1}f\big\|_{L^2_{v,x}}\big\|\big\|w_l(\alpha,\beta)\partial^{\alpha_2}_{\beta_2}g\big\|_{L^2_D}\big\|_{L^\infty_x}\\
&\lesssim\big\|\psi_{|\alpha_1|+|\beta_1|-4}\partial^{\alpha_1}_{\beta_1}f\big\|_{L^2_{v,x}}\sum_{1\le|\alpha'|\le 2}\big\|\psi_{|\alpha_2+\alpha'|+|\beta_2|-4}w_l(\alpha+\alpha',\beta_2)\partial^{\alpha_2+\alpha'}_{\beta_2}g\big\|_{L^2_xL^2_D}\\
&\lesssim\sum_{|\alpha|+|\beta|\le K}\big\|\psi_{|\alpha|+|\beta|-4}\partial^{\alpha}_{\beta}f\big\|_{L^2_{v,x}}\sum_{\substack{|\alpha|\ge 1\\|\alpha|+|\beta|\le K}}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial^{\alpha}_{\beta}g\big\|_{L^2_xL^2_D}.
\end{aligned}\tag{2.8}$$

Secondly, if $|\alpha_1|+|\beta_1|=1$, then $|\alpha_2|+|\beta_2|\le|\alpha|+|\beta|-1$. Using (2.6) to give one x derivative to f, we have

$$\begin{aligned}
I\lesssim&\sum_{|\alpha'|=1}\big\|\psi_{|\alpha_1+\alpha'|+|\beta_1|-4}\partial^{\alpha_1+\alpha'}_{\beta_1}f\big\|_{L^2_{v,x}}\\
&\quad\times\sum_{|\alpha'|\le 1}\big\|\psi_{|\alpha_2+\alpha'|+|\beta_2|-4}w_l(\alpha+\alpha',\beta_2)\partial^{\alpha_2+\alpha'}_{\beta_2}g\big\|_{L^2_xL^2_D}\\
\lesssim&\sum_{\substack{|\alpha|\ge 1\\|\alpha|+|\beta|\le K}}\big\|\psi_{|\alpha|+|\beta|-4}\partial^{\alpha}_{\beta}f\big\|_{L^2_{v,x}}\sum_{|\alpha|+|\beta|\le K}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial^{\alpha}_{\beta}g\big\|_{L^2_xL^2_D}.
\end{aligned}$$

Here we used $\psi\le 1$ and

$$\psi_{|\alpha|+|\beta|-4}\le\psi_{|\alpha_1+\alpha_1'|+|\beta_1|-4}\psi_{|\alpha_2+\alpha_2'|+|\beta_2|-4},\quad\forall\,|\alpha_1'|=1,\quad|\alpha_2'|\le 1.$$

Thirdly, if $|\alpha_1|+|\beta_1|=0$, using (2.6) to give at most two and at least one spatial derivative to f with, we have

$$\begin{aligned}
I&\lesssim\sum_{1\le|\alpha'|\le 2}\big\|\psi_{|\alpha_1+\alpha'|+|\beta_1|-4}\partial^{\alpha_1+\alpha'}_{\beta_1}f\big\|_{L^2_{v,x}}\big\|\psi_{|\alpha_2|+|\beta_2|-4}w_l(\alpha_2,\beta_2)\partial^{\alpha_2}_{\beta_2}g\big\|_{L^2_xL^2_D}\\
&\lesssim\sum_{\substack{|\alpha|\ge 1\\|\alpha|+|\beta|\le K}}\big\|\psi_{|\alpha|+|\beta|-4}\partial^{\alpha}_{\beta}f\big\|_{L^2_{v,x}}\sum_{|\alpha|+|\beta|\le K}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial^{\alpha}_{\beta}g\big\|_{L^2_xL^2_D}.
\end{aligned}\tag{2.9}$$

Here we used

$$\psi_{|\alpha|+|\beta|-4} \le \psi_{|\alpha_1+\alpha'|+|\beta_1|-4}\psi_{|\alpha_2|+|\beta_2|-4}, \quad \forall |\alpha'| \le 2.$$

Combining the above estimate, we have the desired result for I

$$\begin{aligned} I \lesssim & \sum_{|\alpha|+|\beta|\le K} \left\|\psi_{|\alpha|+|\beta|-4}\partial^\alpha_\beta f\right\|_{L^2_{v,x}} \sum_{\substack{|\alpha|\ge 1\\ |\alpha|+|\beta|\le K}} \left\|\psi_{|\alpha|+|\beta|-4} w_l(\alpha,\beta)\partial^\alpha_\beta g\right\|_{L^2_x L^2_D} \\ & + \sum_{\substack{|\alpha|\ge 1\\ |\alpha|+|\beta|\le K}} \left\|\psi_{|\alpha|+|\beta|-4}\partial^\alpha_\beta f\right\|_{L^2_{v,x}} \sum_{|\alpha|+|\beta|\le K} \left\|\psi_{|\alpha|+|\beta|-4} w_l(\alpha,\beta)\partial^\alpha_\beta g\right\|_{L^2_x L^2_D}. \end{aligned}$$

Similarly, using the same discussion on $|\alpha_2|+|\beta_2|$ instead of $|\alpha_1|+|\beta_1|$, we have

$$\begin{aligned} J \lesssim & \sum_{|\alpha|+|\beta|\le K} \left\|\psi_{|\alpha|+|\beta|-4} w_l(\alpha,\beta)\partial^\alpha_\beta f\right\|_{L^2_{v,x}} \sum_{\substack{|\alpha|\ge 1\\ |\alpha|+|\beta|\le K}} \left\|\psi_{|\alpha|+|\beta|-4}\partial^\alpha_\beta g\right\|_{L^2_x L^2_D} \\ & + \sum_{\substack{|\alpha|\ge 1\\ |\alpha|+|\beta|\le K}} \left\|\psi_{|\alpha|+|\beta|-4} w_l(\alpha,\beta)\partial^\alpha_\beta f\right\|_{L^2_{v,x}} \sum_{|\alpha|+|\beta|\le K} \left\|\psi_{|\alpha|+|\beta|-4}\partial^\alpha_\beta g\right\|_{L^2_x L^2_D}. \end{aligned}$$

For the term K, the idea is similar to I. If $|\alpha_1|+|\beta_1|=0$, we use the first term in the minimum of K and apply (2.6) to give at most two and at least one spatial derivative to f. Noticing

$$\psi_{|\alpha|+|\beta|-4} \le \psi_{|\alpha_1+\alpha'|+|\beta_1+\beta'|-4}\psi_{|\alpha_2|+|\beta_2|-4}, \quad \forall 1 \le |\alpha'| \le 2, \quad |\beta'| \le 2,$$

we have

$$\begin{aligned} K \lesssim & \psi_{|\alpha|+|\beta|-4} \sum_{\substack{1\le|\alpha'|\le 2\\ |\beta'|\le 2}} \left\|w^{-m}\partial^{\alpha_1+\alpha'}_{\beta_1+\beta'} f\right\|_{L^2_{v,x}} \left\|w_l(\alpha,\beta)\partial^{\alpha_2}_{\beta_2} g\right\|_{L^2_x L^2_D} \\ \lesssim & \sum_{\substack{|\alpha|\ge 1\\ |\alpha|+|\beta|\le K}} \left\|\psi_{|\alpha|+|\beta|-4}\partial^\alpha_\beta f\right\|_{L^2_{v,x}} \sum_{|\alpha|+|\beta|\le K} \left\|\psi_{|\alpha|+|\beta|-4} w_l(\alpha,\beta)\partial^\alpha_\beta g\right\|_{L^2_x L^2_D}. \end{aligned}$$

Similarly, if $|\alpha_1|+|\beta_1|=1$, we apply (2.6) to give at least one x derivative to f, at most one x derivative to g and deduce the same bound. If $|\alpha_1|+|\beta_1|=2$, we apply (2.6) to give at most two and at least one spatial derivative to g. Noticing

$$\psi_{|\alpha|+|\beta|-4} \le \psi_{|\alpha_1|+|\beta_1+\beta'|-4}\psi_{|\alpha_2+\alpha'|+|\beta_2|-4}, \quad \forall 1 \le |\alpha'| \le 2, \quad |\beta'| \le 2,$$

we have

$$
\begin{aligned}
K &\lesssim \psi_{|\alpha|+|\beta|-4} \sum_{|\beta'|\le 2} \|w^{-m}\partial^{\alpha_1}_{\beta_1+\beta'} f\|_{L^2_{v,x}} \sum_{1\le|\alpha'|\le 2} \|w_l(\alpha,\beta)\partial^{\alpha_2+\alpha'}_{\beta_2} g\|_{L^2_x L^2_D} \\
&\lesssim \sum_{|\alpha|+|\beta|\le K} \|\psi_{|\alpha|+|\beta|-4}\partial^{\alpha}_{\beta} f\|_{L^2_{v,x}} \sum_{\substack{|\alpha|\ge 1\\ |\alpha|+|\beta|\le K}} \|\psi_{|\alpha|+|\beta|-4} w_l(\alpha,\beta)\partial^{\alpha}_{\beta} g\|_{L^2_x L^2_D}.
\end{aligned}
$$

If $|\alpha_1|+|\beta_1|=3$, we will use the second term in the minimum of K. Applying (2.6) to give at least one x derivative to f and at most one x derivative to g, noticing

$$
\begin{aligned}
&\psi_{|\alpha|+|\beta|-4} \le \psi_{|\alpha_1+\alpha_1'|+|\beta_1|-4}\psi_{|\alpha_2+\alpha_2'|+|\beta_2+\beta'|-4}, \\
&w_l(\alpha,\beta) \le w_l(\alpha_2+\alpha_2',\beta_2+\beta')
\end{aligned}
$$

for any $|\alpha_1'|=1, |\alpha'|\le 1, |\beta'|\le 2$, we have

$$
\begin{aligned}
K &\lesssim \psi_{|\alpha|+|\beta|-4} \sum_{|\alpha_1'|=1} \|w^{-m}\partial^{\alpha_1+\alpha_1'}_{\beta_1} f\|_{L^2_{v,x}} \sum_{\substack{|\alpha_2'|\le 1\\ |\beta'|\le 2}} \|w_l(\alpha,\beta)\partial^{\alpha_2+\alpha_2'}_{\beta_2+\beta'} g\|_{L^2_x L^2_D} \\
&\lesssim \sum_{\substack{|\alpha|\ge 1\\ |\alpha|+|\beta|\le K}} \|\psi_{|\alpha|+|\beta|-4}\partial^{\alpha}_{\beta} f\|_{L^2_{v,x}} \sum_{|\alpha|+|\beta|\le K} \|\psi_{|\alpha|+|\beta|-4} w_l(\alpha,\beta)\partial^{\alpha}_{\beta} g\|_{L^2_x L^2_D}.
\end{aligned}
$$

If $4\le|\alpha_1|+|\beta_1|\le K$, then applying (2.6) to give two x derivatives to g and noticing

$$
\begin{aligned}
&\psi_{|\alpha|+|\beta|-4} \le \psi_{|\alpha_1|+|\beta_1|-4}\psi_{|\alpha_2+\alpha'|+|\beta_2+\beta'|-4}, \\
&w_l(\alpha,\beta) \le w_l(\alpha_2+\alpha',\beta_2+\beta')
\end{aligned}
$$

for any $1\le|\alpha'|\le 2, |\beta'|\le 2$, we have

$$
\begin{aligned}
K &\lesssim \psi_{|\alpha|+|\beta|-4} \|w^{-m}\partial^{\alpha_1}_{\beta_1} f\|_{L^2_{v,x}} \sum_{\substack{1\le|\alpha'|\le 2\\ |\beta'|\le 2}} \|w_l(\alpha,\beta)\partial^{\alpha_2+\alpha'}_{\beta_2+\beta'} g\|_{L^2_x L^2_D} \\
&\lesssim \sum_{|\alpha|+|\beta|\le K} \|\psi_{|\alpha|+|\beta|-4}\partial^{\alpha}_{\beta} f\|_{L^2_{v,x}} \sum_{\substack{|\alpha|\ge 1\\ |\alpha|+|\beta|\le K}} \|\psi_{|\alpha|+|\beta|-4} w_l(\alpha,\beta)\partial^{\alpha}_{\beta} g\|_{L^2_x L^2_D}.
\end{aligned}
$$

Substituting all the above estimates into (2.7), we have the desired bound. A similar discussion on the indices $|\alpha_1|+|\beta_1|$ will be used frequently later and will not be mentioned for brevity. □

A direct consequence of Lemma 2.7 is the following estimate, see also [17, Lemma 3.1].

Lemma 2.8. *Let* $K\geq 4, |\alpha|+|\beta|\leq K, l\geq 0$. *Then*

$$\left|\left(\partial^\alpha\Gamma_\pm(f,f),\psi_{2|\alpha|-8}\partial^\alpha f_\pm\right)_{L^2_{v,x}}\right|\lesssim\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}(t), \tag{2.10}$$

$$\left|\left(w_l^2(\alpha,\beta)\partial^\alpha_\beta\Gamma_\pm(f,f),\psi_{2|\alpha|+2|\beta|-8}\partial^\alpha_\beta f\right)_{L^2_{v,x}}\right|\lesssim\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}(t)+\mathcal{E}_{K,l}\mathcal{D}_{K,l}^{\frac{1}{2}}(t). \tag{2.11}$$

Also, for any smooth function $\zeta(v)$ *satisfying* $|\zeta(v)|\approx e^{-\lambda|v|^2}$ *with some* $\lambda>0$, *we have*

$$\left(\partial^\alpha\Gamma_\pm(f,f),\psi_{2|\alpha|-8}\zeta(v)\right)_{L_{v,x}}\lesssim\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}^{\frac{1}{2}}(t). \tag{2.12}$$

Proof. For (2.11), notice that

$$\begin{aligned}&\left(w_l^2(\alpha,\beta)\partial^\alpha_\beta\Gamma_\pm(f,f),\psi_{2|\alpha|+2|\beta|-8}\partial^\alpha_\beta f_\pm\right)_{L^2_{v,x}}\\&=\left(w_l^2(\alpha,\beta)\partial^\alpha_\beta\Gamma_\pm(f,f),\psi_{2|\alpha|+2|\beta|-8}\partial^\alpha_\beta(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right)_{L^2_{v,x}}\\&\quad+\left(w_l^2(\alpha,\beta)\partial^\alpha_\beta\Gamma_\pm(f,f),\psi_{2|\alpha|+2|\beta|-8}\partial^\alpha_\beta\mathbf{P}_\pm f\right)_{L^2_{v,x}}.\end{aligned}$$

The first term on the right-hand side, by directly using Lemma 2.7 and the definition of $\mathcal{E}_{K,l}$ and $\mathcal{D}_{K,l}$, is bounded above by $\mathcal{E}_{K,l}^{1/2}\mathcal{D}_{K,l}(t)$, since there is zero x derivative on $(\mathbf{I}-\mathbf{P})f$ in the definition of $\mathcal{D}_{K,l}$. But there is no such term for $\mathbf{P}f$ in $\mathcal{D}_{K,l}$. Hence, the second right-hand side term can only be bounded above by $\mathcal{E}_{K,l}\mathcal{D}_{K,l}^{1/2}(t)$. This proves (2.11).

Similarly, noticing $P_\pm\Gamma(f,f)=0$, one can obtain (2.10). The proof of (2.12) is directly from Lemma 2.7. This conclude Lemma 2.8. ☐

For later use, we need the following estimate on $v\cdot\nabla_x\phi f_\pm$ and $\nabla_x\phi\cdot\nabla_v f_\pm$. We always assume that $\|\phi\|_{L^\infty_x}\leq C$, which follows from the a priori assumption on energy $\mathcal{E}_{K,l}$ given in (1.12) and hence, $|e^{\pm\phi}|\approx 1$. The proof here is different from [17, Lemmas 3.4, 3.6], since we will cover the full range $0<s<1$.

Lemma 2.9. *Let* $1\leq|\alpha|\leq K, |\alpha|+|\beta|\leq K$ *and* $l\geq 0$. *Then, for* $\alpha_1\leq\alpha,\beta_1\leq\beta$ *with* $|\alpha_1|\geq 1$, *it holds that*

$$\begin{aligned}&\left|\left(v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right)_{L^2_{v,x}}\right|\lesssim\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l},\\&\left|\left(\partial_{\beta_1}v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f_\pm,\psi_{2|\alpha|+2|\beta|-8}e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta f_\pm\right)_{L^2_{v,x}}\right|\lesssim\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}.\end{aligned}$$

Proof. For $|\alpha_1|\geq 1$ with $\alpha_1\leq\alpha$, by using $-3<\gamma\leq -2s$ and $0<s<1$, we have from (1.10) that $|v_i|w_l(|\alpha|,0)\leq\langle v\rangle^\gamma w_l(|\alpha|-1,0)$. Thus,

$$
\begin{aligned}
&\left|\left(v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\right)_{L^2_{v,x}}\right|\\
\lesssim&\left\|\psi_{|\alpha|-4}\partial^{\alpha_1}\nabla_x\phi\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,0)\partial^{\alpha-\alpha_1}f_\pm\right\|_{L^2_{v,x}}\\
&\times\left\|\psi_{|\alpha|-4}\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|,0)\partial^\alpha f_\pm\right\|_{L^2_{v,x}}.
\end{aligned}\tag{2.13}
$$

For the first term on the right hand of (2.13), we discuss its value as the following. If $\alpha_1<\alpha$, then $1\leq|\alpha_1|\leq K-1$ and there is at least one derivative on $f_\pm$ with respect to x. Then by the same discussion on the value of $|\alpha_1|$ as (2.8)-(2.9), one has

$$
\left|\psi_{|\alpha|-4}\partial^{\alpha_1}\nabla_x\phi\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,0)\partial^{\alpha-\alpha_1}f_\pm\right|_{L^2_{v,x}}\lesssim\mathcal{E}^{\frac12}_{K,l}\mathcal{D}^{\frac12}_{K,l},
$$

where we used $\|\langle v\rangle^{\gamma/2+s}(\cdot)\|_{L^2_{v,x}}\lesssim\|\cdot\|_{L^2_xL^2_D}$. If $\alpha_1=\alpha$, then we decompose $f_\pm=\mathbf{P}_\pm f+(\mathbf{I}_\pm-\mathbf{P}_\pm)f$ and give one derivative to $\mathbf{P}_\pm f$ with respect to x by using (2.6). That is,

$$
\begin{aligned}
&\left\|\psi_{|\alpha|-4}\partial^{\alpha}\nabla_x\phi\langle v\rangle^{\frac{\gamma}{2}}w_l(\alpha-\alpha_1,0)\mathbf{P}_\pm f\right\|_{L^2_{v,x}}\\
\lesssim&\left\|\psi_{|\alpha|-4}\partial^{\alpha}\nabla_x\phi\right\|_{L^2_x}\sum_{1\leq|\alpha'|\leq 2}\left\|\psi_{|\alpha'|-4}\partial^{\alpha'}\mathbf{P}_\pm f\right\|_{L^2_{v,x}}\lesssim\mathcal{E}^{\frac12}_{K,l}\mathcal{D}^{\frac12}_{K,l}.
\end{aligned}
$$

For the part $(\mathbf{I}_\pm-\mathbf{P}_\pm)f$, we will use (2.6) to give two derivatives to $(\mathbf{I}_\pm-\mathbf{P}_\pm)f$ when $|\alpha|\geq 3$, one derivative to $(\mathbf{I}_\pm-\mathbf{P}_\pm)f$ when $|\alpha|=2$ and give nothing to $(\mathbf{I}_\pm-\mathbf{P}_\pm)f$ when $|\alpha|=1$. That is,

$$
\begin{aligned}
&\left\|\psi_{|\alpha|-4}\partial^{\alpha}\nabla_x\phi\langle v\rangle^{\frac{\gamma}{2}}w_l(\alpha-\alpha_1,0)(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{v,x}}\\
\lesssim&\sum_{3\leq|\alpha|\leq K}\left\|\psi_{|\alpha|-4}\partial^{\alpha}\nabla_x\phi\right\|_{L^2_x}\\
&\quad\times\sum_{1\leq|\alpha'|\leq 2}\left\|\psi_{|\alpha'|-4}\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,0)\partial^{\alpha'}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{v,x}}\\
&+\sum_{|\alpha|=2}\sum_{|\alpha'|\leq 1}\left\|\psi_{|\alpha+\alpha'|-4}\partial^{\alpha+\alpha'}\nabla_x\phi\right\|_{L^2_x}\\
&\quad\times\sum_{|\alpha_1'|=1}\left\|\psi_{|\alpha_1'|-4}\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,0)\partial^{\alpha_1'}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{v,x}}\\
&+\sum_{|\alpha|=1}\sum_{|\alpha'|\leq 2}\left\|\psi_{|\alpha+\alpha'|-4}\partial^{\alpha+\alpha'}\nabla_x\phi\right\|_{L^2_x}\left\|\langle v\rangle^{\frac{\gamma}{2}}w_l(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{v,x}}\\
\lesssim&\mathcal{E}^{\frac12}_{K,l}\mathcal{D}^{\frac12}_{K,l},
\end{aligned}
$$

where we used -4 in ψ through our argument. Thus, when $\alpha_1=\alpha$,

$$\big\|\psi_{|\alpha|-4}\partial^{\alpha_1}\nabla_x\phi\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,0)\partial^{\alpha-\alpha_1}f_\pm\big\|_{L^2_{v,x}}\lesssim\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}^{\frac{1}{2}}. \tag{2.14}$$

Plugging the above estimate into (2.13), we have

$$\big|\big(v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{v,x}}\big|\lesssim\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}.$$

Similarly, for $|\beta|\le K$ and $\beta_1\le\beta$, we have $|\partial_{\beta_1}v_i|\le\langle v\rangle$ and hence,

$$\begin{aligned}&\big|\big(\partial_{\beta_1}v_i\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f_\pm,\psi_{2|\alpha|+2|\beta|-8}e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta f_\pm\big)_{L^2_{v,x}}\big|\\ \lesssim&\big\|\psi_{|\alpha|+|\beta|-4}\partial^{\alpha_1}\nabla_x\phi\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,|\beta-\beta_1|)\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f_\pm\big\|_{L^2_{v,x}}\\ &\times\big\|\psi_{|\alpha|+|\beta|-4}\langle v\rangle^{\frac{\gamma}{2}}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm\big\|_{L^2_{v,x}}.\end{aligned} \tag{2.15}$$

For the first term on the right-hand side of (2.15), we use the same argument as in (2.13)-(2.14) to find its upper bound $\mathcal{E}_{K,l}^{1/2}\mathcal{D}_{K,l}^{1/2}$. Hence, (2.15) is bounded above by $\mathcal{E}_{K,l}^{1/2}\mathcal{D}_{K,l}$. □

Lemma 2.10. *Let $|\alpha|+|\beta|\le K,l\ge 0$. Then, for $\alpha_1\le\alpha,\beta_1\le\beta$, it holds that*

$$\big|\big(\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{e_i}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{v,x}}\big|\le\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}, \tag{2.16}$$

$$\big|(\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}_{\beta+e_i}f_\pm,\psi_{2|\alpha|+2|\beta|-8}e^{\pm\phi}w_l^2(\alpha,\beta)\partial^\alpha_\beta f_\pm)_{L^2_{v,x}}\big|\le\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}. \tag{2.17}$$

Proof. We firstly prove (2.16). When $\alpha_1=0$, by integration by parts and $\gamma+2s\ge-2$, we have

$$\begin{aligned}&\big|\big(\partial^{e_i}\phi\partial^\alpha_{e_i}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{v,x}}\big|\\ \lesssim&\big|\big(\partial^{e_i}\phi\partial^\alpha f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}\big(\partial_{e_i}w_l^2(\alpha,0)\big)\partial^\alpha f_\pm\big)_{L^2_{v,x}}\big|\\ \lesssim&\big\|\psi_{|\alpha|-4}\nabla_x\phi\langle v\rangle^{\frac{\gamma+2s}{2}}w_l(|\alpha|,0)\partial^\alpha f_\pm\big\|_{L^2_{v,x}}\big\|\psi_{|\alpha|-4}w_l(|\alpha|,0)\partial^\alpha f_\pm\big\|_{L^2_{v,x}}\\ \lesssim&\sum_{|\alpha'|\le2}\big\|\psi_{|\alpha'|-4}\partial^{\alpha'}\nabla_x\phi\big\|_{L^2_x}\sum_{1\le|\alpha|\le K}\big\|\psi_{|\alpha|-4}\langle v\rangle^{\frac{\gamma+2s}{2}}w_l(|\alpha|,0)\partial^\alpha f_\pm\big\|_{L^2_{v,x}}\\ &\times\sum_{|\alpha|\le K}\|\psi_{|\alpha|-4}w_l(|\alpha|,0)\partial^\alpha f_\pm\|_{L^2_{v,x}}\\ \lesssim&\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l},\end{aligned}$$

where we use (2.6) to assure that there is always at least one derivative on the first $f_\pm$. When $|\alpha_1|\geq 1$, we have $|\alpha|\geq 1$. Then we decompose $f_\pm = \mathbf{P}_\pm f + (\mathbf{I}_\pm - \mathbf{P}_\pm) f$ to obtain

$$\big(\partial^{\alpha_1+e_i}\phi\partial_{e_i}^{\alpha-\alpha_1}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{v,x}} = I+J$$

with

$$\begin{aligned}
I &= \big(\partial^{\alpha_1+e_i}\phi\partial_{e_i}^{\alpha-\alpha_1}\mathbf{P}_\pm f,\psi_{2|\alpha|-8}e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{v,x}},\\
J &= \big(\partial^{\alpha_1+e_i}\phi\partial_{e_i}^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f,\psi_{2|\alpha|-8}e^{\pm\phi}w_l^2(\alpha,0)\partial^\alpha f_\pm\big)_{L^2_{v,x}}.
\end{aligned}$$

Now we estimate I and J as the following. For I, noticing there is exponential decay in v, we have

$$\begin{aligned}
|I| &\lesssim \big\|\psi_{|\alpha|-4}\partial^{\alpha_1+e_i}\phi\partial^{\alpha-\alpha_1}\mathbf{P}_\pm f\big\|_{L^2_{v,x}}\big\|\psi_{|\alpha|-4}\langle v\rangle^{\frac{\gamma+2s}{2}}w_l(|\alpha|,0)\partial^\alpha f_\pm\big\|_{L^2_{v,x}}\\
&\lesssim \sum_{|\alpha_1|\leq K}\big\|\psi_{|\alpha_1|-4}\partial^{\alpha_1}\nabla_x\phi\big\|_{L^2_x}\\
&\qquad\times\sum_{1\leq|\alpha|\leq K}\big\|\psi_{|\alpha|-4}\partial^\alpha\mathbf{P}_\pm f\big\|_{L^2_{v,x}}\big\|\psi_{|\alpha|-4}w_l(|\alpha|,0)\partial^\alpha f_\pm\big\|_{L^2_xL^2_D}\\
&\lesssim \mathcal{E}_{K,l}^{\frac12}\mathcal{D}_{K,l},
\end{aligned}$$

where we used same discussion on the value of $|\alpha_1|$ as (2.8)-(2.9) and give at least one derivative to $\mathbf{P}_\pm f$. For J, we first provide some interpolation formulas. For any $k\in\mathbb{R}$, by Young's inequality, we have $\langle\eta\rangle\lesssim\langle\eta\rangle^s\langle v\rangle^k+\langle\eta\rangle^{1+s}\langle v\rangle^{-ks/(1-s)}$ and hence, $\langle\eta\rangle$ is a symbol in $S(\langle\eta\rangle^s\langle v\rangle^k+\langle\eta\rangle^{1+s}\langle v\rangle^{-ks/(1-s)})$, where η is the Fourier variable of v. Then by [12, Lemma 2.3, Corollary 2.5], we have

$$\|f\|_{H^1_v}\lesssim\|f\langle v\rangle^k\|_{H^s}+\big\|f\langle v\rangle^{-\frac{ks}{1-s}}\big\|_{H^{1+s}}. \tag{2.18}$$

By our choice of $w_l(\alpha,\beta)$ in (1.10), we have

$$\begin{aligned}
&w_l(\alpha,0)\leq\langle v\rangle^\gamma w_l(|\alpha|-1,0)^s w(|\alpha|-1,1)^{1-s},\\
&w_l(\alpha,0)=\langle v\rangle^\gamma w_l(|\alpha|-1,0).
\end{aligned}$$

Choosing

$$\langle v\rangle^k = w_l(|\alpha|-1,0)^{1-s}w_l(|\alpha|-1,1)^{-(1-s)}$$

in (2.18), we obtain

$$\begin{aligned}
&\big\|\langle v\rangle^{-\frac{\gamma}{2}}w_l(\alpha,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big\|_{L^2_{x,v}}\\
\lesssim&\big\|\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big\|_{L^2_xH^s_v}\\
&\quad+\big\|\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,1)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big\|_{L^2_xH^{1+s}_v}\leq\sqrt{\mathcal{D}_{K,l}},
\end{aligned}$$

when $|\alpha_1|=1$. When $|\alpha_1|=2$, we have

$$\begin{aligned}&\left\|\langle v\rangle^{-\frac{\gamma}{2}}w_l(\alpha,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^6_xL^2_v}\\ \le&\left\|\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,0)\partial^{\alpha-\alpha_1}\nabla_x(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{x,v}}\le\sqrt{\mathcal{D}_{K,l}}.\end{aligned}$$

When $3\le|\alpha_1|\le K$, we have

$$\begin{aligned}&\left\|\langle v\rangle^{-\frac{\gamma}{2}}w_l(\alpha,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^\infty_xL^2_v}\\ \le&\left\|\langle v\rangle^{\frac{\gamma}{2}}w_l(|\alpha|-1,0)\partial^{\alpha-\alpha_1}\nabla_x(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{H^1_xL^2_v}\le\sqrt{\mathcal{D}_{K,l}}.\end{aligned}$$

Combining the above estimates, we have

$$\begin{aligned}J\lesssim&\Bigg(\sum_{|\alpha_1|=1}\left\|\partial^{\alpha_1}\nabla_x\phi\right\|_{L^\infty_x}\left\|\langle v\rangle^{-\frac{\gamma}{2}}w_l(\alpha,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^2_{x,v}}\\ &\quad+\sum_{|\alpha_1|=2}\left\|\partial^{\alpha_1}\nabla_x\phi\right\|_{L^3_x}\left\|\langle v\rangle^{-\frac{\gamma}{2}}w_l(\alpha,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^6_xL^2_x}\\ &\quad+\sum_{3\le|\alpha_1|\le K}\left\|\partial^{\alpha_1}\nabla_x\phi\right\|_{L^2_x}\left\|\langle v\rangle^{-\frac{\gamma}{2}}w_l(\alpha,0)\partial^{\alpha-\alpha_1}(\mathbf{I}_\pm-\mathbf{P}_\pm)f\right\|_{L^\infty_xL^2_x}\Bigg)\\ &\times\left\|\psi_{2|\alpha|-8}\langle v\rangle^{\frac{\gamma}{2}}w_l(\alpha,0)\partial^\alpha f_\pm\right\|_{L^2_{x,v}}\lesssim\mathcal{E}^{\frac12}_{K,l}\mathcal{D}_{K,l}.\end{aligned}$$

Collecting all the above estimates for I and J, we obtain (2.16). The proof of (2.17) is the same as (2.16), and the details are omitted for brevity. □

Next, we give some illustration for the macroscopic estimate, see also [14]. Recall the projection $\mathbf{P}_\pm$ in (1.7). By multiplying the Eq. (1.4) with $\mu^{1/2},v_j\mu^{1/2}$, $j=1,2,3$, and $(|v|^2-3)/6\mu^{1/2}$ and then integrating them over $\mathbb{R}^3_v$, we have

$$\begin{cases}\partial_t a_\pm+\nabla\cdot b+\nabla_x\cdot\big(v\mu^{\frac12},(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big)_{L^2_v}=0,\\ \partial_t\big(b_j+(v_j\mu^{\frac12},(\mathbf{I}_\pm-\mathbf{P}_\pm)f)_{L^2_v}\big)+\partial_j(a_\pm+2c)\mp E_j\\ \quad+\big(v_j\mu^{\frac12},v\cdot\nabla_x(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big)_{L^2_v}=(L_\pm f+g_\pm,v_j\mu^{\frac12})_{L^2_v},\\ \partial_t\left(c+\dfrac16\big((|v|^2-3)\mu^{\frac12},(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big)_{L^2_v}\right)+\dfrac13\nabla_x\cdot b\\ \quad+\dfrac16\big((|v|^2-3)\mu^{\frac12},v\cdot\nabla(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big)_{L^2_v}=\dfrac16\big(L_\pm f+g_\pm,(|v|^2-3)\mu^{\frac12}\big)_{L^2_v},\end{cases}\tag{2.19}$$

where for brevity, we denote $I=(I_+,I_-)$ with $I_\pm f=f_\pm$ and

$$g_\pm=\pm\nabla_x\phi\cdot\nabla_v f_\pm\mp\frac{1}{2}\nabla_x\phi\cdot v f_\pm+\Gamma_\pm(f,f).$$

Notice that $(\mathbf{P}_\pm f,v\mu^{1/2})_{L^2_v}$ and $(\mathbf{P}_\pm f,(|v|^2-3)\mu^{1/2})_{L^2_v}$ is not 0 in general and similar for $\Gamma_\pm$. Also, we have used

$$\left(\pm\nabla_x\phi\cdot\nabla_v f_\pm\mp\frac{1}{2}\nabla_x\phi\cdot v f_\pm,\mu^{\frac{1}{2}}\right)_{L^2_v}=0,$$

which is obtained by integration by parts on ∇_v. In order to obtain the high-order moments, as in [19], we define for $1\le j,k\le 3$ that

$$\Theta_{jk}(f_\pm)=\big((v_jv_k-1)\mu^{\frac{1}{2}},f_\pm\big)_{L^2_v},\quad \Lambda_j(f_\pm)=\frac{1}{10}\big((|v|^2-5)v_j\mu^{\frac{1}{2}},f_\pm\big)_{L^2_v}.$$

Then multiplying Eq. (1.4) with the high-order moments $(v_jv_k-1)\mu^{1/2}$ and $(|v|^2-5)v_j\mu^{1/2}/10$ and integrating over $\mathbb{R}^3_v$, we have

$$\begin{cases} \quad \partial_t\big(\Theta_{jj}\big((\mathbf{I}_\pm-\mathbf{P}_\pm)f\big)+2c\big)+2\partial_jb_j=\Theta_{jj}(g_\pm+h_\pm), \\ \quad \partial_t\Theta_{jk}\big((\mathbf{I}_\pm-\mathbf{P}_\pm)f\big)+\partial_jb_k+\partial_kb_j+\nabla_x\cdot\big(v\mu^{\frac{1}{2}},(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big)_{L^2_v} \\ =\Theta_{jk}(g_\pm+h_\pm)+\big(\mu^{\frac{1}{2}},g_\pm\big)_{L^2_v},\quad j\neq k, \\ \quad \partial_t\Lambda_j\big((\mathbf{I}_\pm-\mathbf{P}_\pm)f\big)+\partial_jc=\Lambda_j(g_\pm+h_\pm), \end{cases} \tag{2.20}$$

where $h_\pm=-v\cdot\nabla_x(\mathbf{I}_\pm-\mathbf{P}_\pm)f+L_\pm f$. By taking the mean value of every two equations with sign $\pm$ in (2.19), we have

$$\begin{cases} \quad \partial_t\left(\dfrac{a_++a_-}{2}\right)+\nabla_x\cdot b=0, \\ \quad \partial_tb_j+\partial_j\left(\left(\dfrac{a_++a_-}{2}\right)+2c\right)+\dfrac{1}{2}\displaystyle\sum_{k=1}^3\partial_k\Theta_{jk}\big((\mathbf{I}-\mathbf{P})f\cdot[1,1]\big) \\ =\dfrac{1}{2}\big(g_++g_-,v_j\mu^{\frac{1}{2}}\big)_{L^2_v}, \\ \quad \partial_tc+\dfrac{1}{3}\nabla_x\cdot b+\dfrac{5}{6}\displaystyle\sum_{j=1}^3\partial_j\Lambda_j\big((\mathbf{I}-\mathbf{P})f\cdot[1,1]\big)=\dfrac{1}{12}\big(g_++g_-,(|v|^2-3)\mu^{\frac{1}{2}}\big)_{L^2_v} \end{cases}$$

for $1\le j\le 3$. Similarly, taking the mean value with $\pm$ of the equation in (2.20), we have

$$\begin{cases} \partial_t\left(\dfrac{1}{2}\Theta_{jk}\big((\mathbf{I}_\pm-\mathbf{P}_\pm)f\cdot[1,1]\big)+2c\delta_{jk}\right)+\partial_jb_k+\partial_kb_j=\dfrac{1}{2}\Theta_{jk}(g_++g_-+h_++h_-), \\ \dfrac{1}{2}\partial_t\Lambda_j\big((\mathbf{I}_\pm-\mathbf{P}_\pm)f\cdot[1,1]\big)+\partial_jc=\dfrac{1}{2}\Lambda_j(g_++g_-+h_++h_-) \end{cases}$$

for $1\le j,k\le 3$. δ_{jk} is the Kronecker delta. Moreover, for obtaining the dissipation of the electric field E, we take the difference with sign $\pm$ in the first two equations in (2.19), we have

$$\begin{cases} \partial_t(a_+-a_-)+\nabla_x\cdot G=0, & (2.21a)\\ \partial_t G+\nabla_x(a_+-a_-)-2E+\nabla_x\cdot\Theta\big((\mathbf{I}-\mathbf{P})f\cdot[1,-1]\big) \\ =\big((g+Lf)\cdot[1,-1],v\mu^{\frac{1}{2}}\big)_{L^2_v}, & (2.21b)\end{cases}$$

where

$$G=\big(v\mu^{\frac{1}{2}},(\mathbf{I}-\mathbf{P})f\cdot[1,-1]\big)_{L^2_v}.$$

Recall that $E=-\nabla_x\phi$. Then by Eq. (1.5), we have

$$\nabla_x\cdot E=a_+-a_-. \tag{2.22}$$

3 Regularity

In this section, we will prove the smoothing effect of solutions to the Vlasov-Poisson-Boltzmann system with lower-order initial data. Let $K\ge 4$ and $l\ge 0$. The Vlasov-Poisson-Boltzmann system reads

$$\begin{cases} \partial_t f_\pm+v_i\partial^{e_i}f_\pm\pm\frac{1}{2}\partial^{e_i}\phi v_i f_\pm\mp\partial^{e_i}\phi\partial_{e_i}f_\pm\pm\partial^{e_i}\phi v_i\mu^{\frac{1}{2}}-L_\pm f=\Gamma_\pm(f,f),\\ -\Delta_x\phi=\int_{\mathbb{R}^3}(f_+-f_-)\mu^{\frac{1}{2}}dv,\quad \phi\to 0\quad\text{as}\quad |x|\to\infty,\\ f_\pm|_{t=0}=f_{0,\pm}.\end{cases} \tag{3.1}$$

The index appearing in both superscript and subscript means the summation. Our goal is to obtain the a priori estimate from these equations. To extract the smoothing estimate, we let $N=N(\alpha,\beta)>0$ be a large number chosen later. Assume $T\in(0,1],t\in[0,T]$ and

$$\psi=t^N,\quad \psi_k=\begin{cases}1, & \text{if}\quad k\le 0,\\ \psi^k, & \text{if}\quad k>0\end{cases} \tag{3.2}$$

is this section. Then $|\partial_t\psi_k|\lesssim\psi_{k-1/N}$. Let f be the smooth solution to (1.4)-(1.6) over $0\le t\le T$ and assume the a priori assumption

$$\sup_{0\le t\le T}\mathcal{E}_{K,l}(t)\le\delta_0, \tag{3.3}$$

where $\delta_0 \in (0,1)$ is a suitably small constant. Under this assumption, we can derive a simple fact that

$$\|\phi\|_{L^\infty} \lesssim \|\phi\|_{H^2_x} \le \delta_0, \quad \|e^{\pm\phi}\|_{L^\infty} \approx 1.$$

Also, by Eq. (2.21a) and Gagliardo-Nirenberg interpolation inequality (cf. [27, Theorem 12.83]), we have

$$\partial_t \phi = -\Delta_x^{-1} \partial_t (a_+ - a_-) = \Delta_x^{-1} \nabla_x \cdot G, \tag{3.4}$$

$$\begin{aligned} \|\partial_t \phi\|_{L^\infty} &\lesssim \|\nabla_x \partial_t \phi\|^{\frac{1}{2}}_{L^2_x} \|\nabla^2_x \partial_t \phi\|^{\frac{1}{2}}_{L^2_x} \lesssim \|\nabla_x G\|_{H^1_x} \\ &\lesssim \|(\mathbf{I}-\mathbf{P})f\|_{L^2_v H^1_x} \lesssim (\mathcal{E}_{K,l})^{\frac{1}{2}}(t). \end{aligned} \tag{3.5}$$

Theorem 3.1. *Assume* $-3 < \gamma \le -2s, 0 < s < 1, K \ge 4, l \ge 0$. *Let* f *be the solution to* (1.4)-(1.6) *satisfying that*

$$\epsilon_1^2 = \mathcal{E}_{4,l}(0), \quad \sup_{0 \le t \le T} \left\| \langle v \rangle^{C_{K,l}} f(t) \right\|^2_{L^2_{v,x}} < \infty$$

for some large constant $C_{K,l} > 0$ *depending on* K,l. *Then there exists* $t_0 \in (0,1)$ *such that*

$$\sup_{0 \le t \le t_0} \mathcal{E}_{K,l}(t) \le C_{K,l} \epsilon_1^2.$$

The reason of choosing $\psi_{|\alpha|+|\beta|-4}$ in (1.12) is that whenever $K \ge 4$, the initial value $\mathcal{E}_{K,l}(0) = \mathcal{E}_{4,l}(0)$, since $\psi_{|\alpha|+|\beta|-4}|_{t=0} = 0$ whenever $|\alpha|+|\beta| \ge 5$. In order to prove Theorem 3.1, we give the following a priori estimates.

Lemma 3.1. *For any* $l \ge 0$, *there is* $\mathcal{E}_{K,l}$ *satisfying* (1.12) *such that for* $0 \le t \le T$

$$\begin{aligned} \partial_t \mathcal{E}_{K,l}(t) + \lambda \mathcal{D}_{K,l}(t) \lesssim& \|\partial_t \phi\|_{L^\infty_x} \mathcal{E}_{K,l}(t) + \mathcal{E}_{K,l} \\ &+ \sum_{|\alpha|+|\beta| \le K} \left\| \psi_{|\alpha|+|\beta|-4-\frac{1}{2N}} w_l(\alpha,\beta) \partial^\alpha_\beta f \right\|^2_{L^2_{v,x}}, \end{aligned} \tag{3.6}$$

where $D_{K,l}$ *is defined by* (1.13).

Proof. For any $K \ge 4$ being the total derivative of v,x, we let $|\alpha|+|\beta| \le K$. On one hand, we apply ∂^α to Eq. (3.1) to get

$$\begin{aligned} &\partial_t \partial^\alpha f_\pm + v_i \partial^{e_i+\alpha} f_\pm \pm \frac{1}{2} \sum_{\alpha_1 \le \alpha} \partial^{e_i+\alpha_1} \phi v_i \partial^{\alpha-\alpha_1} f_\pm \\ &\mp \sum_{\alpha_1 \le \alpha} \partial^{e_i+\alpha_1} \phi \partial^{\alpha-\alpha_1}_{e_i} f_\pm \pm \partial^{e_i+\alpha} \phi v_i \mu^{\frac{1}{2}} - \partial^\alpha L_\pm f = \partial^\alpha \Gamma_\pm(f,f). \end{aligned} \tag{3.7}$$

On the other hand, we apply ∂^α_β to Eq. (3.1). Then,

$$\begin{aligned}&\partial_t\partial^\alpha_\beta f_\pm+\sum_{\beta_1\le\beta}\partial_{\beta_1}v_i\partial^{e_i+\alpha}_{\beta-\beta_1}f_\pm\pm\frac{1}{2}\sum_{\alpha_1\le\alpha}\sum_{\beta_1\le\beta}\partial^{e_i+\alpha_1}\phi\partial_{\beta_1}v_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f_\pm\\&\quad\mp\sum_{\alpha_1\le\alpha}\partial^{e_i+\alpha_1}\phi\partial^{\alpha-\alpha_1}_{\beta+e_i}f_\pm\pm\partial^{e_i+\alpha}\phi\partial_\beta\big(v_i\mu^{\frac{1}{2}}\big)-\partial^\alpha_\beta L_\pm f=\partial^\alpha_\beta\Gamma_\pm(f,f).\end{aligned}\tag{3.8}$$

Step 1. Estimate without weight. For the estimate without weight, we take the case $|\alpha|\le K$ and $\beta=0$. This case is for obtaining the term $\|\partial^\alpha\nabla_x\phi\|^2_{L^2_x}$ on the left-hand side of the energy inequality. Taking inner product of Eq. (3.7) with $\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm$ over $\mathbb{R}^3_v\times\mathbb{R}^3_x$, we have

$$\begin{aligned}&\big(\partial_t\partial^\alpha f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}+\big(v_i\partial^{e_i+\alpha}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}\\&\quad\pm\left(\frac{1}{2}\sum_{\alpha_1\le\alpha}\partial^{e_i+\alpha_1}\phi v_i\partial^{\alpha-\alpha_1}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\right)_{L^2_{v,x}}\\&\quad\mp\left(\sum_{\alpha_1\le\alpha}\partial^{e_i+\alpha_1}\phi\partial^{\alpha-\alpha_1}_{e_i}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\right)_{L^2_{v,x}}\\&\quad\pm\big(\partial^{e_i+\alpha}\phi v_i\mu^{1/2},\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}-\big(\partial^\alpha L_\pm f,\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}\\&=\big(\partial^\alpha\Gamma_\pm(f,f),\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}.\end{aligned}$$

Now we take the summation on $\pm$ and real part and denote these resulting terms by I_1 to I_7. In the following, we estimate them term by term. For the term I_1,

$$\begin{aligned}I_1=&\frac{1}{2}\partial_t\sum_\pm\big\|e^{\frac{\pm\phi}{2}}\psi_{|\alpha|-4}\partial^\alpha f_\pm\big\|^2_{L^2_{v,x}}\mp\mathrm{Re}\sum_\pm\frac{1}{2}\big(\partial_t\phi e^{\pm\phi}\partial^\alpha f_\pm,\psi_{2|\alpha|-8}\partial^\alpha f_\pm\big)_{L^2_{v,x}}\\&-\mathrm{Re}\sum_\pm\big(\partial_t(\psi_{|\alpha|-4})\partial^\alpha f_\pm,\psi_{|\alpha|-4}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}.\end{aligned}\tag{3.9}$$

The second term on the right-hand side of (3.9) is estimated as

$$\left|\frac{1}{2}\big(\partial_t\phi\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm,\partial^\alpha f_\pm\big)_{L^2_{v,x}}\right|\lesssim\|\partial_t\phi\|_{L^\infty}\big\|\psi_{|\alpha|-4}\partial^\alpha f_\pm\big\|^2_{L^2_{v,x}}\lesssim\|\partial_t\phi\|_{L^\infty}\mathcal{E}_{K,l}(t).$$

The third right-hand side term of (3.9) is estimated as

$$\big|\big(\partial_t(\psi_{|\alpha|-4})\partial^\alpha f_\pm,\psi_{|\alpha|-4}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}\big|\lesssim\big\|\psi_{|\alpha|-4-\frac{1}{2N}}\partial^\alpha f\big\|^2_{L^2_{v,x}}.$$

For the second term I_2, we will combine it with I_3 and $\alpha_1=0$. It turns out that the sum is zero. This is what $e^{\pm\phi}$ designed for, cf. [24]. Taking integration by parts on x, one has

$$\big(v_i\partial^{e_i+\alpha}f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}\pm\left(\frac{1}{2}\partial^{e_i}\phi v_i\partial^\alpha f_\pm,\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\right)_{L^2_{v,x}}=0. \quad (3.10)$$

For the left terms in I_3, the weight will be used. In this case, $|\alpha_1|\geq 1$ and by Lemma 2.9, it is bounded above by $\mathcal{E}^{1/2}_{K,l}\mathcal{D}_{K,l}$. Using Lemma 2.10, the term I_4 is also bounded above by $\mathcal{E}^{1/2}_{K,l}\mathcal{D}_{K,l}$.

For the term I_5, we will decompose $e^{\pm\phi}$ into $(e^{\pm\phi}-1)$ and 1. Recall Eqs. (2.22) and (2.21). For the part of 1, we have

$$\begin{aligned}
&\sum_\pm \pm\mathrm{Re}\big(\partial^{e_i+\alpha}\phi v_i\mu^{\frac{1}{2}},\psi_{2|\alpha|-8}\partial^\alpha f_\pm\big)_{L^2_{v,x}}\\
&=-\mathrm{Re}\big(\partial^\alpha\phi,\psi_{2|\alpha|-8}\partial^\alpha\nabla_x\cdot G\big)_{L^2_x}\\
&=\mathrm{Re}\big(\partial^\alpha\phi,\psi_{2|\alpha|-8}\partial^\alpha\partial_t(a_+-a_-)\big)_{L^2_x}\\
&=\frac{1}{2}\partial_t\big\|\psi_{|\alpha|-4}\partial^\alpha\nabla_x\phi\big\|^2_{L^2_x}.
\end{aligned}$$

For the part of $(e^{\pm\phi}-1)$, notice that

$$|e^{\pm\phi}-1|\lesssim\|\phi\|_{L^\infty}\lesssim\|\nabla_x\phi\|_{H^1_x}.$$

Then

$$\begin{aligned}
&\left|\sum_\pm \pm\mathrm{Re}\big(\partial^{e_i+\alpha}\phi v_i\mu^{\frac{1}{2}},(e^{\pm\phi}-1)\psi_{2|\alpha|-8}\partial^\alpha f_\pm\big)_{L^2_{v,x}}\right|\\
&\lesssim\|\nabla_x\phi\|_{H^1_x}\sum_{|\alpha|\leq K}\big\|\partial^\alpha\nabla_x\phi\big\|_{L^2_{v,x}}\sum_{|\alpha|\leq K}\|\partial^\alpha(\mathbf{I}-\mathbf{P})f\|_{L^2_{v,x}}\\
&\lesssim\mathcal{E}^{\frac{1}{2}}_{K,l}(t)\mathcal{D}_{K,l}(t).
\end{aligned}$$

For the term I_6, since $L_\pm$ commutes with ∂^α and $e^{\pm\phi}$, by Lemma 2.4, we have

$$I_6=-\sum_\pm\big(\partial^\alpha L_\pm f,\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}\geq\lambda\sum_\pm\|\psi_{|\alpha|-4}\partial^\alpha(\mathbf{I}_\pm-\mathbf{P}_\pm)f\|^2_{L^2_xL^2_D}.$$

For the term I_7, by Lemma 2.8, we have

$$|I_7|=\left|\sum_\pm\big(\partial^\alpha\Gamma_\pm(f,f),\psi_{2|\alpha|-8}e^{\pm\phi}\partial^\alpha f_\pm\big)_{L^2_{v,x}}\right|\lesssim\mathcal{E}^{\frac{1}{2}}_{K,l}(t)\mathcal{D}_{K,l}(t).$$

Therefore, combining all the estimates above and taking the summation on $|\alpha|\le K$, we conclude that,

$$\begin{aligned}
&\frac{1}{2}\partial_t\sum_{\pm}\sum_{|\alpha|\le K}\Big(\big\|\psi_{|\alpha|-4}e^{\frac{\pm\phi}{2}}\partial^\alpha f_\pm\big\|_{L^2_{v,x}}+\big\|\psi_{|\alpha|-4}\partial^\alpha\nabla_x\phi\big\|^2_{L^2_x}\Big)\\
&\quad+\lambda\sum_{\pm}\sum_{|\alpha|\le K}\big\|\psi_{|\alpha|-4}\partial^\alpha(\mathbf{I}_\pm-\mathbf{P}_\pm)f\big\|^2_{L^2_xL^2_D}\\
&\lesssim\|\partial_t\phi\|_{L^\infty}\mathcal{E}_{K,l}(t)+\mathcal{E}^{\frac{1}{2}}_{K,l}(t)\mathcal{D}_{K,l}(t)+\sum_{|\alpha|\le K}\big\|\psi_{|\alpha|-4-\frac{1}{2N}}w_l(|\alpha|,0)\partial^\alpha f\big\|^2_{L^2_{v,x}}. \qquad (3.11)
\end{aligned}$$

Step 2. Estimate with weight on the mixed derivatives. Let $K\ge4,|\alpha|+|\beta|\le K$. Taking inner product of Eq. (3.8) with $\psi_{2|\alpha|+2|\beta|-8}e^{\pm\phi}w^2_l(\alpha,\beta)\partial^\alpha_\beta f_\pm$ over $\mathbb{R}^3_v\times\mathbb{R}^3_x$, one has

$$\begin{aligned}
&\big(\partial_t\partial^\alpha_\beta f,e^{\pm\phi}\psi_{2|\alpha|+2|\beta|-8}w^2_l(\alpha,\beta)\partial^\alpha_\beta f\big)_{L^2_{v,x}}\\
&\quad+\Big(\sum_{\beta_1\le\beta}\partial_{\beta_1}v_i\partial^{e_i+\alpha}_{\beta-\beta_1}f,e^{\pm\phi}\psi_{2|\alpha|+2|\beta|-8}w^2_l(\alpha,\beta)\partial^\alpha_\beta f\Big)_{L^2_{v,x}}\\
&\quad\pm\Big(\frac{1}{2}\sum_{\alpha_1\le\alpha,\beta_1\le\beta}\partial^{e_i+\alpha_1}\phi\partial_{\beta_1}v_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f,e^{\pm\phi}\psi_{2|\alpha|+2|\beta|-8}w^2_l(\alpha,\beta)\partial^\alpha_\beta f\Big)_{L^2_{v,x}}\\
&\quad\mp\Big(\sum_{\alpha_1\le\alpha}\partial^{e_i+\alpha_1}\phi\partial^{\alpha-\alpha_1}_{\beta+e_i}f,e^{\pm\phi}\psi_{2|\alpha|+2|\beta|-8}w^2_l(\alpha,\beta)\partial^\alpha_\beta f\Big)_{L^2_{v,x}}\\
&\quad\pm\big(\partial^{e_i+\alpha}\phi\partial_\beta\big(v_i\mu^{\frac{1}{2}}\big),e^{\pm\phi}\psi_{2|\alpha|+2|\beta|-8}w^2_l(\alpha,\beta)\partial^\alpha_\beta f\big)_{L^2_{v,x}}\\
&\quad-\big(\partial^\alpha_\beta L_\pm f,e^{\pm\phi}\psi_{2|\alpha|+2|\beta|-8}w^2_l(\alpha,\beta)\partial^\alpha_\beta f\big)_{L^2_{v,x}}\\
&=\big(\partial^\alpha_\beta\Gamma_\pm(f,f),e^{\pm\phi}\psi_{2|\alpha|+2|\beta|-8}w^2_l(\alpha,\beta)\partial^\alpha_\beta f\big)_{L^2_{v,x}}.
\end{aligned}$$

Now we denote these terms with summation $\sum_\pm$ by J_1 to J_7 and estimate them term by term. The estimate of J_1 to J_4 are similar to I_1 to I_4. For J_1, we have

$$\begin{aligned}
J_1\ge\,&\partial_t\sum_{\pm}\big\|e^{\frac{\pm\phi}{2}}\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm\big\|_{L^2_{v,x}}-C\|\partial_t\phi\|_{L^\infty}\mathcal{E}_{K,l}(t)\\
&-\sum_{\pm}\big\|\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm\big\|^2_{L^2_{v,x}}.
\end{aligned}$$

Similar to (3.10), J_2 and J_3 with $\alpha_1=0$ are canceled by using integration by parts. Using Lemmas 2.9 and 2.10, the left case $\alpha_1\ne0$ in J_3 together with J_4 are bounded above by $\mathcal{E}_{K,l}^{1/2}(t)\mathcal{D}_{K,l}(t)$. For the term J_5, we only need an upper bound

$$\begin{aligned}|J_5|&=\Big|\sum_{\pm}\pm\big(\partial^{e_i+\alpha}\phi\partial_\beta(v_i\mu^{\frac{1}{2}}),\psi_{2|\alpha|+2|\beta|-8}e^{\pm\phi}w_l^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{v,x}}\Big|\\&\lesssim\eta\sum_{\pm}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha f_\pm\big\|^2_{L^2_xL^2_D}+C_\eta\big\|\psi_{|\alpha|-4}\partial^\alpha\nabla_x\phi\big\|^2_{L^2_{v,x}},\quad\forall\eta>0.\end{aligned}$$

Notice that $\|\psi_{|\alpha|-4}\partial^\alpha\nabla_x\phi\|^2_{L^2_{v,x}}$ is bounded above by $\mathcal{E}_{K,l}$. For the term J_6, since $L_\pm$ commutes with $e^{\pm\phi}$, by Lemma 2.4, we have

$$\begin{aligned}J_6=&-\sum_{\pm}\big(\partial_\beta^\alpha L_\pm f,\psi_{2|\alpha|+2|\beta|-8}e^{\pm\phi}w_l^2(\alpha,\beta)\partial_\beta^\alpha f_\pm\big)_{L^2_{v,x}}\\\geq&\lambda\sum_{\pm}\big\|\psi_{|\alpha|+|\beta|-4}e^{\frac{\pm\phi}{2}}w_l(\alpha,\beta)\partial_\beta^\alpha f_\pm\big\|^2_{L^2_xL^2_D}-C_\eta\sum_{\pm}\|\partial^\alpha f_\pm\|^2_{L^2_xL^2_D}\\&-\eta\sum_{\pm}\sum_{|\beta_1|\leq|\beta|}\big\|\psi_{|\alpha|+|\beta|-4}e^{\frac{\pm\phi}{2}}w_l(\alpha,\beta_1)\partial_{\beta_1}^\alpha f_\pm\big\|^2_{L^2_xL^2_D},\quad\forall\eta>0.\end{aligned}$$

Here we use the fact that $\|w_l(\alpha,\beta)(\cdot)\|_{L^2(B_{C_\eta})}\lesssim\|\cdot\|_{L^2_D}$. The term J_7, by Lemma 2.8, is bounded above by

$$\mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}+\mathcal{E}_{K,l}\mathcal{D}_{K,l}^{\frac{1}{2}}\lesssim\big(\mathcal{E}_{K,l}^{\frac{1}{2}}+\mathcal{E}_{K,l}\big)\mathcal{D}_{K,l}+\mathcal{E}_{K,l}.$$

Combining all the above estimate, taking summation on $|\alpha|+|\beta|\leq K$ and letting η sufficiently small, we have

$$\begin{aligned}&\frac{1}{2}\partial_t\sum_{|\alpha|+|\beta|\leq K,\pm}\big\|e^{\frac{\pm\phi}{2}}\psi_{|\alpha|+|\beta|-4}\partial_\beta^\alpha f_\pm\big\|^2_{L^2_{v,x}}\\&\quad+\lambda\sum_{|\alpha|+|\beta|\leq K,\pm}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial_\beta^\alpha f_\pm\big\|^2_{L^2_xL^2_D}\\&\lesssim\|\partial_t\phi\|_{L^\infty_x}\mathcal{E}_{K,l}(t)+\sum_{|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}w_l(\alpha,\beta)\partial_\beta^\alpha f\big\|^2_{L^2_{v,x}}\\&\quad+\big(\mathcal{E}_{K,l}^{\frac{1}{2}}+\mathcal{E}_{K,l}\big)\mathcal{D}_{K,l}+\mathcal{E}_{K,l}.\end{aligned}\tag{3.12}$$

Together with (3.3), taking combination (3.11)+(3.12), we have

$$\begin{aligned}\partial_t\mathcal{E}_{K,l}(t)+\lambda\mathcal{D}_{K,l}(t)\lesssim&\|\partial_t\phi\|_{L^\infty_x}\mathcal{E}_{K,l}(t)+\mathcal{E}_{K,l}\\&+\sum_{|\alpha|+|\beta|\leq K}\big\|\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}w_l(\alpha,\beta)\partial_\beta^\alpha f\big\|^2_{L^2_{v,x}},\end{aligned}\tag{3.13}$$

where we let

$$\mathcal{E}_{K,l}(t)=\sum_{\pm}\sum_{|\alpha|+|\beta|\le K}\left\|e^{\frac{\pm\phi}{2}}\psi_{|\alpha|+|\beta|-4}\partial^\alpha_\beta f_\pm\right\|^2_{L^2_{v,x}}+\sum_{|\alpha|\le K}\left\|\psi_{|\alpha|-4}\partial^\alpha\nabla_x\phi\right\|^2_{L^2_x}. \tag{3.14}$$

It is straightforward to show that $\mathcal{E}_{K,l}$ satisfies (1.12). Notice that there is a term $\|\psi_{|\alpha|-4}\partial^\alpha E(t)\|^2_{L^2_x}$ in $\mathcal{E}_{K,l}$ on the right-hand side of (3.13), and hence, we can put $\|\psi_{|\alpha|-4}\partial^\alpha E(t)\|^2_{L^2_x}$, which is in $\mathcal{D}_{K,l}$, on the left-hand side. □

Therefore, now it suffices to control the last term in (3.6).

Lemma 3.2. *Let $K\ge 4$ and f to be the solution to* (1.4)-(1.6) *and assume the same assumption as in Lemma* 3.1. *It holds that for any $0<\delta<1$ and multi-indices $|\alpha|+|\beta|\le K$, there exists $N=N(\alpha,\beta)>1$ such that*

$$\begin{aligned}
&\left\|\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}w_l(\alpha,\beta)\partial^\alpha_\beta f\right\|^2_{L^2_{v,x}}\\
&\lesssim\delta^2\partial_t\Big(-\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\\
&\quad+\delta^2\Big(\mathcal{D}_{K,l}+\big(\mathcal{E}^{\frac{1}{2}}_{K,l}+\mathcal{E}_{K,l}\big)\mathcal{D}_{K,l}+\|\partial_t\phi\|_{L^\infty_x}\mathcal{E}_{K,l}(t)+\mathcal{E}_{K,l}\Big)\\
&\quad+C_\delta\left\|\langle v\rangle^{C_{K,l}}f\right\|^2_{L^2_{v,x}},
\end{aligned}\tag{3.15}$$

where $\theta^w=\theta^w(v,D_v)$ and $\theta\in S(1)$ is defined by (3.18).

Proof. The proof will be split into two steps.

Step 1. To deal with the last term of (3.6), we choose constants

$$\begin{aligned}
&\delta_1=\delta_1(\alpha,\beta)\in\big(0,\min\{2s/(1+2s),1/2\}\big],\\
&\delta_2=1-\delta_1\in\big[\max\{1/(1+2s),1/2\},1\big),\\
&l_0=\gamma\delta_2<0
\end{aligned}\tag{3.16}$$

to be determined later. Let χ_0 to be a smooth cutoff function such that $\chi_0(z)$ equal to 1 when $|z|<1/2$ and equal to 0 when $|z|\ge 1$. Define

$$\tilde{b}(v,y)=\langle v\rangle^{l_0}|y|^{\delta_1}, \tag{3.17}$$

$$\chi(v,\eta)=\chi_0\left(\frac{\langle\eta\rangle\langle v\rangle^{l_0}}{|y|^{\delta_2}}\right),$$

$$\theta(v,\eta)=\langle v\rangle^{l_0}|y|^{-1-\delta_2}y\cdot\eta\chi(v,\eta). \tag{3.18}$$

If $|\alpha|>4$, we choose $N=N(\alpha)>1$ such that

$$-\frac{2N\times(|\alpha|-4)-1}{2}=-\frac{|\alpha|}{\delta_1}. \tag{3.19}$$

In this case, we have

$$\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}=t^{N(|\alpha|+|\beta|-4-\frac{1}{2N})}=t^{N(|\alpha|-4-\frac{1}{2N})}\times t^{N|\beta|}=\psi_{|\alpha|-4-\frac{1}{2N}}\psi_{|\beta|}. \tag{3.20}$$

Then, by the definition (3.17) of $\tilde{b}$ and Young's inequality,

$$\begin{aligned}\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}\lesssim&\,\psi_{|\beta|}\Bigg(\delta\bigg(\big(\tilde{b}^{\frac{1}{2}}\big)^{\frac{|\alpha|-4-1/(2N)}{|\alpha|-4}}\psi_{|\alpha|-4-\frac{1}{2N}}\bigg)^{\frac{|\alpha|-4}{|\alpha|-4-1/(2N)}}\\&\qquad+C_{0,\delta}\bigg(\big(\tilde{b}^{-\frac{1}{2}}\big)^{\frac{|\alpha|-4-1/(2N)}{|\alpha|-4}}\bigg)^{2N(|\alpha|-4)}\Bigg)\\\lesssim&\,\delta\tilde{b}^{\frac{1}{2}}\psi_{|\alpha|+|\beta|-4}+C_{0,\delta}\Big(\langle v\rangle^{-\frac{l_0|\alpha|}{\delta_1}}|y|^{-|\alpha|}\Big)\psi_{|\beta|-4-\frac{1}{2N}},\end{aligned} \tag{3.21}$$

where we used the assumption $t\le t_0<1$ as in Lemma 3.1 to obtain $\psi_{|\beta|}\le\psi_{|\beta|-4-1/2N}$.

If $|\alpha|=0$, we can simply use

$$\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}=\psi_{|\beta|-4-\frac{1}{2N}}.$$

If $0<|\alpha|\le 4$ and $|\beta|>4$, we have

$$\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}=t^{N(|\alpha|+|\beta|-4-\frac{1}{2N})}=\psi_{|\alpha|-\frac{1}{2N}}\psi_{|\beta|-4}.$$

Then we choose $N=N(\alpha)>1$ by

$$-\frac{2N|\alpha|-1}{2}=-\frac{|\alpha|}{\delta_1}, \tag{3.22}$$

and use Young's inequality to obtain

$$\begin{aligned}\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}\lesssim&\,\psi_{|\beta|-4}\Bigg(\delta\bigg(\big(\tilde{b}^{\frac{1}{2}}\big)^{\frac{|\alpha|-1/(2N)}{|\alpha|}}\psi_{|\alpha|-\frac{1}{2N}}\bigg)^{\frac{|\alpha|}{|\alpha|-1/(2N)}}\\&\qquad+C_{0,\delta}\bigg(\big(\tilde{b}^{-\frac{1}{2}}\big)^{\frac{|\alpha|-1/(2N)}{|\alpha|}}\bigg)^{2N|\alpha|}\Bigg)\\\lesssim&\,\delta\tilde{b}^{\frac{1}{2}}\psi_{|\alpha|+|\beta|-4}+C_{0,\delta}\Big(\langle v\rangle^{-\frac{l_0|\alpha|}{\delta_1}}|y|^{-|\alpha|}\Big)\psi_{|\beta|-4-\frac{1}{2N}},\end{aligned}$$

where we also used $\psi_{|\beta|-4}\le\psi_{|\beta|-4-1/2N}$ from $t\le t_0<1$.

If $0<|\alpha|\le 4, |\beta|\le 4$ and $|\alpha|+|\beta|>4$, we choose $N=N(\alpha,\beta)$ such that

$$-\frac{2N(|\alpha|+|\beta|-4)-1}{2}=-\frac{|\alpha|}{\delta_1}. \tag{3.23}$$

Then by the definition (3.17) of $\tilde{b}$ and Young's inequality,

$$\begin{aligned}\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}} &\lesssim \delta\left(\left(\tilde{b}^{\frac{1}{2}}\right)^{\frac{|\alpha|+|\beta|-4-1/(2N)}{|\alpha|+|\beta|-4}}\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}\right)^{\frac{|\alpha|+|\beta|-4}{|\alpha|+|\beta|-4-1/(2N)}} \\ &\quad +C_{0,\delta}\left(\left(\tilde{b}^{-\frac{1}{2}}\right)^{\frac{|\alpha|+|\beta|-4-1/(2N)}{|\alpha|+|\beta|-4}}\right)^{2N(|\alpha|+|\beta|-4)} \\ &\lesssim \delta\tilde{b}^{\frac{1}{2}}\psi_{|\alpha|+|\beta|-4}+C_{0,\delta}\left(\langle v\rangle^{-\frac{l_0|\alpha|}{\delta_1}}|y|^{-|\alpha|}\right),\end{aligned} \tag{3.24}$$

where $C_{0,\delta}$ is a large constant depending on $\delta>0, |\alpha|$ and $|\beta|$.

If $0<|\alpha|\le 4, |\beta|\le 4$ and $|\alpha|+|\beta|\le 4$, we simply choose $N>1$, use $t\le t_0<1$ to obtain

$$\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}\le 1,$$

and choose $\eta\in[0,1)$ such that $-\eta/(2(1-\eta))=-|\alpha|/\delta_1$. Then

$$\begin{aligned}\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}=1 &\lesssim \left(\delta^{\eta}\tilde{b}^{\frac{\eta}{2}}\right)^{\frac{1}{\eta}}+\left(\delta^{-\eta}\tilde{b}^{-\frac{\eta}{2}}\right)^{\frac{1}{1-\eta}} \\ &\lesssim \delta\tilde{b}^{\frac{1}{2}}+C_{0,\delta}\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}|y|^{-|\alpha|}.\end{aligned}$$

Therefore, for any $|\alpha|\ge 0$, taking the Fourier transform $(\cdot)^{\wedge}$ with respect to x, we have

$$\begin{aligned}&\left\|\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}w_l(\alpha,\beta)\partial^{\alpha}_{\beta}f\right\|_{L^2_{v,x}} \\ =&\left\|\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}w_l(\alpha,\beta)\left(\partial^{\alpha}_{\beta}f\right)^{\wedge}(v,y)\right\|_{L^2_{v,y}} \\ \lesssim&\,\delta\left\|\psi_{|\alpha|+|\beta|-4}\tilde{b}^{\frac{1}{2}}w_l(\alpha,\beta)\left(\partial^{\alpha}_{\beta}f\right)^{\wedge}(v,y)\right\|_{L^2_{v,y}} \\ &+C_{0,\delta}\left\|\psi_{|\beta|-4-\frac{1}{2N}}w_l(\alpha,\beta)\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}\partial_{\beta}f\right\|_{L^2_{v,x}}.\end{aligned} \tag{3.25}$$

To deal with the second right-hand side term of (3.25), we use a similar interpolation on $\tilde{a}^{1/2}$.

In fact, if $|\beta|>4$, we have

$$\begin{aligned}\psi_{|\beta|-4-\frac{1}{2N}}\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}\lesssim&\left(\psi_{|\beta|-4-\frac{1}{2N}}\left(\frac{\delta}{C_{0,\delta}}\tilde{a}^{\frac{1}{2}}\right)^{\frac{|\beta|-4-1/(2N)}{|\beta|-4}}\right)^{\frac{|\beta|-4}{|\beta|-4-1/(2N)}}\\&+\left(\left(C_{0,\delta}\delta^{-1}\tilde{a}^{-\frac{1}{2}}\right)^{\frac{|\beta|-4-1/(2N)}{|\beta|-4}}\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}\right)^{2N(|\beta|-4)}\\\lesssim&\frac{\delta}{C_{0,\delta}}\psi_{|\beta|-4}\tilde{a}^{\frac{1}{2}}+C_\delta\tilde{a}^{-\frac{1}{2}(2N(|\beta|-4)-1)}\langle v\rangle^{C_K},\end{aligned}\tag{3.26}$$

where $C_{0,\delta}$ comes from (3.21) and C_K depends only on K. Here, K is the total order of derivatives. Recalling the definition (1.8) of $\tilde{a}$, we have

$$\tilde{a}^{-\frac{1}{2}(2N(|\beta|-4)-1)}\lesssim\left(\langle v\rangle^{-\frac{\gamma}{2}}\langle\eta\rangle^{-s}\right)^{2N(|\beta|-4)-1}.$$

When $|\alpha|>4, N$ is given by (3.19), i.e. $2N=(2|\alpha|/\delta_1+1)/(|\alpha|-4)$, and hence,

$$\tilde{a}^{-\frac{1}{2}(2N(|\beta|-4)-1)}\lesssim\left(\langle v\rangle^{-\frac{\gamma}{2}}\langle\eta\rangle^{-s}\right)^{\frac{|\beta|-4}{|\alpha|-4}\left(\frac{2|\alpha|}{\delta_1}+1\right)-1}.$$

Then we choose $\delta_1=\delta_1(\alpha,\beta)>0$ sufficiently small such that

$$-s\left(\frac{|\beta|-4}{|\alpha|-4}\left(\frac{2|\alpha|}{\delta_1}+1\right)-1\right)\leq-|\beta|$$

and hence,

$$\tilde{a}^{-\frac{1}{2}(2N(|\beta|-4)-1)}\lesssim\langle v\rangle^{C_K}\langle\eta\rangle^{-|\beta|}.\tag{3.27}$$

When $0<|\alpha|\leq 4, N$ is given by (3.22), i.e. $2N=(2|\alpha|/\delta_1+1)/|\alpha|$, and hence,

$$\tilde{a}^{-\frac{1}{2}(2N(|\beta|-4)-1)}\lesssim\left(\langle v\rangle^{-\frac{\gamma}{2}}\langle\eta\rangle^{-s}\right)^{\frac{|\beta|-4}{|\alpha|}\left(\frac{2|\alpha|}{\delta_1}+1\right)-1}.$$

Then we choose $\delta_1=\delta_1(\alpha,\beta)>0$ sufficiently small such that

$$-s\left(\frac{|\beta|-4}{|\alpha|}\left(\frac{2|\alpha|}{\delta_1}+1\right)-1\right)\leq-|\beta|$$

and hence, we still have (3.27) in this case. When $|\alpha|=0, N$ can be arbitrarily large. Then we choose N sufficiently large that (3.27) is valid. Combining the above three cases, (3.26) becomes

$$\psi_{|\beta|-4-\frac{1}{2N}}\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}\lesssim\frac{\delta}{C_{0,\delta}}\psi_{|\beta|-4}\tilde{a}^{\frac{1}{2}}+C_\delta\langle v\rangle^{C_K}\langle\eta\rangle^{-|\beta|}.$$

If $|\beta|\le 4$, we choose $\eta\in(0,1)$ such that $-\eta/(2(1-\eta))=-|\beta|/(2s)$. Then

$$\begin{aligned}\psi_{|\beta|-4-\frac{1}{2N}}\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}=\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}&\lesssim\frac{\delta}{C_{0,\delta}}\tilde{a}^{\frac{1}{2}}+C_\delta\left(\tilde{a}^{-\frac{1}{2}}\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}\right)^{\frac{\eta}{(1-\eta)}}\\&\lesssim\frac{\delta}{C_{0,\delta}}\tilde{a}^{\frac{1}{2}}+C_\delta\langle v\rangle^{C_K}\langle\eta\rangle^{-|\beta|}.\end{aligned}$$

Thus, whenever $|\beta|\le 4$ or $|\beta|>4$, we have

$$\psi_{|\beta|-4-\frac{1}{2N}}\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}\in S\left(\frac{\delta}{C_{0,\delta}}\tilde{a}^{\frac{1}{2}}+C_\delta\langle v\rangle^{C_K}\langle\eta\rangle^{-|\beta|}\right)$$

uniformly in δ, as a symbol in (v,η). Then using Lemma 2.2 with respect to v, we have

$$\begin{aligned}&\left\|\psi_{|\beta|-4-\frac{1}{2N}}w_l(\alpha,\beta)\langle v\rangle^{\frac{-l_0|\alpha|}{\delta_1}}\partial_\beta f\right\|_{L^2_{v,x}}\\\lesssim&\frac{\delta}{C_{0,\delta}}\|\psi_{|\beta|-4}(\tilde{a}^{\frac{1}{2}})^w w_l(0,\beta_1)\partial_\beta f\|_{L^2_{v,x}}+C_\delta\|\langle v\rangle^{C_{K,l}}f\|_{L^2_{v,x}}\\\lesssim&\frac{\delta}{C_{0,\delta}}\mathcal{D}^{\frac{1}{2}}_{K,l}+C_\delta\|\langle v\rangle^{C_{K,l}}f\|_{L^2_{v,x}}.\end{aligned}$$

Plugging this into (3.25), we have

$$\begin{aligned}&\|\psi_{|\alpha|+|\beta|-4-\frac{1}{2N}}w_l(\alpha,\beta)\partial^\alpha_\beta f\|^2_{L^2_{v,x}}\\\lesssim&\delta^2\|\psi_{|\alpha|+|\beta|-4}\tilde{b}^{\frac{1}{2}}w_l(\alpha,\beta)(\partial^\alpha_\beta f)^\wedge(v,y)\|^2_{L^2_{v,y}}\\&+\delta^2\mathcal{D}_{K,l}+C_\delta\|\langle v\rangle^{C_{K,l}}f\|^2_{L^2_{v,x}}.\end{aligned}\tag{3.28}$$

Now it suffices to eliminate the first right-hand side term of (3.28).

Step 2. Recalling (3.18), we regard θ as a symbol in (v,η) with parameter y. Then,

$$|\theta(v,\eta)|=\langle v\rangle^{l_0}|y|^{-1-\delta_2}|y\cdot\eta|\chi(v,\eta)\lesssim 1.$$

Direct calculation gives that $\partial^\alpha_v\partial^\beta_\eta\theta\lesssim 1$ and hence, $\theta\in S(1)$ as a symbol on (v,η). On the other hand, regarding the Poisson bracket on (v,η) we have

$$\begin{aligned}\{\theta,v\cdot y\}=&\langle v\rangle^{l_0}|y|^{1-\delta_2}+\langle v\rangle^{l_0}|y|^{1-\delta_2}(\chi(v,\eta)-1)\\&+\langle v\rangle^{l_0}|y|^{-1-\delta_2}y\cdot\eta\partial_\eta\chi\cdot y\\=:&\tilde{b}+R_1+R_2.\end{aligned}$$

Now we claim that $R_1,R_2\in S(\tilde{a})$. Indeed, noticing the support of $\chi-1$, by (3.16) we have

$$|R_1|\le\langle v\rangle^{l_0}\langle\eta\rangle^{\frac{1-\delta_2}{\delta_2}}\langle v\rangle^{l_0\frac{1-\delta_2}{\delta_2}}\le\langle v\rangle^{\gamma}\langle\eta\rangle^{2s}\le\tilde{a}.$$

For R_2, since $1-2\delta_2\le 0$, we have

$$|R_2|\le\langle v\rangle^{2l_0}|y|^{1-2\delta_2}|\eta|\mathbf{1}_{\langle\eta\rangle\langle v\rangle^{l_0}\le|y|^{\delta_2}}\le\langle v\rangle^{\frac{l_0}{\delta_2}}\langle\eta\rangle^{\frac{1-\delta_2}{\delta_2}}\le\tilde{a}.$$

Higher derivative estimate can be calculated by Leibniz's formula and hence, $R_1,R_2\in S(\tilde{a})$. Thus, by Lemma 2.3 and (2.3), we have

$$\begin{aligned}\|\tilde{b}^{1/2}\widehat{g}(v,y)\|^2_{L^2_{v,y}}&=\big(\tilde{b}(v,y)\widehat{g},\widehat{g}\big)_{L^2_{v,y}}\\&=\mathrm{Re}\big(\{\theta,v\cdot y\}^w(v,D_v)\widehat{g},\widehat{g}\big)_{L^2_{v,y}}+\mathrm{Re}\big((R_1+R_2)^w(v,D_v)\widehat{g},\widehat{g}\big)_{L^2_{v,y}}\\&\le 2\pi\mathrm{Re}\big(iv\cdot y\widehat{g},\theta^w(v,D_v)\widehat{g}\big)_{L^2_{v,y}}+C\|(\tilde{a}^{\frac12})^w g\|^2_{L^2_{v,x}}\\&\le 2\pi\mathrm{Re}\big(v\cdot\nabla_x g,(\theta^w\widehat{g})^\vee\big)_{L^2_{v,x}}+C\|(\tilde{a}^{\frac12})^w g\|^2_{L^2_{v,x}}\end{aligned}\tag{3.29}$$

for any g in a suitable smooth space. Here and after, we write $\theta^w=\theta^w(v,D_v)$. Note that

$$\begin{aligned}&\mathrm{Re}2\pi\big(iv\cdot y\widehat{g},\theta^w(v,D_v)\widehat{g}\big)_{L^2_{v,y}}\\&=2\pi\big(iv\cdot y\widehat{g},\theta^w(v,D_v)\widehat{g}\big)_{L^2_{v,y}}+2\pi\big(\theta^w(v,D_v)\widehat{g},iv\cdot y\widehat{g}\big)_{L^2_{v,y}}\\&=2\pi\big(i[\theta(v,D_v),v\cdot y]^w\widehat{g},\widehat{g}\big)_{L^2_{v,y}}\\&=\big(\{\theta,v\cdot y\}^w(v,D_v)\widehat{g},\widehat{g}\big)_{L^2_{v,y}},\end{aligned}$$

and the Weyl quantization $(\cdot)^w$ is acting on (v,η) with parameter y.

Now we let $g=\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm e^{\pm\phi/2}$ in (3.29), then

$$\begin{aligned}&\big\|\tilde{b}^{\frac12}\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm\big)^\wedge(v,y)e^{\frac{\pm\phi}{2}}\big\|_{L^2_{v,x}}\\&\lesssim\mathrm{Re}\Big(v\cdot\nabla_x\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial^\alpha_\beta fe^{\frac{\pm\phi}{2}},\big(\theta^w\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}+\mathcal{D}_{K,l}\\&=:K_0+\mathcal{D}_{K,l}.\end{aligned}\tag{3.30}$$

By Eq. (1.4), we have

$$\begin{aligned}v\cdot\nabla_x\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)&=v_i\partial^{\alpha+e_i}_\beta f_\pm e^{\frac{\pm\phi}{2}}\pm\frac12 v_i\partial^{e_i}\phi e^{\frac{\pm\phi}{2}}\partial^\alpha_\beta f_\pm\\&=\partial_\beta\big(v_i\partial^{\alpha+e_i}f_\pm e^{\frac{\pm\phi}{2}}\big)-\sum_{0\ne\beta_1\le\beta}\partial_{\beta_1}v_i\partial^{\alpha+e_i}_{\beta-\beta_1}f_\pm e^{\frac{\pm\phi}{2}}\pm\frac12 v_i\partial^{e_i}\phi e^{\frac{\pm\phi}{2}}\partial^\alpha_\beta f_\pm\end{aligned}$$

$$
\begin{aligned}
=&-\partial_t\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\mp\frac{1}{2}\sum_{\alpha_1\le\alpha}\sum_{\beta_1\le\beta}\partial^{e_i+\alpha_1}\phi\partial_{\beta_1}v_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f_\pm e^{\frac{\pm\phi}{2}}\\
&\pm\sum_{\alpha_1\le\alpha}C^{\alpha_1}_\alpha\partial^{e_i+\alpha_1}\phi\partial^{\alpha-\alpha_1}_{\beta+e_i}f_\pm e^{\frac{\pm\phi}{2}}\mp\partial^{e_i+\alpha}\phi\partial_\beta(v_i\mu^{\frac{1}{2}})e^{\frac{\pm\phi}{2}}+\partial^\alpha_\beta L_\pm f e^{\frac{\pm\phi}{2}}\\
&+\partial^\alpha_\beta\Gamma_\pm(f,f)e^{\frac{\pm\phi}{2}}-\sum_{0\ne\beta_1\le\beta}\partial_{\beta_1}v_i\partial^{\alpha+e_i}_{\beta-\beta_1}f_\pm e^{\frac{\pm\phi}{2}}\pm\frac{1}{2}v_i\partial^{e_i}\phi e^{\frac{\pm\phi}{2}}\partial^\alpha_\beta f_\pm.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
K_0=\psi_{2|\alpha|+2|\beta|-8}\Bigg(&\mathrm{Re}\left(-w_l(\alpha,\beta)\partial_t\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\left(\theta^w w_l(\alpha,\beta)\left(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\right)^\wedge\right)^\vee\right)_{L^2_{v,x}}\\
&\mp\mathrm{Re}\left(w_l(\alpha,\beta)\frac{1}{2}\sum_{\substack{\alpha_1\le\alpha\\ \beta_1\le\beta}}\partial^{e_i+\alpha_1}\phi\partial_{\beta_1}v_i\partial^{\alpha-\alpha_1}_{\beta-\beta_1}f_\pm e^{\frac{\pm\phi}{2}},\left(\theta^w w_l(\alpha,\beta)\left(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\right)^\wedge\right)^\vee\right)_{L^2_{v,x}}\\
&\pm\mathrm{Re}\left(w_l(\alpha,\beta)\sum_{\alpha_1\le\alpha}C^{\alpha_1}_\alpha\partial^{e_i+\alpha_1}\phi\partial^{\alpha-\alpha_1}_{\beta+e_i}f_\pm e^{\frac{\pm\phi}{2}},\left(\theta^w w_l(\alpha,\beta)\left(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\right)^\wedge\right)^\vee\right)_{L^2_{v,x}}\\
&\mp\mathrm{Re}\left(w_l(\alpha,\beta)\partial^{e_i+\alpha}\phi\partial_\beta(v_i\mu^{\frac{1}{2}})e^{\frac{\pm\phi}{2}},\left(\theta^w w_l(\alpha,\beta)\left(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\right)^\wedge\right)^\vee\right)_{L^2_{v,x}}\\
&+\mathrm{Re}\left(w_l(\alpha,\beta)\partial^\alpha_\beta L_\pm f e^{\frac{\pm\phi}{2}},\left(\theta^w w_l(\alpha,\beta)\left(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\right)^\wedge\right)^\vee\right)_{L^2_{v,x}}\\
&+\mathrm{Re}\left(w_l(\alpha,\beta)\partial^\alpha_\beta\Gamma_\pm(f,f)e^{\frac{\pm\phi}{2}},\left(\theta^w w_l(\alpha,\beta)\left(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\right)^\wedge\right)^\vee\right)_{L^2_{v,x}}\\
&-\mathrm{Re}\left(w_l(\alpha,\beta)\sum_{0\ne\beta_1\le\beta}\partial_{\beta_1}v_i\partial^{\alpha+e_i}_{\beta-\beta_1}f_\pm e^{\frac{\pm\phi}{2}},\left(\theta^w w_l(\alpha,\beta)\left(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\right)^\wedge\right)^\vee\right)_{L^2_{v,x}}\\
&\pm\mathrm{Re}\left(w_l(\alpha,\beta)\frac{1}{2}v_i\partial^{e_i}\phi e^{\frac{\pm\phi}{2}}\partial^\alpha_\beta f_\pm,\left(\theta^w w_l(\alpha,\beta)\left(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\right)^\wedge\right)^\vee\right)_{L^2_{v,x}}\Bigg).
\end{aligned}
$$

Denote these terms by K_1 to K_8. Noticing that there is coefficient δ in (3.28), we only need to obtain an upper bound for these terms. For K_1, noticing that θ^w is self-adjoint, we have

$$
K_1\le\frac{1}{2}\partial_t\left(-\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\left(\theta^w w_l(\alpha,\beta)\left(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\right)^\wedge\right)^\vee\right)_{L^2_{v,x}}
$$

$$
\begin{aligned}
&+C\Big|\Big(-\psi_{2|\alpha|+2|\beta|-8-\frac{1}{N}}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\Big|\\
&+C\Big|\Big(\partial_t\phi\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\Big|.
\end{aligned}
$$

We denote the second and third terms on the right-hand side by $K_{1,1}$ and $K_{1,2}$. Since $\theta\in S(1)$, θ^w is a bounded operator on $L^2_{v,y}$. The boundedness of θ^w will be frequently used in the following without further mention. Using the trick from (3.21)-(3.28) to the term for the first $f_\pm$ in $K_{1,1}$, we have

$$
K_{1,1}\lesssim\delta^2\big\|\psi_{|\alpha|+|\beta|-4}\tilde{b}^{\frac{1}{2}}w_l(\alpha,\beta)\big(\partial^\alpha_\beta f\big)^\wedge\big\|^2_{L^2_{v,y}}+\delta^2\mathcal{D}_{K,l}+C_\delta\|\langle v\rangle^{C_{K,l}}f\|^2_{L^2_{v,x}}+\mathcal{E}_{K,l}.
$$

The term $K_{1,2}$ is similar to the case I_1, i.e.

$$
K_{1,2}\lesssim\|\partial_t\phi\|_{L^\infty_x}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial^\alpha_\beta f\big\|^2_{L^2_{v,x}}\lesssim\|\partial_t\phi\|_{L^\infty_x}\mathcal{E}_{K,l}(t).
$$

For the term K_2 with $\alpha_1=\beta_1=0$, a nice observation is that it is the same as K_8 except for the sign and hence, they are eliminated. For K_2 with $\alpha_1+\beta_1\neq0$, the order of derivatives for the first $f_\pm$ is less or equal to $K-1$ and hence, the weight can be controlled as $w_l(\alpha,\beta)\partial_{\beta_1}v_i\lesssim\langle v\rangle^\gamma w_l(\alpha-\alpha_1,\beta-\beta_1)$. Then similar to Lemma 2.9, by noticing θ^w is bounded on $L^2_{v,y}$, we have

$$
|K_2+K_8|\lesssim\mathcal{E}^{\frac{1}{2}}_{K,l}\mathcal{D}_{K,l}.
$$

For K_3, when $\alpha_1=0$, noticing θ^w is self-adjoint, we use integration by parts over v to obtain

$$
\begin{aligned}
|K_3|=&\Big|\Big(\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^{e_i}\phi\partial^\alpha_{\beta+e_i}f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\Big|\\
\lesssim&\Big|\Big(\psi_{2|\alpha|+2|\beta|-8}\partial_{e_i}\big(w_l(\alpha,\beta)\big)\partial^{e_i}\phi\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\Big|\\
&+\Big|\Big(\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^{e_i}\phi\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\underbrace{[\partial_{e_i},\theta^w]}_{\in S(1)}w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\Big|\\
&+\Big|\Big(\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^{e_i}\phi\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w\partial_{e_i}\big(w_l(\alpha,\beta)\big)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\Big|\\
\lesssim&\|\partial^{e_i}\phi\|_{H^2_x}\big\|\psi_{|\alpha|+|\beta|-4}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm\big\|^2_{L^2_{v,x}}\\
\lesssim&\delta_0\mathcal{E}_{K,l}(t)
\end{aligned}
$$

with the help of (2.6) and $\theta\in S(1)$. When $\alpha_1\neq 0$, then $\alpha\neq 0$, the total number of derivatives on the first $f_\pm$ is less or equal to K and there is at least one derivative on the second $f_\pm$ with respect to x. Thus,

$$|K_3|\lesssim \mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}.$$

For K_4, there is exponential decay in v and hence, $|K_4|\lesssim\mathcal{E}_{K,l}$. For K_5, recalling that we only need an upper bound and using Lemma 2.7 with $\partial^\alpha\mu=0$ for $|\alpha|\geq 1$, we have $|K_5|\lesssim\mathcal{E}_{K,l}+\mathcal{D}_{K,l}$. For K_6, we use Lemma 2.8 to obtain

$$|K_6|\lesssim \mathcal{E}_{K,l}^{\frac{1}{2}}\mathcal{D}_{K,l}+\mathcal{E}_{K,l}\mathcal{D}_{K,l}^{\frac{1}{2}}\lesssim\big(\mathcal{E}_{K,l}^{\frac{1}{2}}+\mathcal{E}_{K,l}\big)\mathcal{D}_{K,l}+\mathcal{E}_{K,l}.$$

For K_7, since $\beta_1\neq 0$, one has $|\partial_{\beta_1}v_i|\lesssim 1$ and the total number of derivatives on the first $f_\pm$ is less or equal to K. Also, $w(\alpha,\beta)=\langle v\rangle^\gamma w(|\alpha|+1,|\beta|-1)$. These yield that $|K_7|\lesssim\mathcal{E}_{K,l}$. Combining the above estimate with (3.30) and choosing $\delta>0$ sufficiently small, we have

$$\begin{aligned}
&\|\psi_{|\alpha|+|\beta|-4}\tilde{b}^{\frac{1}{2}}w_l(\alpha,\beta)(\partial^\alpha_\beta f)^\wedge(v,y)\|^2_{L^2_{v,y}}\\
\lesssim&\frac{1}{2}\partial_t\Big(-\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\\
&+\big(\mathcal{E}_{K,l}^{\frac{1}{2}}+\mathcal{E}_{K,l}\big)\mathcal{D}_{K,l}+C_\delta\|\langle v\rangle^{C_{K,l}}f\|^2_{L^2_{v,x}}+\|\partial_t\phi\|_{L^\infty_x}\mathcal{E}_{K,l}(t)+\delta^2\mathcal{D}_{K,l}+\mathcal{E}_{K,l}.
\end{aligned}$$

Substituting this into (3.28), we have the desired estimate (3.15). This completes the proof of Lemma 3.2. □

Proof of Theorem 3.1. Substituting (3.15) into (3.6), we have that for $0<\delta<1$,

$$\begin{aligned}
&\partial_t\mathcal{E}_{K,l}(t)+\lambda D_{K,l}(t)\\
\lesssim&\delta^2\sum_{|\alpha|+|\beta|\leq K}\partial_t\Big(-\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\\
&+\|\partial_t\phi\|_{L^\infty_x}\mathcal{E}_{K,l}(t)+\delta^2\big(\mathcal{D}_{K,l}+\big(\mathcal{E}_{K,l}^{\frac{1}{2}}+\mathcal{E}_{K,l}\big)\mathcal{D}_{K,l}\big)+\mathcal{E}_{K,l}+C_\delta\|\langle v\rangle^{C_{K,l}}f\|^2_{L^2_{v,x}}.
\end{aligned}$$

By (3.5) and (3.3), we have $\|\partial_t\phi\|_{L^\infty_x}\lesssim\mathcal{E}_{K,l}^{1/2}\lesssim\delta_0^{1/2}$. Using the a priori assumption (3.3) and choosing $\delta,\delta_0>0$ sufficiently small, we have

$$\begin{aligned}
&\partial_t\mathcal{E}_{K,l}(t)+\lambda\mathcal{D}_{K,l}(t)\lesssim\mathcal{E}_{K,l}(t)+\|\langle v\rangle^{C_{K,l}}f\|^2_{L^2_{v,x}}\\
&\quad+\delta^2\sum_{|\alpha|+|\beta|\leq K}\partial_t\Big(-\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}.
\end{aligned}$$

By solving this ODE with neglecting $\lambda\mathcal{D}_{K,l}(t)$ and noticing

$$\Big|\Big(-\psi_{2|\alpha|+2|\beta|-8}w_l(\alpha,\beta)\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}},\big(\theta^w w_l(\alpha,\beta)\big(\partial^\alpha_\beta f_\pm e^{\frac{\pm\phi}{2}}\big)^\wedge\big)^\vee\Big)_{L^2_{v,x}}\Big|\lesssim\mathcal{E}_{K,l}(t),$$

we have that for $0\le t\le t_0$,

$$\begin{aligned}&\mathcal{E}_{K,l}(t)\lesssim\mathcal{E}_{K,l}(0)+\delta^2\mathcal{E}_{K,l}(t)+\delta^2\mathcal{E}_{K,l}(0)+\int_0^t\big(\mathcal{E}_{K,l}+\|\langle v\rangle^{C_{K,l}}f\|_{L^2_{v,x}}\big)d\tau,\\&\mathcal{E}_{K,l}(t)\lesssim\epsilon_1^2,\end{aligned}\tag{3.31}$$

by choosing $\delta>0$ and $t_0=t_0(\epsilon_1,\|\langle v\rangle^{C_{K,l}}f\|_{L^2_{v,x}})>0$ sufficiently small. Here we used $\mathcal{E}_{K,l}(0)\le\mathcal{E}_{4,l}(0)$. This completes the proof of Theorem 3.1. □

Proof of Theorem 1.1. We prove Theorem 1.1 in four steps.

Step 1. It follows immediately from the a priori estimate (3.3) and Theorem 3.1 that

$$\sup_{0\le t\le t_0}\mathcal{E}_{K,l}\le C_{K,l}\epsilon_1^2$$

holds true for some small $t_0>0$, as long as ϵ_1 is sufficiently small. The rest is to prove the local existence and uniqueness of solutions in terms of the energy norm $\mathcal{E}_{K,l}$. One can use the iteration on the system

$$\begin{cases}\quad\partial_t f^{n+1}_\pm+v\cdot\nabla_x f^{n+1}_\pm\mp\nabla_x\phi^n\cdot\nabla_x f^{n+1}_\pm\pm\dfrac{1}{2}\nabla_x\phi^n\cdot v f^{n+1}_\pm\pm v\mu^{\frac12}\cdot\nabla_x\phi^n-L_\pm f\\=\Gamma_\pm(f^n,f^{n+1}),\\\quad-\Delta_x\phi^{n+1}=\displaystyle\int_{\mathbb{R}^3}\mu^{\frac12}(f^{n+1}_+-f^{n+1}_-)dv,\\\quad f^{n+1}|_{t=0}=f_0\end{cases}$$

to find the local existence and the details of proof are omitted for brevity, see [20,24,31].

Step 2. Notice that the constants in Theorem 3.1 are independent of time t and hence, we can apply Theorem 3.1 to any time interval with length less than t_0 to obtain that, for $0<\tau<T$,

$$\sup_{\tau\le t\le T}\mathcal{E}_{K,l}(t)\le\epsilon_1^2C_{\tau,T,K,l}.\tag{3.32}$$

Recalling Definition 1.12 of $\mathcal{E}_{K,l}$ and the choice (3.2) of ψ, we have that, for any $0<\tau<T, l\geq 0$ and $K\geq 4$,

$$\sup_{\tau\leq t\leq T}\sum_{|\alpha|+|\beta|\leq K}\left\|w_l(\alpha,\beta)\partial^\alpha_\beta f\right\|^2_{L^2_{v,x}}+\sup_{\tau\leq t\leq T}\sum_{|\alpha|\leq K}\left\|\partial^\alpha\nabla_x\phi\right\|^2_{L^2_x}\leq C_{\tau,T,l}<\infty. \tag{3.33}$$

Notice that $\psi^{-1}_{|\alpha|+|\beta|-4}$ is singular near $t=0$ when $|\alpha|+|\beta|>4$, so the constant is necessarily depending on τ. This proves (1.16).

Let $l\geq 0, K\geq 4$ and assume additionally $\mathcal{E}_{4,C_{K,l}}(0)$ is sufficiently small for some large constant $C_{K,l}>0$ to be chosen later. Then by (3.32), we have

$$\sup_{\tau\leq t\leq T}\mathcal{E}_{K,C_{K,l}}(t)\leq \epsilon_1^2 C_{\tau,T,K,l}. \tag{3.34}$$

For the regularity on t, the technique above is not applicable and we only make a rough estimate. For any $t>0$, applying $\langle v\rangle^l\partial_t^k\partial^\alpha_\beta$ with $k,l\geq 0, |\alpha|+|\beta|\leq K$ to Eq. (1.4) and taking $L^2_{v,x}$ norms, we have

$$\begin{aligned}
&\left\|\langle v\rangle^l\partial_t^{k+1}\partial^\alpha_\beta f_\pm\right\|^2_{L^2_{v,x}}\\
\lesssim&\left\|\langle v\rangle^l v\cdot\nabla_x\partial_t^k\partial^\alpha_\beta f_\pm\right\|^2_{L^2_{v,x}}+\left\|\langle v\rangle^l\sum_{k_1\leq k}\partial^\alpha_\beta\left(\partial_t^{k_1}\nabla_x\phi\cdot v\partial_t^{k-k_1}f_\pm\right)\right\|^2_{L^2_{v,x}}\\
&+\left\|\langle v\rangle^l\sum_{k_1\leq k}\partial^\alpha\left(\partial_t^{k_1}\nabla_x\phi\cdot\nabla_v\partial_t^{k-k_1}\partial_\beta f_\pm\right)\right\|^2_{L^2_{v,x}}+\left\|\langle v\rangle^l\partial_t^k\partial^\alpha\nabla_x\phi\cdot\partial_\beta(v\mu^{\frac12})\right\|^2_{L^2_{v,x}}\\
&+\left\|\langle v\rangle^l\partial^\alpha_\beta L_\pm\partial_t^k f_\pm\right\|^2_{L^2_{v,x}}+\left\|\langle v\rangle^l\sum_{k_1\leq k}\partial^\alpha_\beta\Gamma_\pm\left(\partial_t^{k_1}f,\partial_t^{k-k_1}f\right)\right\|^2_{L^2_{v,x}}.
\end{aligned}\tag{3.35}$$

Denoting

$$\mathcal{E}_{K,l,k}=\sum_{|\alpha|+|\beta|\leq K,k_1\leq k}\left\|\langle v\rangle^l\partial^\alpha_\beta\partial_t^{k_1}f\right\|_{L^2_{v,x}},$$

we estimate the right-hand side terms of (3.35) one by one. The first term on the right-hand side is bounded above by $\mathcal{E}_{K+1,l+1,k}$. Applying the trick in Lemma 2.9, the second right-hand side term of (3.35) is bounded above by

$$\sum_{|\alpha|+|\beta|\leq K,k_1\leq k}\left\|\partial_t^{k_1}\partial^\alpha_\beta\nabla_x\phi\right\|^2_{L^2_x}\sum_{|\alpha|+|\beta|\leq K,k_1\leq k}\left\|\langle v\rangle^{l+1}\partial_t^{k_1}\partial^\alpha_\beta f_\pm\right\|^2_{L^2_{v,x}}\lesssim\mathcal{E}^2_{K,l+1,k}.$$

Similarly, applying the trick in Lemma 2.10, the third term of (3.35) is bounded above by $\mathcal{E}^2_{K+1,l+1,k}$. For the fourth term, when $k=0$, it is bounded above by $\mathcal{E}_{K,l,0}$.

When $k\geq 1$, by using (3.4), it is bounded above by $\mathcal{E}_{K,l,k-1}$. For the fifth term, noticing $L_\pm\in S(\tilde{a})\subset S(\langle v\rangle^{\gamma+2s}\langle\eta\rangle^{2s})$ and $s\in(0,1)$, we have

$$\big\|\langle v\rangle^l\partial^\alpha_\beta L_\pm\partial^k_t f_\pm\big\|^2_{L^2_{v,x}}\lesssim\big\|\langle v\rangle^{l+\gamma+2s}\langle D_v\rangle^2\langle(D_x,D_v)\rangle^K\partial^k_t f_\pm\big\|^2_{L^2_{v,x}}\lesssim\mathcal{E}_{K+2,l+\gamma+2s,k}.$$

For the last term, using (2.5), it is bounded above by

$$\sum_{|\alpha|+|\beta|\leq K+2,k_1\leq k}\big\|\langle v\rangle^{l+\frac{\gamma+2s}{2}}\partial^\alpha_\beta\partial^{k_1}_t f\big\|^2_{L^2_{v,x}}\lesssim\mathcal{E}^2_{K+2,l+\frac{\gamma+2s}{2},k}.$$

Combining the above estimate and taking summation of (3.35) over $|\alpha|+|\beta|\leq K$, $k\leq k_0$ for any $k_0\geq 0$, we have

$$\begin{aligned}\mathcal{E}_{K,l,k_0+1}(t)\lesssim&\,\mathcal{E}_{K,l,0}+\mathcal{E}_{K,l,k_0-1}+\mathcal{E}_{K,l+1,k_0}+\mathcal{E}^2_{K+1,l+1,k_0}\\&+\mathcal{E}_{K+2,l+\gamma+2s,k_0}+\mathcal{E}^2_{K+2,l+\frac{\gamma+2s}{2},k_0}.\end{aligned}$$

The t derivative on the right hand is less than the left hand. Hence, noticing (3.34), for any $T>\tau>0$, we have

$$\sup_{\tau\leq t\leq T}\mathcal{E}_{K,l,k_0}(t)\leq C_{\tau,T,l,k_0}.$$

For the time derivatives on $\nabla_x\phi$, we apply (3.4) to obtain

$$\sup_{\tau\leq t\leq T}\sum_{|\alpha|\leq K,k\leq k_0}\big\|\partial^\alpha\partial^k_t\nabla_x\phi\big\|^2_{L^2_x}\lesssim\sup_{\tau\leq t\leq T}\mathcal{E}_{K,l,k_0}(t)\leq C_{\tau,T,l,k_0}.$$

Then we obtain (1.17). Consequently, by Sobolev embedding, $f\in C^\infty(\mathbb{R}^+_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v)$.

Step 3. Now we additionally assume (1.18) is sufficiently small. Noticing $\psi=1$ in Theorem 2.1, (2.2) shows that for any $\tau_0\geq\tau$,

$$\begin{aligned}&\sum_{|\alpha|\leq 4}\|\partial^\alpha E(\tau_0-\tau)\|^2_{L^2_x}+\sum_{|\alpha|\leq 4}\|\partial^\alpha\mathbf{P}f(\tau_0-\tau)\|^2_{L^2_{v,x}}\\&\quad+\sum_{|\alpha|+|\beta|\leq 4}\big\|w_l(\alpha,\beta)\partial^\alpha_\beta(\mathbf{I}-\mathbf{P})f(\tau_0-\tau)\big\|^2_{L^2_{v,x}}\lesssim\epsilon_0^2,\end{aligned}\tag{3.36}$$

which is the global-in-time estimate. Using this as the initial data instead of (1.15), we can apply the above calculation on any time interval $[\tau_0-\tau,\tau_0+t_0]$ to obtain the same estimate as in (3.33) with constants independent of T. In fact, in this case, we can deduce from Theorem 3.1 that

$$\sup_{\tau_0\le t\le \tau_0+t_0}\sum_{|\alpha|+|\beta|\le K}\left\|w_l(\alpha,\beta)\partial^\alpha_\beta f\right\|^2_{L^2_{v,x}}+\sup_{\tau_0\le t\le \tau_0+t_0}\sum_{|\alpha|\le K}\left\|\partial^\alpha\nabla_x\phi\right\|^2_{L^2_x}\le C_{\tau,l}<\infty, \quad (3.37)$$

where the constant $C_{\tau,l}$ is independent of τ_0,T. We then give some comments on why the constant $C_{\tau,l}$ is independent of T. In (3.34) we start at the initial point $\tau>0$ and go to the endpoint $T>0$, so that it contains $[T/t_0]+1$ steps of length t_0 and hence, the constant depends on $T>0$. Here, $[T/t_0]$ is the largest integer less than T/t_0. However, in estimate (3.37), we start at the initial point $\tau_0-\tau$ and go to the endpoint τ_0+t_0, whose length is $t_0+\tau>0$ and is independent of $T>0$. Here, the choice of

$$t_0=t_0\big(\epsilon_1,\|\langle v\rangle^{C_{K,l}}f\|_{L^2_{v,x}}\big)>0$$

in (3.31) is uniform in any time t and is independent of $T>0$. Thus, starting from the point $\tau_0-\tau$ and ending at τ_0+t_0 takes $[t_0-\tau]+1$ steps of length $t_0>0$ that is independent of T. Therefore, the constant $C_{\tau,l}$ in (3.37) is independent of $T>0$. Since the starting point $\tau_0\ge\tau$ is arbitrary, we deduce a uniform estimate independent of time T in (3.37). This completes the proof of Theorem 1.1(3).

Notice that the estimate (3.33) is necessarily depending on τ since $\psi^{-1}_{|\alpha|+|\beta|-4}$ is singular near $t=0$ when $|\alpha|+|\beta|>4$.

Step 4. If we assume (1.19) is sufficiently small for some large enough $C_{K,l}>0$, then by Theorem 2.1, we can obtain the estimate (3.36) with l replaced by $C_{K,l}$. Then the result follows from the same argument as Steps 2 and 3; the proof of regularity is the same as Step 2 while the same proof in Step 3 shows that the constant is independent of $T>0$. This completes the proof of Theorem 1.1. □

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