COMMUNICATIONS IN MATHEMATICAL RESEARCH
25(1)(2009), 19–29

Reducing Subspaces of Toeplitz Operators on N_φ-type Quotient Modules on the Torus*

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Communicated by Ji You-qing

Abstract: In this paper, we prove that the Toeplitz operator with finite Blaschke product symbol $S_{\psi(z)}$ on N_φ has at least m non-trivial minimal reducing subspaces, where m is the dimension of $H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega)$. Moreover, the restriction of $S_{\psi(z)}$ on any of these minimal reducing subspaces is unitary equivalent to the Bergman shift M_z.

Key words: module, N_φ-type quotient module, the analytic Toeplitz operator, reducing subspace, finite Blaschke product
2000 MR subject classification: 47B35, 47A15
Document code: A
Article ID: 1674-5647(2009)01-0019-11

1 Introduction

Let D denote the open unit disk in the complex plane $\mathbb{C}$ and T^2 be cartesian product of two copies of T, where T is the unit circle. It is well known that T^2, as usually is endowed with the rotation invariant Lebesgue measure, is the distinguished boundary of D^2. Let $dm(z)$ denote the normalized Lebesgue measure on T and $\mathrm{d}m(z)\mathrm{d}m(\omega)$ be the product measure on the torus T^2. The Bergman space is denoted by $L_a^2(D)$ and Bergman shift is denoted by M_z. Let $H^2(\Gamma^2)$ be the Hardy space on the two dimensional torus T^2. We denote by z and ω the coordinate functions. Shift operators T_z and T_ω on $H^2(\Gamma^2)$ are defined by $T_z f = zf$ and $T_\omega f = \omega f$ for $f \in H^2(\Gamma^2)$. Clearly, both T_z and T_ω have infinite multiplicity. A closed subspace M of $H^2(\Gamma^2)$ is called a submodule (over the algebra $H^\infty(D^2)$), if it is invariant under multiplications by functions $H^\infty(D^2)$. Equivalently, M is a submodule if it is invariant for both T_z and T_ω. The quotient space $N:\ H^2(\Gamma^2) \ominus M$ is called a quotient module. Clearly, $T_z^*N \subset N$ and $T_\omega^*N \subset N$. In the study here, it is necessary to distinguish the classical Hardy space in the variable z and that in the variable ω, for which we denote

*Received date: Oct. 29, 2007.
Foundation item: The NSF (10671083) of China.

by $H^2(\Gamma_z)$ and $H^2(\Gamma_\omega)$, respectively. In this paper, we look at submodules of the form $[z\text{-}\varphi(\omega)]$, where φ is an inner function in $H^2(\Gamma_\omega)$ and $[z\text{-}\varphi(\omega)]$ is the closure of $(z\text{-}\varphi)H^\infty(\Gamma^2)$ in $H^2(\Gamma^2)$. For simplicity we denote $[z\text{-}\varphi(\omega)]$ by M_φ. $N_\varphi = H^2(\Gamma^2) \ominus M_\varphi$ denote N_φ-type quotient modules on the torus. For a function $\psi \in H^\infty(D^2)$, we define the Toeplitz operator S_ψ on N_φ with symbol ψ by

$$S_\psi(f) = P_{N_\varphi}(\psi f), \qquad \forall f \in N_\varphi,$$

where P_{N_φ} is a projection from $H^2(\Gamma^2)$ to N_φ.

The quotient module N_φ has a very rich structure. In deed, when φ is inner, N_φ can be identified with the tensor product of two well-known classical spaces, namely the quotient space $H^2(\Gamma) \ominus \varphi H^2(\Gamma)$ and the Bergman space $L_a^2(D)$. Clearly, when $\varphi(\omega) = \omega$, N_φ is unitary equivalent to $L_a^2(D)$. In fact, it is shown in [1] that $\{T_z, T_\omega, H^2(\Gamma^2)\}$ is the minimal super-isometrical dilation of M_z. Then the reducible problem of Toeplitz operator with finite Blaschke product on the Bergman space is turned to the reducible problem of Toeplitz operator with finite Blaschke product on N_ω. It is obtained in [1] that Toeplitz operator with finite Blaschke product $S_{\psi(z)}$ on N_ω has at least a reducing subspace M, moreover, $S_\psi|_M \cong M_z$. In this paper, we prove that when φ is a non-constant inner function, the conclusion like that in [1] is also true.

2 Preliminaries

In order to prove the main theorem, we need the following lemma.

Lemma 2.1[2] *Let $\varphi(\omega)$ be a one variable non-constant inner function and $\{\lambda_k(\omega):\ k = 1, 2, \cdots, m\}$ be an orthonormal basis of $H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega)$, and*

$$e_j(z, \omega) = \frac{\omega^j + \omega^{j-1}z + \cdots + z^j}{\sqrt{j+1}} \qquad (j = 0, 1, \cdots).$$

Let

$$E_{k,j} = \lambda_k(\omega)e_j(z,\ \varphi(\omega)).$$

Then $\{E_{k,j}:\ k = 1, 2, \cdots, m;\ j = 0, 1, \cdots\}$ is an orthonormal basis for N_φ.

Lemma 2.2[2] *There exists a unitary operator U,*

$$U:\ N_\varphi \longrightarrow (H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega)) \otimes L_a^2(D),$$
$$E_{k,j} \longmapsto \lambda_k(\omega)\sqrt{j+1}\xi^j$$

such that

$$US_z = (I \bigotimes M_z)U,$$

where I is an identity map on $H^2(\Gamma_\omega) \ominus \varphi(\omega)H^2(\Gamma_\omega)$.

Lemma 2.3[1] *Suppose that*

$$\varphi(\omega) = \omega, \qquad \psi(z) = z\prod_{l=1}^{N-1}\frac{z-\alpha_l}{1-\bar{\alpha}_l z} \quad (|\ \alpha_l\ | > 0,\ \alpha_l \neq \alpha_k (\forall l \neq k),\ 1 \leq l, k \leq N-1).$$