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Commuting Toeplitz Operators with Harmonic Symbols on $A^2(\overline{\mathbb{D}}, \mathrm{d}\mu)$*

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Abstract: For any given symmetric measure μ on the closed unit disk $\overline{\mathbb{D}}$, we apply the Berezin transform to characterizing semi-commuting and commuting Toeplitz operators with bounded harmonic symbols on $A^2(\overline{\mathbb{D}}, \mathrm{d}\mu)$.

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1 Introduction

Let $\mathbb{D}, \overline{\mathbb{D}}$ and $\partial\mathbb{D}$ denote the open unit disk, the closed unit disk and the unit circle in the complex plane $\mathbb{C}$, respectively. A finite positive Borel measure μ supported on $\overline{\mathbb{D}}$ is said to be a symmetric measure if there is a finite positive Borel measure ν supported on the closed unit interval [0, 1] such that

$$\int_{\overline{\mathbb{D}}} f(z)\mathrm{d}\mu(z) = (2\pi)^{-1}\int_0^{2\pi}\mathrm{d}\theta\int_0^1 f(re^{\mathrm{i}\theta})\mathrm{d}\nu(r)$$

for any continuous function $f(z)$ on $\overline{\mathbb{D}}$. In this case, the measure ν is called the radial component of μ. In order that μ is a normalized measure on $\overline{\mathbb{D}}$, we suppose $\nu([0,1]) = 1$. Throughout this paper, μ always denotes a normalized symmetric measure on $\overline{\mathbb{D}}$ and ν always denotes the radial component of μ. For $1 \le p < +\infty$, let $L^p(\overline{\mathbb{D}}, \mathrm{d}\mu)$ denote the space of measurable functions f on $\overline{\mathbb{D}}$ such that

$$\|f\|_p = \Big(\int_{\overline{\mathbb{D}}} |f(z)|^p \mathrm{d}\mu(z)\Big)^{1/p} < +\infty.$$

$A^p(\overline{\mathbb{D}}, \mathrm{d}\mu)$ is the set of those functions in $L^p(\overline{\mathbb{D}}, \mathrm{d}\mu)$ which are analytic on $\mathbb{D}$. $L^\infty(\overline{\mathbb{D}}, \mathrm{d}\mu)$, $L^\infty(\mathbb{D}, \mathrm{d}A)$ and $L^\infty(\partial\mathbb{D}, \mathrm{d}\theta)$ denote the set of the essentially bounded measurable functions on $\overline{\mathbb{D}}$, $\mathbb{D}$ and $\partial\mathbb{D}$ with respect to $\mathrm{d}\mu$, $\mathrm{d}A$ and $\mathrm{d}\theta$, respectively, where $\mathrm{d}A$ is the normalized

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area measure on $\mathbb{D}$. As usual, the space of bounded analytic functions on $\mathbb{D}$ is denoted by $H^\infty(\mathbb{D})$.

Suppose that

$$\nu([\epsilon, 1]) > 0, \qquad 0 \le \epsilon < 1.$$

Frankfurt[1] obtained the following results.

(1) Let

$$\omega(n) = \|z^n\|_2^2 = \int_0^1 r^{2n} \mathrm{d}\nu(r).$$

For $z \in \mathbb{D}$,

$$K_z(w) = \sum_{n=0}^{\infty} \frac{(\overline{z}w)^n}{\omega(n)}$$

is the reproducing kernel of $A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$, that is,

$$f(z) = \langle f,\ K_z \rangle \qquad \text{for any } f \text{ in } A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu).$$

In fact, if σ is a compact subset of $\mathbb{D}$, $\{K_z : z \in \sigma\}$ is a uniformly bounded set of linear functionals on $A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$. Moreover, $A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$ is a Hilbert space with the inner product

$$\langle f,\ g \rangle = \int_{\overline{\mathbb{D}}} f(z)\overline{g(z)} \mathrm{d}\mu(z).$$

(2) The set of analytic polynomials is dense in $A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$.

(3) If $\nu(\{1\}) > 0$, then

$$A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu) = H^2(\partial\mathbb{D})$$

with equivalent norms, where $H^2(\partial\mathbb{D})$ is the classical Hardy space on $\partial\mathbb{D}$.

It follows from (3) that if $\nu(\{1\}) > 0$, then for any $f \in A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$, the radial limit $\lim\limits_{r\to 1} f(re^{\mathrm{i}\theta})$ exists for almost all $\theta \in [0, 2\pi]$ (see Chapter 1 of [2]).

In addition, if, for any z in $\mathbb{D}$, $K_z(w)$ is the reproducing kernel of $A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$, then

$$\nu([\epsilon, 1]) > 0, \qquad 0 \le \epsilon < 1$$

(see Proposition 3.19 of [3]). In this paper we always suppose that $\nu([\epsilon, 1]) > 0$ for every $0 \le \epsilon < 1$.

In particular, if $\mathrm{d}\nu(r) = 2r\mathrm{d}r$, then $A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$ is just the Bergman space $A^2(\mathbb{D},\ \mathrm{d}A)$. For more results on the Bergman space $A^2(\mathbb{D},\ \mathrm{d}A)$, the reader is referred to [4] and [5]. If ν is a single unit point mass at 1, then $A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$ is just the classical Hardy space $H^2(\partial\mathbb{D})$. A reference for the classical Hardy space is Duren's book [2]. Moreover, given any symmetric measure μ on $\overline{\mathbb{D}}$, if $f \in H^2(\partial\mathbb{D})$ then $f \in A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$. In fact, if the Taylor series of $f \in H^2(\partial\mathbb{D})$ is $\sum\limits_{n=0}^{\infty} a_n z^n$, then

$$\sum_{n=0}^{\infty} |a_n|^2 \omega(n) \le \sum_{n=0}^{\infty} |a_n|^2 < \infty$$

since $\omega(n) \le 1$.

Let P be the orthogonal projection from $L^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$ onto $A^2(\overline{\mathbb{D}},\ \mathrm{d}\mu)$. Then

$$Pf(z) = \langle Pf,\ K_z \rangle = \langle f,\ K_z \rangle = \int_{\overline{\mathbb{D}}} f(w)\overline{K_z(w)} \mathrm{d}\mu(w) = \int_{\overline{\mathbb{D}}} f(w) K_w(z) \mathrm{d}\mu(w), \qquad z \in \mathbb{D}$$