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A Continuation Method for Solving Fixed Point Problems in Unbounded Convex Sets*

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Abstract: In this paper, an unbounded condition is presented, under which we are able to utilize the interior point homotopy method to solve the Brouwer fixed point problem on unbounded sets. Two numerical examples in $\mathbf{R}^3$ are presented to illustrate the results in this paper.
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1 Introduction

If $\Omega \subset \mathbf{R}^n$ is a bounded closed convex set, $\Phi(x)$ is a continuous mapping in $\mathbf{R}^n$ and $\Phi(\Omega) \subset \Omega$, then $\Phi(x)$ has a fixed point in Ω. This is the well-known Brouwer fixed point theorem. In 1976, Kellogg *et al.*[1] presented a homotopy method for computing the Brouwer fixed point of a twice continuously differentiable mapping. From then on, this method has become a powerful tool in dealing with fixed point problems (see [2]–[4] etc. and the references therein). In 1978, Chow *et al.*[2] constructed the homotopy

$$(1-\mu)(x-\Phi(x))+\mu(x-x^0) \tag{1.1}$$

for the bounded closed convex set. This homotopy is used by many authors to compute fixed points and solutions of nonlinear systems. In 1996, for a class of nonconvex bounded closed subsets in $\mathbf{R}^n$, Yu and Lin[5] proposed an interior point homotopy method which can numerically solve the fixed points of $\Phi(x)$ in them. Xu and Li[6] extended the results in [5] to more general non-convex cases. But all their results required that the set Ω is bounded.

In general, the boundedness on Ω is a theoretical ideality, certainly it is also very interesting and useful to solve numerically the fixed points of $\Phi(x)$ in general unbounded subsets

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in $\mathbf{R}^n$. However, to our knowledge, there has been hardly any result in this area, in this paper, we are devoted to completing this work.

Throughout this paper, let

$$\Omega = \{x \in \mathbf{R}^n : \ g_i(x) \le 0, \ i = 1, \cdots, m\},$$
$$\Omega^0 = \{x \in \mathbf{R}^n : \ g_i(x) < 0, \ i = 1, \cdots, m\}$$

and let $\partial\Omega = \Omega \backslash \Omega^0$ be the boundary of Ω. In addition, denote the nonnegative and positive orthant of $\mathbf{R}^m$ by $\mathbf{R}^m_+$ and $\mathbf{R}^m_{++}$, respectively. For any $x \in \partial\Omega$, denote the active index set at x by

$$B(x) = \{i \in (1, \cdots, m) : \ g_i(x) = 0\}.$$

2 Main results

In the following we use the homotopy method to solve the fixed points of $\Phi(x)$ in some unbounded sets under the following assumptions:

(C_1) Ω^0 is nonempty;

(C_2) For any $x \in \partial\Omega$, the matrix $\{\nabla g_i(x) : \ i \in B(x)\}$ is of full column rank;

(C_3) There exists some $\xi \in \Omega$ such that

$$\lim_{\|x\| \to \infty} (x - \xi)^T (x - \Phi(x)) = +\infty;$$

(C_4) $g_i(x), \ i = 1, \cdots, m$ are convex.

Remark 2.1 It should be pointed out that the boundedness assumption is removed and replaced by assumption (C_3), i.e., if assumption (C_3) is satisfied, then we are able to utilize the homotopy method to solve the fixed points of $\Phi(x)$ in some unbounded subsets in $\mathbf{R}^n$.

Under assumptions (C_1)–(C_4), to solve the Brouwer fixed point problem, we construct the following homotopy:

$$H(w, w^{(0)}, \mu) = \begin{pmatrix} (1-\mu)(x - \Phi(x) + \nabla g(x) y) + \mu(x - x^{(0)}) \\ Y g(x) - \mu Y^{(0)} g(x^{(0)}) \end{pmatrix} = 0, \tag{2.1}$$

where

$$w = (x, y) \in \mathbf{R}^{n+m}$$

and

$$w^{(0)} = (x^{(0)}, y^{(0)}) \in \Omega^0 \times \mathbf{R}^m_{++}.$$

Note that under assumptions (C_1)–(C_4), we get the global convergence of the interior point homotopy method similar to the analysis in [5] except for the boundedness of the component x of w, hence we list it in the following Lemma 2.2. Before proving the Lemma 2.2, we give the following notations and the Lemma 2.1 which will be used in the proof of Lemma 2.2. Let

$$\Omega^+(\xi) = \{x \in \Omega : \ (x - \xi)^T (x - \Phi(x)) > 0\}, \qquad \Omega^-(\xi) = \Omega \backslash \Omega^+(\xi).$$