COMMUNICATIONS IN MATHEMATICAL RESEARCH
25(4)(2009), 379–384

Non-simultaneous Blow-up Criteria for Localized Parabolic Equations*

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Abstract: This paper deals with blow-up solutions for parabolic equations coupled via localized exponential sources, subject to homogeneous Dirichlet boundary conditions. The criteria are proposed to identify simultaneous and non-simultaneous blow-up solutions. The related classification for the four nonlinear parameters in the model is optimal and complete.
Key words: non-simultaneous blow-up, simultaneous blow-up, critical exponent
2000 MR subject classification: 35K05, 35K60, 35B40, 35B33
Document code: A
Article ID: 1674-5647(2009)04-0379-06

1 Introduction and Main Results

In the present paper, we consider the following parabolic system coupled via localized sources

$$\begin{cases} u_t = \Delta u + \mathrm{e}^{mu(0,t)+pv(0,t)},\ v_t = \Delta v + \mathrm{e}^{qu(0,t)+nv(0,t)}, & (x,t) \in \Omega \times (0,T), \\ u = v = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where $\Omega = B_R = \{|x| < R\} \subset \mathbf{R}^N$, m, n, p, q are nonnegative constants, and the continuous functions $u_0(x)$, $v_0(x)$ are nonnegative, nontrivial, radially symmetric, non-increasing, and vanish on ∂B_R. The existence and uniqueness of local classical solutions to (1.1) are well known (see, for example, [1]). Nonlinear parabolic systems like (1.1) come from population dynamics, chemical reactions, heat transfer, etc., where u and v represent the densities of two biological populations during a migration, the thicknesses of two kinds of chemical reactants, the temperatures of two different materials during a propagation, etc.

In 2002, Zheng *et al.*[2] discussed the local problem

$$u_t = \Delta u + \mathrm{e}^{mu+pv}, \quad v_t = \Delta v + \mathrm{e}^{qu+nv}, \qquad (x,t) \in \Omega \times (0,T) \tag{1.2}$$

with homogeneous Dirichlet boundary conditions. The simultaneous blow-up rates are obtained for radially symmetric blow-up solutions in the region $\{0 \le m < q,\ 0 \le n < p\}$.

*__Received date:__ April 27, 2009.
Foundation item: The NSF (10471013, 10771024) of China.

Later, Zhao and Zheng[3], Li and Wang[4] studied the localized problem (1.1) with more general $\Omega \subset \mathbf{R}^N$ and $x_0 \in \Omega$. The critical blow-up exponents were obtained. Uniform blow-up profiles for simultaneous blow-up solutions were proved in the exponent region $\{0 \le m \le q,\ 0 \le n \le p\}$. If simultaneous blow-up happens in $\{0 \le q < m,\ 0 \le p < n\}$, the uniform blow-up profiles hold also. The other studies for parabolic systems with power or exponential nonlinearities can be found, e.g., in [5]–[7], where critical blow-up exponents, blow-up rates, and blow-up profiles were obtained.

Under the nonlinear source e^{mu} (e^{nv}), the component u (v) can blow up by itself for large initial data if $m > 0$ ($n > 0$). So there may be non-simultaneous blow-up as well, defined as, e.g.,

$$\limsup_{t\to T} \|u(\,\cdot\,,t)\|_\infty = +\infty, \qquad \sup_{t\in[0,T)} \|v(\,\cdot\,,t)\|_\infty < +\infty.$$

In contrast, the simultaneous blow-up means that

$$\limsup_{t\to T} \|u(\,\cdot\,,t)\|_\infty = \limsup_{t\to T} \|v(\,\cdot\,,t)\|_\infty = +\infty.$$

Motivated by the works above, in the present paper, we propose a complete and optimal classification for the simultaneous and non-simultaneous blow-up solutions of (1.1). Assume the initial data satisfy

$$\Delta u_0 + (1-\varepsilon\varphi)\mathrm{e}^{mu_0(0)+pv_0(0)}, \quad \Delta v_0 + (1-\varepsilon\varphi)\mathrm{e}^{qu_0(0)+nv_0(0)} \ge 0 \qquad \text{in } B_R \tag{1.3}$$

for some constant $\varepsilon \in (0,1)$, where φ is the first eigenfunction of

$$-\Delta\varphi = \lambda\varphi \qquad \text{in } B_R$$

with $\varphi = 0$ on ∂B_R, normalized by

$$\|\varphi\|_\infty = 1, \quad \varphi > 0 \qquad \text{in } B_R.$$

It is easy to check that u_t, $v_t \ge 0$ by the comparison principle.

The main results of the paper are the following criteria for identifying simultaneous and non-simultaneous blow-up in (1.1).

Theorem 1.1 *There exists initial data such that non-simultaneous blow-up occurs in* (1.1) *if and only if $m > q$ or $n > p$ (for u or v blowing up alone, respectively).*

Corollary 1.1 *Any blow-up in* (1.1) *is simultaneous if and only if $m \le q$ and $n \le p$.*

Theorem 1.2 *Any blow-up in* (1.1) *is non-simultaneous if and only if $m > q$ with $n \le p$ (for u blowing up alone), or $n > p$ with $m \le q$ (for v blowing up alone).*

Theorem 1.3 *Both simultaneous and non-simultaneous blow-up may occur in* (1.1) *if and only if $m > q$ and $n > p$.*

In summary, the complete and optimal classification for simultaneous and non-simultaneous blow-up solutions of (1.1) can be shown by the following figure: