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KKM Theorems on FC-spaces with Their Applications to Variational Inequalities*

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Abstract: A new concept of finitely continuous topological spaces (in short, FC-space) without convexity structure and linear structure was introduced. Some KKM type theorems in noncompact FC-spaces were obtained, and from which, some section theorems and variational inequality theorems were proved under much weak assumptions. Our results improve and generalize the corresponding conclusions in recent literature.

Key words: FC-space, FC-subspace, KKM map, lower semicontinuous, variational inequality
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1 Introduction and Preliminaries

Recently, Ding[1,2] introduced the concept of finitely continuous topological spaces (simply, FC-spaces) without convexity structure and linear structure, which is a far-reaching generalization of convex subsets of Hausdorff topological vector spaces, convex space (see [3]), hyperconvex space (see [4]), H-space (see [5]), generalized convex space (see [6]) and other convex structures. On the other hand, many authors obtained a lot of important results on FC-spaces, see [1], [2], [7] and others.

Our aim in this paper is to obtain two KKM type theorems for multimap on noncompact FC-spaces, establish some generalized section theorems, and discuss the existence of solutions for extended variational inequalities as applications of KKM type theorems obtained in Section 2. These results improve and generalize the corresponding conclusions in recent literature.

Firstly, we list some definitions and notations.

Let X and Y be two topological spaces, the multimap $T : X \multimap Y$ be a map from X to

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the power set 2^Y of Y. $T(x)$ is called the value at $x \in X$. Let

$$T(A) = \bigcup_{x\in A} T(x), \qquad A \subset X.$$

Let X be a topological space, $\mathbf{R}$ a real number space, $f : X \to \mathbf{R}$. f is said to be a lower semicontinuous map (in short, l.s.c) if $\{x \in X : f(x) > \gamma\}$ is open in X for each $\gamma \in \mathbf{R}$.

Definition 1.1[1] *$(X, D, \{\phi_N\})$ is said to be an FC-space, if X is a topological space, D is a nonempty subset of X, and for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$, there exists a continuous map $\phi_N : \triangle_n \to X$, where $\langle D\rangle$ denotes the set of all nonempty finite subset of D. If $D = X$, then $(X, D, \{\phi_N\})$ is denoted by $(X, \{\phi_N\})$.*

Definition 1.2[2] *Let $(X, D, \{\phi_N\})$ be an FC-space, A and B two subsets of X. B is said to be an FC-subspace of X with respect to A, if for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$ and for each $J = \{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\} \subset A \bigcap \{x_0, x_1, \cdots, x_n\}$,*

$$\phi_N(\triangle_k) \subset B,$$

where $\triangle_k = \mathrm{co}(\{e_{i_0}, e_{i_1}, \cdots, e_{i_k}\})$ is the face of $\triangle_n$ corresponding to J. If $A = B$, then call B be an FC-subspace of X.

Definition 1.3[1] *Let $(X, D, \{\phi_N\})$ be an FC-space. $F : D \multimap X$ is said to be a KKM map, if for each $N = \{x_0, x_1, \cdots, x_n\} \in \langle D\rangle$ and any $\{x_{i_0}, x_{i_1}, \cdots, x_{i_k}\} \subset \{x_0, x_1, \cdots, x_n\}$,*

$$\phi_N(\triangle_k) \subset \bigcup_{j=0}^{k} F(x_{i_j}),$$

where $\triangle_k = \mathrm{co}(\{e_{i_j} : j = 0, 1, \cdots, k\})$.

Definition 1.4 *Let X be a nonempty set, Y a topological space. A map $F : X \multimap Y$ is said to be transfer closed (resp. transfer open) valued if for $x \in X$ and $y \in Y$ with $y \notin F(x)$ (resp. $y \in F(x)$), there exists an $x' \in X$ such that $y \notin \overline{F(x')}$ (resp. $y \in \mathrm{int}F(x')$).*

Theorem 1.1[8] *If X is a nonempty set, Y is a topological space, and $F : X \multimap Y$ is a map, then*

(i) *F is transfer closed valued map if and only if $\bigcap_{x\in X} F(x) = \bigcap_{x\in X} \overline{F(x)}$;*

(ii) *F is transfer open valued map if and only if $\bigcup_{x\in X} F(x) = \bigcup_{x\in X} \mathrm{int}F(x)$,*

where $\overline{A}$ and $\mathrm{int}A$ denote the closure and interior of A respectively for $A \subset Y$.

Theorem 1.2[9] *Let F_i $(0 \le i \le n)$ be $n + 1$ closed (resp. open) subsets of an n-simplex $v_0v_1 \cdots v_n$. If*

$$v_{i_0}v_{i_1} \cdots v_{i_k} \subset F_{i_0} \bigcup F_{i_1} \bigcup \cdots \bigcup F_{i_k}$$

holds for all faces $v_{i_0}v_{i_1} \cdots v_{i_k}$, $0 \le k \le n$, $0 \le i_0 < i_1 < \cdots < i_k \le n$, then

$$\bigcap_{i=0}^{n} F_i \ne \emptyset.$$