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$H(2)$-unknotting Number of a Knot*

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Abstract: An $H(2)$-move is a local move of a knot which is performed by adding a half-twisted band. It is known an $H(2)$-move is an unknotting operation. We define the $H(2)$-unknotting number of a knot K to be the minimum number of $H(2)$-moves needed to transform K into a trivial knot. We give several methods to estimate the $H(2)$-unknotting number of a knot. Then we give tables of $H(2)$-unknotting numbers of knots with up to 9 crossings.

Key words: knot, $H(2)$-move, $H(2)$-unknotting number, signature, Arf invariant, Jones polynomial, Q polynomial

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1 Introduction

An $H(2)$-move is a change in a knot projection as shown in Fig. 1.1(a). Note that both diagrams are taken to represent single component knots, and so the strings are connected as shown in dotted arcs. Since we obtain the diagram by adding a twisted band to each of these knots as shown in Fig. 1.1(b), it can be said that each of the knots is obtained from the other by adding a twisted band. It is easy to see that an $H(2)$-move is an unknotting operation (see Theorem 1 of [1]). We call the minimum number of $H(2)$-moves needed to transform a knot K into another knot K' the $H(2)$-Gordian distance from K to K', denoted by $\mathrm{d}_2(K, K')$. In particular, the $H(2)$-unknotting number of K is the $H(2)$-Gordian distance from K to a trivial knot, denoted by $\mathrm{u}_2(K)$.

In this paper, we give several criteria on the $H(2)$-unknotting number and then we give tables of the $H(2)$-unknotting numbers of knots with up to 9 crossings.

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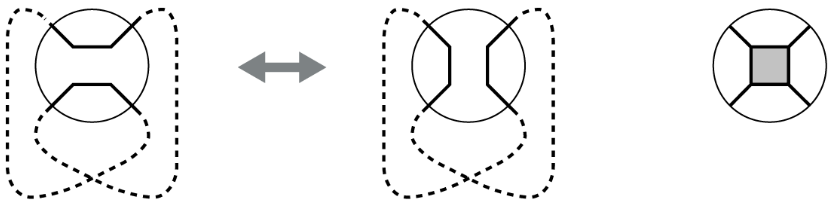

(a) (b)

Fig. 1.1 $H(2)$-move

Lickorish[2] was the first to consider an $H(2)$-unknotting number one knot and he has given a criterion for the $H(2)$-unknotting number one knot using the linking form on the first homology group of the double cover of S^3 branched along the knot. As an application he showed that the $H(2)$-unknotting number of the figure-eight knot is two, which had been conjectured by Riley. Similarly, he showed that among the knots in the prime knot table with at most seven crossings 4_1, 6_3, 7_2, 7_5 and 7_7 are the only knots with $H(2)$-unknotting number greater than one.

Then, Hoste *et al.*[1] have given an inequality estimating a lower bound of the $H(2)$-unknotting number, which uses the minimum number of generators of the first homology group of the cyclic branched covering space of the knot (Theorem 2.1); they studied an $H(2)$-move in a more general context. Besides, Yasuhara[3] has given a criterion on an $H(2)$-unknotting number one knot using the signature and the Arf invariant as an application of the theorem on a surface in a 4-dimensional manifold (Theorem 4.1). He mentions that this criterion proves $\mathrm{u}_2(4_1)$, $\mathrm{u}_2(3_1\#3_1) > 1$. We prove Theorem 4.1 from 3-dimensional point of view using polynomial invariants. This leads to a further criterion for the $H(2)$-unknotting number one knot which does not cover Theorem 4.1 (Theorem 5.1); however, it requires that the determinant $\equiv 0 \pmod 3$. The proof uses some relations among the Jones polynomial, the signature, and the Conway polynomial in [4], which is based on the Casson invariant of the double branched covering space of a knot. Furthermore, using the Jones-Rong value [5, 6] of the Brandt-Lickorish-Millett-Ho Q polynomial [7, 8] we introduce another method to calculate the $H(2)$-unknotting number (Theorem 8.1), which is motivated by Stoimenow[9], where he calculated the unknotting number.

On the other hand, Nakajima[10] has listed the $H(2)$-unknotting numbers of prime knots with up to 10 crossings. He uses the above-mentioned criteria due to Hoste *et al.* and Yasuhara to give a lower bound and a relation with the usual unknotting number (Theorem 3.1) to give an upper bound. In this paper, we list the $H(2)$-unknotting numbers of knots with up to 9 crossings including composite knots (Tables 9.1–9.3), which improves Nakajima's table.

This paper is organized as follows: In Section 2, we review a criterion for the $H(2)$-unknotting number using the first homology group of the cyclic branched covering space, and give the definitions and some properties of the polynomial invariants and the signature. In Section 3, we give a relation between the $H(2)$-unknotting number and usual unknotting