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The Class Number of Derived Subgroups and the Structure of Camina Groups*

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Abstract: A finite group G is called a Camina group if G has a proper normal subgroup N such that gN is precisely a conjugacy class of G for any $g \in G - N$. In this paper, the structure of a Camina group G is determined when N is a union of $2, 3$ or 4 conjugacy classes of G.

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1 Introduction

All groups considered in this paper are finite. Let G be a group and N a proper normal subgroup of G. If for any $g \in G - N$, we always have $gN \subseteq g^G$, then we call N a Camina kernel of G, where g^G denotes the conjugacy class of g in G. A group with a non-trivial Camina kernel is said to satisfy F2-condition. Groups with F2-condition, first introduced by Camina[1], have been studied by Macdonald, Chillag, Mann and Scoppola (see [2]–[7]). Of particular importance among this class of groups are the so-called Camina groups.

Definition 1.1 *A group G is called a Camina group if there exists a proper normal subgroup N of G such that $gN = g^G$ for all $g \in G - N$.*

Clearly, a Camina group has a Camina kernel such that each non-trivial coset of the Camina kernel is precisely a conjugacy class. Now let G be a group. If G is a Camina group with respect to the Camina kernel N, then G/N is abelian and $G' \leq N$. Also since $gN = g^G \subseteq gG'$ for any $g \in G - N$, we have $N \leq G'$. Thus $N = G'$, and G' is a Camina kernel. Conversely, noticing that $g^G \subseteq gG'$ always holds for any $g \in G$, we see that if G' is a Camina kernel of G, then G is a Camina group. Hence G is a Camina group if and only if

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G' is a Camina kernel. So Definition 1.1 is equivalent to the definition of Camina groups in [8]. For the sequel use, we gather the facts about Camina groups together in the following lemma, which already have been indicated in our above argument.

Lemma 1.1 *Let G be a group. Then the following conditions are equivalent*:

(a) *G is a Camina group*;

(b) *G' is a Camina kernel*;

(c) *$G' < G$, and $gG' = g^G$ for any $g \in G - G'$*;

(d) *$G' < G$, and $|G'| = |g^G|$ for any $g \in G - G'$.*

Let G be a Camina group. How the class number of G' influences the structure of G is an interesting question. If G' is a union of k conjugacy classes of G, then G is called a Canima(k)-group, or $C(k)$-group for short. Obviously, G is a $C(1)$-group if and only if G is a non-trivial abelian group. Also G satisfies F2-condition if G is a $C(k)$-group with $k > 1$. In this paper, we determine the structure of $C(k)$-groups with k a small number. Firstly in Section 2, we give some lemmas which are useful in the sequel. Then in the next two sections, $C(k)$-groups with $k = 2, 3$ and 4 are completely classified. Moreover, an example is given at the end in order to show that one kind of $C(4)$-groups characterized by Theorem 4.2 exist.

In what follows, $GF(q)$ denotes a finite field of q elements. For a group G, a G-class means a conjugacy class of G, and G^* and $\pi(G)$ denote the set $G - \{1\}$ and the set of primes dividing $|G|$ respectively. If $H \leq G$, $g \in G$, then g^H denotes the set $\{g^h \mid h \in H\}$; especially g^G is the G-class containing g. Other notations and terminologies not mentioned here agree with standard usage.

2 Lemmas

Lemma 2.1 *Let G be a non-abelian Camina group. Then*

(a) *$Z(G) \leq G'$, and $|Z(G)| \leq k$ when G is a $C(k)$-group*;

(b) *G is non-nilpotent if $|\pi(G)| > 1$.*

Proof. (a) If there exists $g \in Z(G) - G'$, then

$$|g^G| = 1 < |G'|,$$

a contradiction. Hence

$$Z(G) \leq G',$$

and it must be $|Z(G)| \leq k$ when G is a $C(k)$-group.

(b) Assume that G is nilpotent and let

$$G = P_1 \times P_2 \times \cdots \times P_k,$$

where $P_i \in \mathrm{Syl}_{p_i}(G)$, $k > 1$. Since G is non-abelian, we may suppose that $P_1' \neq 1$. Choose $x \in P_2 - P_2'$. Then $x \notin G'$, and

$$|P_1'||P_2'| \cdots |P_k'| = |G'| = |x^G| = |x^{P_2}| = |P_2 : C_{P_2}(x)|.$$

It follows that $|P_1'|$ divides $|P_2|$, a contradiction.