Communications
in
Mathematical
Research
25(4)(2009), 289–298

Weighted Approximation by Left Quasi-interpolants of Derivatives of Gamma Operators*

Jiang Hong-Biao
(*Department of mathematics, Lishui University, Lishui, Zhejiang, 323000*)

Communicated by Ma Fu-ming

Abstract: In order to obtain much faster convergence, Müller introduced the left Gamma quasi-interpolants and obtained an approximation equivalence theorem in terms of $\omega_{\varphi}^{2r}(f,t)_p$. Guo extended the Müller's results to $\omega_{\varphi^\lambda}^{2r}(f,t)_\infty$. In this paper we improve the previous results and give a weighted approximation equivalence theorem.

Key words: Gamma operator, quasi-interpolant, weighted approximation, modulus of smoothness, derivative

2000 MR subject classification: 41A25, 41A35, 41A36

Document code: A

Article ID: 1674-5647(2009)04-0289-10

1 Introduction

Gamma operators are given by

$$G_n(f,x) = \int_0^\infty g_n(x,t) f\left(\frac{n}{t}\right) \mathrm{d}t, \qquad x \in I = [0,\infty), \tag{1.1}$$

with the kernel

$$g_n(x,t) = \frac{x^{n+1}}{n!} \mathrm{e}^{-xt} t^n.$$

Sometimes, we also use an alternate representation

$$G_n(f,x) = \frac{1}{n!} \int_0^\infty \mathrm{e}^{-t} t^n f\left(\frac{nx}{t}\right) \mathrm{d}t, \qquad x \in I. \tag{1.2}$$

These operators have been investigated in a series of papers (see [1]–[4] and the references therein). The optimal degree of approximation of the method of Gamma operators G_n in L_p space is $O(n^{-1})$. In order to obtain much faster convergence, quasi-interpolants $G_n^{(k)}$ of G_n in the sense of Sablonnière[5] are considered in [6]–[9].

***Received date:** Oct. 19, 2006.
Foundation item: The NSF (102005) of Zhejiang Province, China.

We first recall the construction of the left Gamma quasi-interpolants (see [8])

$$G_n^{(k)}(f,x) = \sum_{j=0}^{k} \alpha_j^n(x) D^j G_n(f,x), \qquad 0 \le k \le n, \tag{1.3}$$

where $D^j = \dfrac{\mathrm{d}^j}{\mathrm{d}x^j}$, $D^0 = id$ and $\alpha_j^n(x) \in \Pi_j$ (the space of algebraic polynomials of degree at most j). In [8] Müller obtained an approximation equivalence theorem: for $f \in L_p(I)$, $1 \le p \le \infty$, $\varphi(x) = x$, $n \ge 4r$, $r \in N$, and $0 < \alpha < r$, one has

$$\|G_n^{(2r-1)} f - f\|_p = O(n^{-\alpha}) \Longleftrightarrow \omega_\varphi^{2r}(f,t)_p = O(t^{2\alpha}). \tag{1.4}$$

In [7] Guo *et al.* gave a weighted approximation equivalence theorem for $G_n^{(2r-1)}(f,x)$ in L_∞ space: for $f \in L_\infty(I)$, $0 \le \lambda \le 1$, $\varphi(x) = x$, $w(x) = x^a(1+x)^b$ ($a \ge 0$, b is arbitrary), $n \ge 4r$, and $0 < \alpha < 2r$, one has

$$|w(x)(G_n^{(2r-1)}(f,x) - f(x))| = O((n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^\alpha) \Longleftrightarrow \omega_{\varphi^\lambda}^{2r}(f,t)_w = O(t^\alpha). \tag{1.5}$$

In this paper we consider a weighted approximation for $G_{n,s}^{(2r-1)}(f^{(s)},x)$ in L_∞ space.

Theorem 1.1 *For $f^{(s)}, wf^{(s)} \in L_\infty(I)$, $0 \le \lambda \le 1$, $\varphi(x) = x$, $w(x) = x^a(1+x)^b$, $n - s \ge 4r$, $0 < \alpha - s < 2r$, $s \in N_0 = N \bigcup \{0\}$, one has*

$$|w(x)(G_{n,s}^{(2r-1)}(f^{(s)},x) - f^{(s)}(x))| = O((n^{-\frac{1}{2}}\varphi^{1-\lambda}(x))^{\alpha-s}) \Longleftrightarrow \omega_{\varphi^\lambda}^{2r}(f^{(s)},t)_w = O(t^{\alpha-s}). \tag{1.6}$$

In L_∞ space, when $s = 0$, (1.6) is (1.5); when $s = 0$, $a = b = 0$, $\lambda = 1$, (1.6) is (1.4).

Throughout this paper $\|\cdot\|$ denotes $\|\cdot\|_\infty$, and C denotes a positive constant independent of n, x and not necessarily the same at each occurrence.

2 Preliminaries and Lemmas

In this section, for $0 \le \lambda \le 1$, $0 < \alpha - s < 2r$, $s \in N_0$, we first give some notations as follows:

$$\|f\|_0 = \sup_{x\in(0,\infty)} |w(x)\varphi^{(\alpha-s)(\lambda-1)}(x) f(x)|,$$

$$C_{\lambda,w}^0 = \{f | wf \in L_\infty, \|f\|_0 < \infty\},$$

$$\|f\|_{2r} = \sup_{x\in(0,\infty)} |w(x)\varphi^{2r+(\alpha-s)(\lambda-1)}(x) f^{(2r)}(x)|,$$

$$C_{\lambda,w}^{2r} = \{f \in C_{\lambda,w}^0 : f^{(2r-1)} \in A.C._{\mathrm{loc}}, \|f\|_{2r} < \infty\}.$$

The modulus of smoothness and K-functional we will use later are defined as follows (see [1]):

$$\omega_{\varphi^\lambda}^r(f,t)_w = \begin{cases} \sup\limits_{0<h\le t} \|w\triangle_{h\varphi^\lambda}^r f\|, & a = 0; \\ \sup\limits_{0<h\le t} \|w\triangle_{\varphi^\lambda}^r f\|_{[t^*,\infty)} + \sup\limits_{0<h\le t^*} \|w\vec{\triangle}_h^r f\|_{[0,12t^*]}, & a > 0, \end{cases}$$

where

$$t^* = \begin{cases} (rt)^{\frac{1}{1-\lambda}}, & 0 < t < \dfrac{1}{8r},\ 0 \le \lambda < 1; \\ 0, & \lambda = 1, \end{cases}$$