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Commutators of Multilinear Singular Integrals with Lipschitz Functions*

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Abstract: The boundedness of commutators of multilinear singular integrals with Lipschitz functions in product Lebesgue spaces is obtained.

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1 Introduction

In recently years, multilinear singular integrals of Calderón-Zygmund type have attracted much attention. Many results parallel to the linear theroy of classical Calderón-Zygmund operators have been obtained. For details, one can see [1]–[5]. Meanwhile, commutators of singular integral operators continue to attract many authors' attention, see [6]–[12] and the references therein. Indeed, the multilinear commutator as a generalization of commutator was introduced by Perez and Trujillo-Gonzalez in [11] recently. And in [12], Perez and Trujillo-Gonzalez obtained sharp weighted estimates for vector valued singular integral operators and commutators. Motivated by these results mentioned above, we consider commutators of multilinear singular integrals with Lipschitz functions.

Let T be a multilinear operator which is initially defined on the m-fold product of Schwartz space $\mathscr{S}(\mathbf{R}^n)$ and take its values into the space of tempered distributions $\mathscr{S}'(\mathbf{R}^n)$. We assume that the distributional kernel on $(\mathbf{R}^n)^{m+1}$ of the operators coincides away from the diagonal $y_0 = y_1 = \cdots = y_m$ in $(\mathbf{R}^n)^{m+1}$ with a function K so that

$$\begin{aligned} T\boldsymbol{f}(x) &= T(f_1, \ \cdots, \ f_m)(x) \\ &= \int_{(\mathbf{R}^n)^m} K(x, y_1, \cdots, y_m) f_1(y_1) \cdots f_m(y_m) \mathrm{d}y_1 \cdots \mathrm{d}y_m, \end{aligned} \tag{1.1}$$

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whenever $f_1, \cdots, f_m$ are $L_C^\infty(\mathbf{R}^n)$ and $x \notin \bigcap_{j=1}^m \text{supp} f_j$, where as usual, $L_C^\infty(\mathbf{R}^n)$ denotes all $L^\infty(\mathbf{R}^n)$ functions with compact support. Moreover, we assume that the kernel function K satisfies the standard estimates

$$|K(y_0, y_1, \cdots, y_m)| \le A\Big(\sum_{k,l=0}^m |y_k - y_l|\Big)^{-mn} \tag{1.2}$$

when $y_0, y_1, \cdots, y_m$ are not all equal, and for some $\epsilon > 0$,

$$|K(y_0, y_1, \cdots, y_j, \cdots, y_m) - K(y_0, y_1, \cdots, y_j', \cdots, y_m)| \le \frac{A|y_j - y_j'|^\epsilon}{\Big(\sum\limits_{k,l=0}^m |y_k - y_l|\Big)^{mn+\epsilon}}, \tag{1.3}$$

provided that $0 \le j \le m$ and

$$|y_j - y_j'| \le \frac{1}{2} \max_{0 \le k \le m} |y_j - y_k|.$$

Such kernels are called m-linear Calderón-Zygmund kernels and the collection of such functions is denoted by $m\text{-}CZK(A, \epsilon)$ in [1].

For these operators, Grafakos and Torres[1] obtained a boundedness estimate

$$T : L^{q_1} \times \cdots \times L^{q_m} \to L^q$$

for some $1 < q_1, \cdots, q_m < \infty$ with

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}, \tag{1.4}$$

which implies the boundedness of the operator for all possible exponents in such range of values, and an endpoint estimate

$$T : L^{q_1} \times \cdots \times L^{q_m} \to L^{q,\infty}$$

for $1 < q_1, \cdots, q_m < \infty$ satisfying (1.4), where L^p and $L^{p,\infty}$ denote Lebesgue spaces and weak Lebesgue spaces for $0 < p < \infty$ respectively. In particular, it holds that

$$T : L^1 \times \cdots \times L^1 \to L^{\frac{1}{m},\infty},$$

which extends the classical results to the linear case $T : L^1 \to L^{1,\infty}$.

Let T be as in (1.2) with an $m\text{-}CZK(A, \epsilon)$ kernel. If T is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ to L^q with $1 < q_1, \cdots, q_m < \infty$ and

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q},$$

then we say that T is an $m\text{-}CZO$.

Now we define multilinear commutators generated by Calderón-Zygmund operators and Lipschitz functions. First we recall the following definition of Lipschitz functions.

Definition 1.1 *Let $\beta > 0$ and b be a locally integrable function on $\mathbf{R}^n$. We say b belongs to the space $Lip(\beta)$ if there is a constant $C > 0$ such that*

$$|b(x) - b(y)| \le C|x - y|^\beta \tag{1.5}$$

for almost every x and y in $\mathbf{R}^n$. The minimal constant C appeared in (1.5) is the $Lip(\beta)$ norm of b and is denoted simply by $\|b\|_{Lip(\beta)}$.

Let $\boldsymbol{b} = (b_1, \cdots, b_m)$ with $b_i \in \text{Lip}(\beta_i)$, $0 < \beta_i \le 1$ for $1 \le i \le m$, where $\beta = \sum_{i=1}^m \beta_i$ and $0 < \beta < n$, and $\boldsymbol{f} = (f_1, \cdots, f_m)$ for suitable functions $f_1, \cdots, f_m$. We consider the