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Some Properties of $\mathrm{Aut}_*(X)$ and the Subgroup $\mathrm{Aut}_{\Sigma}(X)$*

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Abstract: Let $\mathrm{Aut}_*(X)$ denote the group of homotopy classes of self-homotopy equivalences of X, which induce identity automorphisms of homology group. We describe a decomposition of $\mathrm{Aut}_*(X_1\vee\cdots\vee X_n)$ as a product of its simpler subgroups. We consider the subgroup $\mathrm{Aut}_{\Sigma}(X)$ of all self homotopy classes α of X such that $\Sigma\alpha = 1_{\Sigma X} : \Sigma X \to \Sigma X$, and also give some properties of $\mathrm{Aut}_{\Sigma}(X)$.

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1 Introduction

Let X be a pointed CW-complex and let $\mathrm{Aut}(X)$ denote the set of homotopy classes of self-maps of X that are homotopy equivalences. This set is a group, called group of self-homotopy equivalences, with respect to the operation induced by the composition of maps. The excellent survey paper [1] gives an idea of the extensive literature on these groups.

In [2] Pavešić proved that under the assumption that the self-equivalences of $X \times Y$ are reducible the group $\mathrm{Aut}(X \times Y)$ decomposes as a product of two subgroups denoted by $\mathrm{Aut}_X(X \times Y)$ and $\mathrm{Aut}_Y(X \times Y)$. In [3] Pavešić proved that the subgroup $\mathrm{Aut}_{\#}(X)$ of $\mathrm{Aut}(X)$ is always reducible, so that $\mathrm{Aut}_{\#}(X \times Y)$ can be decomposed as a product of $\mathrm{Aut}_{\#X}(X \times Y)$ and $\mathrm{Aut}_{\#Y}(X \times Y)$. There is also a considerable interest by many authors concerning subgroups of $\mathrm{Aut}(X)$, which induce identity automorphisms of the homology groups of X, see [4]–[6]. Let $\mathrm{Aut}_*(X)$ denote the subset of $\mathrm{Aut}(X)$ consisting of classes of maps which induce identity automorphisms of the homology groups of X, i.e. the kernel of the obvious representation $\mathrm{Aut}(X) \longrightarrow \bigoplus_{i=1}^{\infty} \mathrm{Aut}(H_i(X))$. $\mathrm{Aut}_*(X)$ is indeed a subgroup of

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$\mathrm{Aut}(X)$ by [4]. Let $\mathrm{Aut}_X(X\vee Y)$ be the subset of $\mathrm{Aut}(X\vee Y)$ whose elements are represented by maps $f : X \vee Y \to X \vee Y$ that are over X, i.e., are of the form $f = (i_X, f_Y)$, therefore

$$f(x_0, y) = f_Y(y), \qquad f(x, y_0) = i_X(x) = (x, y_0).$$

Similarly one can define $\mathrm{Aut}_Y(X \vee Y)$.

In [6] the authors proved that under the assumption that the self-equivalences of $X \vee Y$ are reducible the group $\mathrm{Aut}(X \vee Y)$ decomposes as a product of two subgroups denoted by $\mathrm{Aut}_X(X\vee Y)$ and $\mathrm{Aut}_Y(X\vee Y)$, respectively. In [7] the authors have studied self-homotopy equivalences of sum object. Unfortunately, the reducibility is quite a restrictive condition and is usually difficult to verify, which limits the applicability of the result. Luckily, this problem disappears if we consider $\mathrm{Aut}_*(X \vee Y)$.

In Section 2 we show that the self-equivalences in $\mathrm{Aut}_*(X \vee Y)$ are always reducible, hence there is a corresponding decomposition of $\mathrm{Aut}_*(X \vee Y)$ for arbitrary X and Y. This paves the way for a generalization of our approach to factorizations of self-equivalences of more than two wedge spaces. The authors have studied the properties of $\mathrm{Aut}_\Omega(X)$ in [8] and [9]. In Section 3 we show some properties of $\mathrm{Aut}_\Sigma(X)$ which is the subgroup of $\mathrm{Aut}_*(X)$.

All spaces in this paper are pointed and connected, and they have the based homotopy type of a CW-complex. All maps and homotopy classes are base-point preserving, and we do not distinguish the notation between a map and its homotopy class. A homotopy inverse of a homotopy equivalence f is denoted by f^{-1}. A nilpotent space is one such that the fundamental group is nilpotent and which acts nilpotently on the higher homotopy groups (see [10]).

The identity map of X is denoted id_X or simply id and the constant map is $* : X \to Y$. A space X is an H-space if there is a map $\mu : X \times X \to X$ whose composition with each of the two inclusion $X \to X\times X$ is id_X. If $f : X \to Y$ and $g : Y \to X$ are such that $g\circ f = id_X$ then X is a retract of Y, g is called a retraction of f and f is called a section of g. A map $f : X \to Y$ induces function $f_* : [A, X] \to [A, Y]$ and $f^* : [Y, B] \to [X, B]$ by composition, for all A and B. The homomorphism of homology groups $H_n(X) \to H_n(Y)$ induced by f is denoted by f_* or f_{*n}. The standard notation of homotopy theory will be used: Σ for (reduced) suspension, Ω for loop-space, $\vee$ for wedge and $\wedge$ for smashed product. The natural isomorphism between $[\Sigma X, Y]$ and $[X, \Omega Y]$ is called adjoint isomorphism. Other notations one can see in [11]–[13].

Given a group G and its subgroup A, B we will write $G = A \cdot B$ if every $g \in G$ can be uniquely factorized as $g = ab$ where $a \in A$ and $b \in B$. Equivalently, $G = A \cdot B$ if $G = \{ab|\ a \in A, b \in B\}$ and the intersection of A and B is trivial.

2 Reducibility of Self-equivalences

In this section we show that the self-equivalences of $\mathrm{Aut}_*(X \vee Y)$ are always reducible. We then use this fact to derive a factorization of $\mathrm{Aut}_*(X \vee Y)$ and its generalization to $\mathrm{Aut}_*(X_1 \vee \cdots \vee X_n)$. All spaces in this section are pointed and simply connected.