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Star-shaped Differentiable Functions and Star-shaped Differentials*

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Abstract: Based on the isomorphism between the space of star-shaped sets and the space of continuous positively homogeneous real-valued functions, the star-shaped differential of a directionally differentiable function is defined. Formulas for star-shaped differential of a pointwise maximum and a pointwise minimum of a finite number of directionally differentiable functions, and a composite of two directionally differentiable functions are derived. Furthermore, the mean-value theorem for a directionally differentiable function is demonstrated.
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1 Introduction

In 1950s and 1960s Fenchel and Rockafellar[1] investigated differential theory and optimization theory for nonsmooth convex functions. A convex function is directionally differentiable and its subdifferential is a convex set. In 1970s Clarke[2] made contributions to the differential and optimization theory for Lipschitz continuous functions. The generalized subdifferential for a Lipschitz function is a convex set, but the set of generalized directional derivatives for Lipschitz continuous functions is not a linear space. In 1969, Pshenichyi[3] suggested the concept of quasidifferentiability where the directional derivative is a sublinear function. But the class of quasidifferentiable functions in the sense of Pshenichnyi is insufficient to describe many important situations; for instance, it cannot include the so-called D. C. functions. In 1970s, Demyanov, Rubinov and Polyakova extended the concept of quasidifferentiable functions in the sense of Pshenichnyi by introducing a new definition, which

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ensures many good properties about arithmetic operations of quasidifferentiable functions in the new sense (see [4] and [5]). A quasidifferentiable function in the sense of Demyanov and Rubinov is directionally differentiable and its directional derivative is representable as a difference of two convex functions with the differential as a pair of convex compact sets. The class of quasidifferentiable functions, in the sense of Demyanov and Rubinov, have a very wide practical applied background, such as problems for storages and problems for optimum layout of circuits.

Recently, noticing that many functions appearing in bilevel programs are a new class of important directionally differentiable functions, Zhang *et al.*[6] proposed the concept of generalized quasidifferentiable functions, and Zhang *et al.*[7] explored the optimality conditions for nondifferentiable optimization problems of generalized quasidifferentiable functions. But for a general directionally differentiable function without any structures, there is no suitable defintion for its differential. Observing that if the directionally derivative is a positively homogeneous continuous function in direction, then it can be represented as a difference of two nonnegative positively homogeneous continuous functions, we can express the directional derivative as the difference of two gauge functions of star-shaped sets. A directionally differentiable function whose directional derivatives is continuous in direction is defined as a star-shaped differentiable function. A star-shaped differential of a star-shaped differentiable function is a pair of star-shaped sets, but not a pair of convex compact sets. We can verify that any quasidifferentiable function in the sense of Demyanov and Rubinov is star-shaped differentiable.

In this paper, we first present some preliminaries about the space of star-shaped sets. Secondly, we give the concept of star-shaped differential and verify arithmetic operations of star-shaped differentials, and derive formulas for star-shaped differentials of a pointwise maximum and of a pointwise minimum of a finite number of directionally differentiable functions, and a composite of two directionally differentiable functions. Finally, we demonstrate the mean-value theorem for a star-shaped differentiable function.

2 Preliminaries

In this section we recall some results about the space of star-shaped sets and the space of positively homogeneous continuous functions (see [8]).

Definition 2.1[9] *A closed subset A of $\mathbf{R}^n$ is called a star-shaped set if it contains the origin as an interior point and every ray*

$$L_x = \{\lambda x \mid \lambda \geq 0\}, \qquad \forall\ x \in \mathbf{R}^n \backslash \{0\}$$

does not intersect the boundary of A more than once.

Define

$$K = \{A \subseteq \mathbf{R}^n \mid A \text{ is a star-shaped set}\}.$$