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A New Proof of Diophantine Equation $\binom{n}{2} = \binom{m}{4}$*

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Abstract: By using algebraic number theory and p-adic analysis method, we give a new and simple proof of Diophantine equation $\binom{n}{2} = \binom{m}{4}$.

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1 Introduction

In the section D3 of [1], there is a famous problem: does

$$\binom{n}{2} = \binom{m}{4} \tag{1.1}$$

have any other nontrivial solutions besides $(m, n) = (10, 21)$? In 1995, Pinter[2], and in 1996, de Weger[3] solved this problem, respectively by using the package KANT and the Baker-Davenport reduction algorithm, and the transcendence result of Baker and Wüstholz that yield absolute upper bounds for m, n. In 1997, Stroeker and de Weger[4], and in 2000, Hajdu and Pinter[5] resolved it respectively by using linear forms in elliptic logarithms, and using Pari and the program package SIMATH. In 2004, Li and Cao[6] gave an elementary solution of this problem by using difference recursion sequence method. In this note, we give a new and simple proof by using algebraic number theory and p-adic analysis method.

2 Main Theorem and Proof

Theorem 2.1 *All the integral points of Diophantine equation* (1.1) *are* $(m, n) = (-7, -20)$, $(-7, 21)$, $(-3, -5)$, $(-3, 6)$, $(-1, -1)$, $(-1, 2)$, $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, 0)$, $(2, 1)$,

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$(4,-1)$, $(4,2)$, $(6,-5)$, $(6,6)$, $(10,-20)$, $(10,21)$.

Proof. Equation (1.1) is equivalent to

$$\frac{n(n-1)}{2!}=\frac{m(m-1)(m-2)(m-3)}{4!},$$

that is,

$$(m^2-3m+1)^2+2=3(2n-1)^2.$$

We factorize the above equation in the quadratic algebraic field $\boldsymbol{Q}(\sqrt{-2})$ (see [7]–[9]). Since the class-number of the ring of integer is 1 and the unit roots of $\boldsymbol{Q}(\sqrt{-2})$ are ±1, then

$$(m^2-3m+1)\pm\sqrt{-2}=\pm(1+\sqrt{-2})A^2,$$
$$(2m-3)^2\pm(1+\sqrt{-2})(2A)^2=5\pm4\sqrt{-2},$$

where A, A' are two conjugate algebraic integers and

$$2n-1=AA'.$$

Put

$$A_1=2A.$$

Then

$$(2m-3)^2-(1+\sqrt{-2})A_1^2=5\pm4\sqrt{-2}, \tag{2.1}$$

or

$$(2m-3)^2+(1+\sqrt{-2})A_1^2=5\pm4\sqrt{-2}. \tag{2.2}$$

Case 1 First, we solve equation (2.1). Let

$$\rho=\sqrt{-2}.$$

Write (2.1) as

$$(2m-3)^2-(1+\rho)A_1^2=5\pm4\rho. \tag{2.3}$$

Put

$$\theta=\sqrt{1+\rho}.$$

Then θ is an algebraic integer in a totally complex quartic field $\boldsymbol{Q}(\theta)$ and satisfies

$$\theta^2=1+\rho.$$

From [10], we know that $\{1,\ \theta,\ \rho,\ \theta\rho\}$ is a basis of $\boldsymbol{Q}(\theta)$. $\{\theta,\ -\theta,\ \theta'=\sqrt{1-\rho},\ -\theta'\}$ denotes all conjugates of θ in $\boldsymbol{Q}(\theta)$.

Write

$$\alpha=(a,b,c,d)$$

as a shorthand for

$$\alpha=a+b\theta+c\rho+d\theta\rho.$$

The conjugates of α are

$$\alpha_-=a-b\theta+c\rho-d\theta\rho=(a,-b,c,-d),$$
$$\alpha'=a+b\theta'-c\rho-d\theta'\rho,$$
$$\alpha'_-=a-b\theta'-c\rho+d\theta'\rho.$$