COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
25(2)(2009), 177–192

Rational-slice Knots via Strongly Negative-amphicheiral Knots*

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Abstract: We show that certain satellite knots of every strongly negative-amphicheiral rational knot are rational-slice knots. This proof also shows that the 0-surgery manifold of a certain strongly negative amphicheiral knot such as the figure-eight knot bounds a compact oriented smooth 4-manifold homotopy equivalent to the 2-sphere such that a second homology class of the 4-manifold is represented by a smoothly embedded 2-sphere if and only if the modulo two reduction of it is zero.
Key words: rational-slice knot, strongly negative-amphicheiral knot, 0-surgery, rational cobordism, 4-manifold
2000 MR subject classification: 57M25, 57Q60
Document code: A
Article ID: 1674-5647(2009)02-0177-16

1 Statement of Result

A knot K in the 3-sphere $\mathbf{S}^3$ is a slice knot if K bounds a smooth proper disk D in the 4-disk $\mathbf{B}^4$ bounded by $\mathbf{S}^3$. In this paper, we generalize the concept of a slice knot to a concept on a rational knot, i.e., a knot K in a rational-homology 3-sphere S (= a smooth oriented 3-manifold with the rational-homology of $\mathbf{S}^3$). A rational $(4, 2)$-disk pair is a $(4, 2)$-dimensional manifold pair (B, D) such that B is a rational 4-disk, namely a compact smooth oriented 4-manifold with the rational-homology of the 4-disk $\mathbf{B}^4$, and D is a smooth proper disk in B. The boundary pair $(S, K) = (\partial B, \partial D)$ is a rational knot, which we call a weakly rational-slice knot. We need a more detailed concept of a weakly rational-slice knot. To state it, we note that there is a natural isomorphism

$$H_2(S, S\backslash K) \to H_2(B, B\backslash D)$$

on infinite cyclic groups which can be seen by taking a relative tubular neighborhood of (D, K) in (B, S) and then considering excision isomorphisms. We denote by $\mathrm{b}H_*(\bullet)$ the quotient group of the integral homology group $H_*(\bullet)$ by the torsion subgroup $\mathrm{t}H_*(\bullet)$. Then we see that the natural homomorphism

$$\mathrm{b}H_1(S\backslash K) \to \mathrm{b}H_1(B\backslash D)$$

*Received date: Dec. 31, 2008.
This paper is dedicated to Professor Jose Maria Montesinos on the occasion of his 65th birthday.

is a monomorphism on infinite cyclic groups.

For an integer $d \geqq 1$, the knot (S, K) is a d-rational-slice knot if it bounds a rational $(4, 2)$-disk pair (B, D) such that the cokernel of the natural monomorphism

$$\mathrm{b}H_1(S \backslash K) \to \mathrm{b}H_1(B \backslash D)$$

is isomorphic to Z_d $(= Z/dZ)$.

Let $o(K)$ denote the homological order of the element $[K] \in H_1(S)$, where the zero element is understood to have the order 1. A rational-slice knot is a 1-rational-slice knot (S, K) with $o(K) = 1$, meaning that the knot (S, K) bounds a rational $(4, 2)$-disk pair (B, D) which induces a meridian-preserving natural isomorphsim $\mathrm{b}H_1(S \backslash K) \to \mathrm{b}H_1(B \backslash D)$ on the infinite cyclic groups with meridian generators. We see that any rational-slice knot (S, K) is an algebraic-slice knot, that is, a knot with a null-cobordant Seifert matrix in the sense of Levine[1]. In fact, we can construct a Seifert surface F for K in S since $o(K) = 1$ and hence a compact smooth oriented 3-manifold A in B bounded by the closed surface $F \cup (-D)$ by applying the Pontrjagin-Thom construction to the natural isomorphism

$$H^1(B \backslash D) \cong H^1(S \backslash K) \cong Z.$$

The existence of this 3-manifold A means that K is an algebraic-slice knot (cf. [2, Theorem 12.2.3]).

Let O be a link with components O_i $(i = 1, 2, \cdots, s)$ in the 3-sphere $\mathbf{S}^3$. We deform the link O into a link $\tilde{O} = \cup_{i=1}^{s} \tilde{O}_i$ in an unknotted solid torus $V \subset \mathbf{S}^3$. There are infinitely many ways of constructing links $\tilde{O} \subset V$ from O. The link $\tilde{O}$ in V is an m-satellite link and denoted by $\tilde{O}(m)$ if m is the greatest common divisor of the integers $m_i \geqq 0$ $(i = 1, 2, \cdots, s)$ such that the cokernel of the natural homomorphism $H_1(\tilde{O}_i) \to H_1(V)$ is isomorphic to Z_{m_i} for every i.

Let $V(K)$ be a tubular neighborhood of a knot K in S. An m-satellite link of a link O in $\mathbf{S}^3$ along a knot K in S is a link in S which is the image $\tilde{O}(m; K) \subset V(K) \subset S$ of an m-satellite link $\tilde{O}(m) \subset V$ under a (meridian, longitude)-preserving and orientation-preserving homeomorphism (called a faithful homeomorphism) $V \to V(K)$.

A knot K in S is strongly negative-amphicheiral if there is an orientation-reversing involution τ on S such that $\tau(K) = K$ and the fixed point set

$$\mathrm{Fix}(\tau) = \mathbf{S}^0 \subset K.$$

In this case, it turns out that there are two types of strongly negative-amphicheiral knots. To state it, let (S_τ, K_τ) be the orbit pair of the pair (S, K) under the action τ, and

$$\tau^* : H_1(S_\tau \backslash K_\tau) \to Z_2$$

the monodromy map of the double covering $S \backslash K \to S_\tau \backslash K_\tau$. We say that K is of type I or II according to whether the restriction of τ^* to the torsion subgroup $\mathrm{t}H_1(S_\tau \backslash K_\tau)$ is non-trivial or trivial, respectively. If S is a Z_2-homology 3-sphere, then K is always of type II, as we shall show in Corollary 2.1. In Example 2.1, we shall give an example of a strongly negative-amphicheiral knot K with $o(K) = 2$ of type I in a rational-homology 3-sphere S with $H_1(S) = Z_2 \oplus Z_2$. The following theorem is our main result.