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Spectral Method for a Class of Cahn-Hilliard Equation with Nonconstant Mobility*

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Abstract: In this paper, we propose and analyze a full-discretization spectral approximation for a class of Cahn-Hilliard equation with nonconstant mobility. Convergence analysis and error estimates are presented and numerical experiments are carried out.

Key words: Cahn-Hilliard equation, spectral method, error estimate

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1 Introduction

The Cahn-Hilliard (C-H for short) equation was originally proposed by Cahn and Hilliard to simulate binary alloys. It has subsequently been adopted to model many physical situations such as phase transitions and interface dynamics in multi-phase fluids (see [1]). In this paper, we consider an initial-boundary value problem for a class of C-H equation which is of the form

$$\frac{\partial u}{\partial t} + D[m(x,t)(D^3u - DA(u))] = 0, \qquad (t,x) \in Q_T = (0,T] \times (0,1), \tag{1.1}$$

$$Du(x,t) = D^3u(x,t) = 0, \qquad x = 0, 1, \quad 0 \le t \le T, \tag{1.2}$$

$$u(x,0) = u_0, \qquad x \in (0,1), \tag{1.3}$$

where $D = \dfrac{\partial}{\partial x}$ and typically

$$A(s) = -s + \gamma_1 s^2 + \gamma_2 s^3, \qquad \gamma_2 > 0.$$

Here $u(x,t)$ represents a relative concentration of one component in binary mixture. The function $m(x,t)$ is the mobility, which restricts diffusion of both components to the interfacial region only. Throughout this paper, we assume that

$$0 < m_0 \le m(x,t) \le M_0, \quad |m'_x(x,t)| \le M_1, \qquad \forall (x,t) \in Q_T, \tag{1.4}$$

where m_0, M_0 and M_1 are positive constants.

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In the past years, the C-H equation with constant mobility has been intensively studied, and there have been many outstanding results concerning the existence, regularity and special properties of the solution in [2] and [3]. Numerical methods for C-H equation can be found in [3]–[10]. These numerical computation techniques contained the finite element methods (see [5] and [11]), the finite difference methods (see [6] and [8]), and spectral the formulations (see [4] and [10]). In recent years, the equation with concentration dependent mobility has also caused much attention.

The layout of the paper is as follows: in Section 2, we consider a full-discretization implicit scheme and some corresponding estimates. In Section 3, we study the convergence property of this fully discrete spectral approximation. Finally, we perform some numerical experiments which illustrate our results in Section 4.

2 Full-discretization Spectral Method and Some Estimates

In this section we set up a full-discretization scheme for equation (1.1) and analyze the boundedness of its solution. Let $\|\cdot\|_k$ and $|\cdot|_k$ be the norm and semi-norm of the Soboliv spaces $H^k(0,1)(k\in\mathbb{N})$, respectively. Let $(\,\cdot\,,\,\cdot\,)$ be the standard L^2 inner product over $(0,1)$. Define

$$L^\infty(0,1)=\{v;\ \|v\|_\infty=\operatorname*{esssup}_{x\in(0,1)}|u|<+\infty\},$$

$$H_E^2(0,1)=\{v\in H^2(0,1);\ Dv\big|_{x=0,1}=0\},$$

$$L^2(0,T;H^m(0,1))=\{u\in H^m(0,1);\int_0^T\|u\|_m^2\mathrm{d}t<+\infty\}.$$

Denote by

$$S_N=\operatorname{span}\{\cos k\pi x, k=0,1,2,\cdots,N\}$$

for any integer $N>0$. Define an orthogonal projection operator $P_N: H_E^2\mapsto S_N$ by

$$(P_Nu,v_N)=(u,v_N),\qquad \forall v_N\in S_N. \tag{2.1}$$

The weak solution for the initial boundary value problem (1.1)–(1.3) is equivalent to the solution of the following equations

$$(u_t,v)+(D^2u-A(u),D(mDv))=0,\qquad \forall v\in H_E^2(0,1), \tag{2.2}$$

$$(u(\,\cdot\,,0),v)=(u_0,v),\qquad \forall v\in H_E^2(0,1). \tag{2.3}$$

Moreover, the existence of the weak solution of this problem was introduced and some boundedness about the weak solution was given in [12].

Theorem 2.1[12] *Assume that $u_0\in H_E^2(0,1)$ and* (1.4) *is satisfied. Then there exists a unique weak solution $u\in H^{4,1}(Q_T)$ of the initial-boundary value problem* (1.1)–(1.3). *Furthermore, we have*

$$\|Du(x,t)\|\le C,\quad \|u\|_\infty\le C,\qquad 0\le t\le T, \tag{2.4}$$

$$\|D^2u(x,t)\|^2\le C,\quad \int_0^t\|D^4u(x,t)\|^2\mathrm{d}s\le C,\qquad 0\le t\le T, \tag{2.5}$$