Communications
in
Mathematical
Research
25(5)(2009), 402–410

Forward-backward Stochastic Differential Equations and Backward Linear Quadratic Stochastic Optimal Control Problem*

Zhang De-tao
(*School of Mathematics, Shandong University, Jinan, 250100*)

Communicated by Shi Shao-yun

Abstract: In this paper, we use the solutions of forward-backward stochastic differential equations to get the optimal control for backward stochastic linear quadratic optimal control problem. And we also give the linear feedback regulator for the optimal control problem by using the solutions of a group of Riccati equations.

Key words: backward stochastic differential equations, optimal control, Riccati equation

2000 MR subject classification: 60H10, 93E

Document code: A

Article ID: 1674-5647(2009)05-0402-09

1 Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let B_t be a d-dimensional Brownian motion in this space. We denote the natural filtration of this Brownian motion by $\mathcal{F}_t$. We consider the following forward-backward stochastic differential equations:

$$\begin{cases} x_t = \Phi(y_0) + \displaystyle\int_0^t b(s,\, x_s,\, y_s,\, z_s)\mathrm{d}s + \int_0^t \sigma(s,\, x_s,\, y_s,\, z_s)\mathrm{d}B_s, \\ y_t = \xi + \displaystyle\int_t^T f(s,\, B_s^\tau x_s,\, y_s,\, z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}B_s, \qquad t \in [0, T], \end{cases} \tag{1.1}$$

where $(x,\, y,\, z)$ takes values in $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^{n\times d}$, b, σ and f are mappings with appropriate dimensions and are, for each fixed $(x,\ y,\ z)$, $\mathcal{F}_t$-progressively measurable, and $B_t(\omega)$ is an $n \times k$ bounded progressively measurable matrix-valued process. We assume that they are Lipschitz with respect to $(x,\, y,\, z)$. $T > 0$ is an arbitrarily prescribed number and the time interval is called the time duration. This is one kind of fully coupled forward-backward stochastic differential equations (FBSDEs, in short).

*__Received date:__ Nov. 14, 2007.
Foundation item: The NSF (10671112) of China, National Basic Research Program (973 Program) (2007CB814904) of China, the NSF (Z2006A01) of Shandong Province, and the Chinese New Century Young Teachers Program.

FBSDEs can be encountered in stochastic optimization problem, especially when one applies the stochastic maximum principle. It also has useful applications in mathematical finance, especially in derivative security pricing models that involve large investors. The study of fully coupled FBSDEs started in early 1990s. In his Ph.D thesis, Antonelli[1] obtained the first result on the solvability of an FBSDE using the fixed point theorem. Today, several methods have been established for solving a (coupled) FBSDE. Among them two are considered effective: the Four Step Scheme by Ma *et al.*[2] and the Method of Continuation by Hu *et al.*[3]. The former provides explicit relations between the forward and backward components of the adapted solution via a quasilinear partial differential equation, but requires the non-degeneracy of the forward diffusion and the non-randomness of the coefficients; while the latter relaxs these conditions, but requires essentially some monotonicity assumptions, and that both forward and backward components are of the same dimension. The result was later extended by Peng and Wu[4] to the case when different dimensions are allowed, and the monotonicity assumptions are weakened so that the results can be widely used especially in Hamiltionian systems arising in stochastic optimal problem (stochastic maximum principle, in particular). A more general version of the continuation method was given in Yong[5]. Along the lines of monotonicity condition, but combined with the dual methods, Wu[6] obtained a comparison theorem for FBSDEs.

Stochastic linear quadratic optimal control problems have been first studied by Wonham[7] and then it has been discussed in [8]–[10]. And the backward linear quadratic stochastic control problems which have important applications in mathematical finance have been studied by Lim and Zhou[11] using square complete technique with determined coefficients. In section 3 we use the solution of FBSDE to give an explicit form of optimal control when the system coefficients are random. For deterministic case we get the optimal feedback control which is same to that in [11], however the method is quite simple.

The results of this paper are clear and easy to understand. We wish it can be applied in practice, especially in mathematical finance.

2 Existence and Uniqueness of FBSDE

In this section let us give one existence and uniqueness result of FBSDE which is useful to study stochastic control problems.

We consider the following special kind of FBSDE:

$$\left\{\begin{array}{l} \mathrm{d}x_t = b(t,\, x_t,\, y_t,\, z_t)\mathrm{d}t + \sigma(t,\, x_t,\, y_t,\, z_t)\mathrm{d}B_t, \\ -\mathrm{d}y_t = f(t,\, B_t^\tau x_t,\, y_t,\, z_t)\mathrm{d}t - z_t\,\mathrm{d}B_t, \\ x_0 = \Phi(y_0), \quad y_T = \xi, \qquad t\in[0,T]. \end{array}\right. \tag{2.1}$$

For notation simplification, we assume d=1; here $(x,y,z)\in\mathbf{R}^{n+n+n\times 1}$, b, σ and f are mappings with appropriate dimensions and are $\mathcal{F}_t$-progressively measurable, for each fixed (x,y,z). We denote

$$u=\left(\begin{array}{c} x\\ y\\ z\end{array}\right), \qquad A(t,u)=\left(\begin{array}{c} -f\\ b\\ \sigma\end{array}\right)(t,\,u),$$