Communications
in
Mathematical
Research
25(1)(2009), 30–36

Eigenvalue Problem of Doubly Stochastic Hamiltonian Systems with Boundary Conditions*

Han Yue-cai[1,2] and Ma Yong[1]

(*1. School of Mathematics, Jilin University, Changchun, 130012*)

(*2. School of Mathematics, Shandong University, Jinan, 250100*)

Communicated by Li Yong

Abstract: In this paper, we investigate the eigenvalue problem of forward-backward doubly stochastic differential equations with boundary value conditions. We show that this problem can be represented as an eigenvalue problem of a bounded continuous compact operator. Hence using the famous Hilbert-Schmidt spectrum theory, we can characterize the eigenvalues exactly.

Key words: doubly stochastic Hamiltonian system, eigenvalue problem, spectrum theory

2000 MR subject classification: 60H10, 62P05

Document code: A

Article ID: 1674-5647(2009)01-0030-07

1 Introduction

Stochastic Hamiltonian systems were introduced in the theory of stochastic optimal control as a necessary condition of an optimal control, known as the stochastic version of the maximum principle of Pontryagin's type (see [1]–[6]). In fact, those stochastic Hamiltonian systems with boundary conditions are forward-backward stochastic differential equations (FBSDE for short). These have been extensively investigated by Antonelli[7], Ma *et al.*[8], Hu and Peng[9], Peng and Wu[10], Yong[11]. Recently, combining the FBSDE and the backward doubly stochastic differential equations introduced by Pardoux and Peng[12], Peng and Shi[13] have investigated a type of time-symmetric FBSDE. They showed the uniqueness and existence of solutions for these equations under certain monotonicity conditions.

In this paper, we study a special type of time-symmetric FBSDE, namely doubly stochastic Hamiltonian systems (DSHS for short). We discuss the eigenvalue problem of this type

*__Received date:__ July 4, 2008.

Foundation item: The NSF (10601019 and J0630104) of China, Chinese Postdoctoral Science Foundation and 985 Program of Jilin University.

of stochastic Hamiltonian system in a standard functional analysis way.

The rest of this paper is organized as follows. The next section begins with a general formulation of time-symmetric FBSDE, then a special case, DSHS with boundary conditions. In Section 3, we give the proof of the main results.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $T > 0$ be fixed throughout this paper. Let $\{W_t : 0 \leq t \leq T\}$ and $\{B_t : 0 \leq t \leq T\}$ be two mutually independent standard Brownian motions which are $\mathbf{R}^d$-valued processes defined on $(\Omega, \mathcal{F}, P)$. Without loss of generality, we assume that $d = 1$. Let $\mathcal{N}$ denote the class of P-null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define

$$\mathcal{F}_t \triangleq \mathcal{F}_t^w \vee \mathcal{F}_{t,T}^B,$$

where

$$\mathcal{F}_t^w = \mathcal{N} \vee \sigma\{W_r - W_0 : 0 \leq r \leq t\},$$

$$\mathcal{F}_{t,T}^B = \mathcal{N} \vee \sigma\{B_r - B_t : t \leq r \leq T\}.$$

Note that the collection $\{\mathcal{F}_t : t \in [0, T]\}$ is neither increasing nor decreasing. Thus it does not constitute a filtration.

Let $M^2(0, T; \mathbf{R}^n)$ denote the set of all classes ($dt \times dP$ is equal a.e.) $\mathcal{F}_t$-measurable stochastic processes $\{\varphi_t : t \in [0, T]\}$ which satisfy

$$\mathrm{E} \int_0^T |\varphi_t|^2 \mathrm{d}t < +\infty.$$

For a given $\varphi_t, \psi_t \in M^2(0, T; \mathbf{R}^n)$, one can define the forward Itô integration $\int_0^\cdot \varphi_s \mathrm{d}W_s$ and the backward Itô integration $\int_\cdot^T \psi_s \mathrm{d}B_s$. They are both in $M^2(0, T; \mathbf{R}^n)$.

Let $H(y, Y, z, Z) : \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}$ and $\Phi(y) : \mathbf{R}^n \to \mathbf{R}$ be C^1 functions. Find a triple

$$(y, Y, z, Z) \in M^2(0, T; \mathbf{R}^n)$$

such that a boundary problem for a doubly stochastic Hamiltonian system satisfies the following form

$$\begin{cases} dy_t = H_Y(t, y_t, Y_t, z_t, Z_t)dt + H_Z(t, y_t, Y_t, z_t, Z_t)dW_t - z_t dB_t, \\ y(0) = y_0, \\ -dY_t = H_y(t, y_t, Y_t, z_t, Z_t)dt + H_z(t, y_t, Y_t, z_t, Z_t)dB_t - Z_t dW_t, \\ Y_T = \Phi_y(y_T), \end{cases} \tag{2.1}$$

where H_y, H_Y, H_z, H_Z are gradients of the function H with respect to y, Y, z, Z respectively.

This is a sort of time-symmetric FBSDE introduced by Peng and Shi[13]. Let

$$\xi = (y, Y, z, Z)^\top,$$

$$\Lambda(t, \xi) = (-H_y, H_Y, -H_z, H_Z)^\top(t, \xi).$$

We assume the following: