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The Lie Algebras in which Every Subspace Is Its Subalgebra*

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Abstract: In this paper, we study the Lie algebras in which every subspace is its subalgebra (denoted by HB Lie algebras). We get that a nonabelian Lie algebra is an HB Lie algebra if and only if it is isomorphic to $g\dot{+}\mathbb{C}id_g$, where g is an abelian Lie algebra. Moreover we show that the derivation algebra and the holomorph of a nonabelian HB Lie algebra are complete.

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1 Introduction

The classification of Lie algebras is the most important work in Lie theory. There are two ways to get the classification of Lie algebras: by dimension, or by structure. The dimension approach has got a lot of useful results and some interesting applications in general relativity. However, it seems to be neither feasible, nor fruitful to proceed by dimension in the classification of Lie algebras when its dimension is beyond 6. We then turn to the structure approach. In this paper we study a special class of Lie algebras.

A subspace η of a Lie algebra is its subalgebra with $[\eta,\ \eta] \subset \eta$. The algebras in which every subalgebra is its ideal have been studied in [1], and the algebras in which every subspace is a subalgebra have been studied in [2]. In this paper, we study the Lie algebras in which every subspace is its subalgebra. We also study the derivation algebra and the holomorph of an HB Lie algebra.

Complete Lie algebras (i.e., centerless with only inner derivations: $H^0(g,\ g) = H^1(g,\ g) = 0$) first appeared in 1951, in the context of Schenkman's theory of subinvariant Lie algebras (see [3]). In recent years, different authors have concentrated on classifications and structural properties of complete Lie algebras (see [4]–[9]). We prove that the holomorph of an HB Lie algebra is complete.

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In this paper, all Lie algebras discussed are finite dimensional complex Lie algebras.

2 The Structure of an HB Lie Algebra

Lemma 2.1 *If H is a Lie algebra, then the following assertions are equivalent:*

(1) *H is an HB Lie algebra*;

(2) *For any basis $\{x_1, x_2, \cdots, x_n\}$ of H, $[x_i,\ x_j] \in \mathbb{C}x_i + \mathbb{C}x_j$, $1 \le i, j \le n$.*

Proof. (1)⇒(2). By the definition of an HB Lie algebra, it is obvious.

(2)⇒(1). For any basis $\{x_1, x_2, \cdots, x_k\}$ of a subspace H_1, let $\{x_1, x_2, \cdots, x_k, x_{k+1}, \cdots, x_n\}$ be a basis of H. By (2), $[x_i, x_j] \in \mathbb{C}x_i + \mathbb{C}x_j$, $1 \le i, j \le k$, and we may assume $[x_i, x_j] = a_{ij}x_i + b_{ij}x_j$. For any $x = \sum\limits_{i=1}^{k} a_i x_i$, $y = \sum\limits_{j=1}^{k} b_j x_j$ in this subspace, we have

$$[x, y] = \Big[\sum_{i=1}^{k} a_i x_i,\ \sum_{j=1}^{k} b_j x_j\Big] = \sum_{i=1}^{k}\sum_{j=1}^{k} a_i b_j [x_i, x_j] = \sum_{i=1}^{k}\sum_{j=1}^{k} (a_i b_j a_{ij} x_i + a_i b_j b_{ij} x_j).$$

Hence this subspace is a subalgebra. By the definition of an HB Lie algebra, H is an HB Lie algebra.

Let L be a 2-dimensional Lie algebra. For any basis $\{x,\ y'\}$ of L, there exists another basis $\{x,\ y\}$ of L, such that $[x,\ y] \in \mathbb{C}y$ or $[x,\ y] \in \mathbb{C}x$. In fact, if L is abelian, then $0 = [x,\ y'] = 0y$. If L is nonabelian, as $[x,\ y'] \in \mathbb{C}x + \mathbb{C}y'$, we may assume $[x,\ y'] = ax + by'$. If $b \neq 0$, let $y = ax + by'$, and then

$$[x, y] = [x,\ ax + by'] = b(ax + by') = by;$$

if $b = 0$, let $y = y'$, and then

$$[x, y] = ax.$$

Lemma 2.2 *Let H be an HB Lie algebra. Then H has a decomposition*:

$$H = H_1 \dot{+} H_2 \dot{+} \cdots \dot{+} H_s,$$

where H_i is a subspace of H which has a basis $\{x_{i1},\ x_{i2},\ \cdots,\ x_{in_i}\}$, such that

$$[x_{ip},\ x_{jq}] = \lambda(ip,\ jq)x_{ip}, \qquad i < j,\ 1 \le p \le n_i,\ 1 \le q \le n_j;$$
$$[x_{ip_1},\ x_{ip_2}] = 0, \qquad 1 \le p_1, p_2 \le n_i.$$

Proof. When $\dim H = 2$, the lemma holds.

In fact, there exists a basis $\{x_1,\ x_2\}$ of 2-dimensional Lie algebra such that

$$[x_1,\ x_2] = \lambda x_2$$

or

$$[x_1,\ x_2] = \lambda x_1.$$

We assume that the lemma holds for $\dim H < n$ to prove the lemma holds for $\dim H = n$. For any basis $\{x_1,\ x_2,\ \cdots,\ x_n\}$ of H, by the definition of an HB Lie algebra, we obtain that $\mathbb{C}x_1 + \mathbb{C}x_i$, $2 \le i \le n$, is a 2-dimensional Lie algebra. We choose a basis $\{x_1,\ y_i\}$ of this Lie algebra such that

$$[x_1,\ y_i] = \lambda_i y_i$$