COMMUNICATIONS IN MATHEMATICAL RESEARCH
26(2)(2010), 176–182

On the Asymmetry for Convex Domains of Constant Width*

JIN HAI-LIN AND GUO QI
(*Department of Mathematics, Suzhou University of Technology and Science, Suzhou, Jiangsu, 215009*)

Communicated by Yin Jing-xue

Abstract: The extremal convex bodies of constant width for the Minkowski measure of asymmetry are discussed. A result, similar to that of H. Groemer's and of H. Lu's, is obtained, which states that, for the Minkowski measure of asymmetry, the most asymmetric convex domains of constant width in $\mathbf{R}^2$ are Reuleaux triangles.

Key words: asymmetry measure, reuleaux polygon, constant width
2000 MR subject classification: 52A38
Document code: A
Article ID: 1674-5647(2010)02-0176-07

1 Introduction

Measures of (central) symmetry, or as we prefer, asymmetry for convex bodies have been extensively investigated. Among these researches, it is a popular topic to determine the extremal bodies in a class of convex bodies for a given asymmetry measure (see [1]–[2] etc.). Recently, Groemer and Wallen[3] and Lu and Pan[4] determined the extremal bodies in the class of convex domains in $\mathbf{R}^2$ of constant width, respectively for an asymmetry measure they are concerned with. Concretely, they both showed that the most asymmetric domains are Reuleaux triangles under the concerned asymmetry measure.

In this paper, we prove a similar result for the well-known Minkowski measure of asymmetry. Precisely, the following theorem is obtained:

Theorem 1.1 *Let K be a convex domain of constant width. Then*

$$1 \leq As(K) \leq \frac{\sqrt{3}+1}{2},$$

where $As(\cdot)$ denotes the Minkowski measure of convex bodies. Equality holds on the left-hand side precisely if and only if K is a circular disc, and on the right-hand side precisely if and only if K is a Reuleaux triangle.

***Received date:** Sept. 8, 2009.
Foundation item: The NSF (08KJD110016) of Jiangsu Hight Education.

2 Prelimilary

$\mathbf{R}^n$ denotes the usual n-dimensional Euclidean space with the canonical inner product $\langle\cdot,\ \cdot\rangle$. A bounded closed convex set $C \subset \mathbf{R}^n$ is called a convex body (a convex domain for $n = 2$) if it has non-empty interior (int for brief). The family of all convex bodies in $\mathbf{R}^n$ is denoted by $\mathcal{K}^n$. Given a convex body $C \in \mathcal{K}^n$ and a point $x \in \text{int}(C)$, for a hyperplane H through x and the pair H_1, H_2 of support hyperplanes of C parallel to H, let $r(H, x)$ be the ratio, not less than 1, in which H divides the distance between H_1 and H_2. Note

$$r(C, x) = \max\{r(H, x) : H \ni x\},$$

and define the Minkowski measure $As(C)$ of asymmetry of C by (see [3])

$$As(C) = \min_{x\in\text{int}(C)} r(C, x).$$

A point $x \in \text{int}(C)$ satisfying $r(C, x) = As(C)$ is called a critical point (of C). The set of all critical points of C is denoted by C^*.

Thus, if for a given body $C \in \mathbf{R}^n$ and an $x \in \mathbf{R}^n$, we define the support functional of C at x as, for all $u \in S^{n-1}$ (the $(n-1)$-dimensional unit sphere),

$$h_x(C, u) = \sup_{y\in C}\langle y - x, u\rangle,$$

then it is easy to see that

$$r(C, x) = \sup_{u\in S^{n-1}} \frac{h_x(C, -u)}{h_x(C, u)}$$

and

$$As(C) = \sup_{u\in S^{n-1}} \frac{h_{x_0}(C, -u)}{h_{x_0}(C, u)},$$

where x_0 is a critical point of C.

The following is another equivalent definition of the Minkowski measure (see [1] and [5]). Let $C \in \mathcal{K}^n$ and $x \in \text{int}(C)$. For a chord D of C through x, let $r_1(D, x)$ be the ratio, not less than 1, in which x divides the length of D, and note that

$$r_1(C, x) = \max\{r_1(D, x) : D \ni x\}.$$

Then

$$r_1(C, x) = r(C, x),$$

and so the Minkowski measure

$$As(C) = \min_{x\in\text{int}(C)} r_1(C, x).$$

A chord D satisfying $r_1(D, x) = As(C)$ is called a critical chord (of C).

Note that

$$S_C(x) = \{p \in bd(C) :\ \text{the chord } pq \text{ passes through } x \text{ and satisfies } \frac{xp}{xq} = r_1(C, x)\},$$

where bd denotes the boundary, and pq denotes the segment with endpoints p, q or its length freely if no confusing is caused.