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New Explicit and Exact Solutions for the Klein-Gordon-Zakharov Equations*

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Abstract: In this paper, based on the generalized Jacobi elliptic function expansion method, we obtain abundant new explicit and exact solutions of the Klein-Gordon-Zakharov equations, which degenerate to solitary wave solutions and triangle function solutions in the limit cases, showing that this new method is more powerful to seek exact solutions of nonlinear partial differential equations in mathematical physics.

Key words: generalized Jacobi elliptic functions expansion method, doubly periodic solution, exact solution, Klein-Gordon-Zakharov equation

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1 Introduction

In this paper, we consider the following Klein-Gordon-Zakharov equations:

$$\begin{cases} u_{tt} - u_{xx} + u + \alpha n u + \gamma |u|^2 u = 0, \\ v_t + n_x + \beta(|u|^2)_x = 0, \\ n_t + v_x = 0. \end{cases} \tag{1.1}$$

After eliminating the real value function $v = v(x,t)$, (1.1) are reduced to the following system:

$$\begin{cases} u_{tt} - u_{xx} + u + \alpha n u + \gamma |u|^2 u = 0, \\ n_{tt} - n_{xx} - \beta(|u|^2)_{xx} = 0. \end{cases} \tag{1.2}$$

(1.2) is a coupled nonlinear wave model which describes the interaction of the Langmuir wave and the ion acoustic wave in a high frequency plasma and so on (see [1]–[4]). The complex value function $u = u(x,t)$ is the fast time scale component of electric field raised by electrons and the real value function $n = n(x,t)$ is the deviation of ion density from its equilibrium, where α, β, γ are three real parameters.

In recent years, there have been many works on the qualitative research of the global solutions for the Klein-Gordon-Zakharov equations (1.2) (see [5]–[8]). Chen considered orbital

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stability of solitary waves for the Klein-Gordon-Zakharov equations in [9]. More recently, some exact solutions for the Klein-Gordon-Zakharov equations (1.2) and their special case when $\gamma = 0$ or $\gamma = 1$ are obtained by using different methods (see [10]–[12]).

The aim of this paper is to find new and more general explicit and exact Jacobian elliptic function solutions of the Klein-Gordon-Zakharov equations (1.2) by using the generalized Jacobian elliptic function expansion method.

2 Summary of the Generalized Jacobi Elliptic Function Expansion Method

Let a partial differential equation, for instance, with two variables x and t, be given as follows:

$$P(u, u_t, u_x, u_{xx}, \cdots) = 0, \tag{2.1}$$

where P is a differential polynomial formula with the variables x and t usually.

We seek formal solutions of the following form of (2.1):

$$u(\xi) = \sum_{i=1}^{n} a_i F^i(\xi) + \sum_{i=1}^{n} b_i F^{i-1}(\xi)E(\xi) + \sum_{i=1}^{n} c_i F^{i-1}(\xi)G(\xi) + \sum_{i=1}^{n} d_i F^{i-1}(\xi)H(\xi) + a_0, \tag{2.2}$$

where a_0, a_i, b_i, c_i, d_i $(i = 1, 2, \cdots, n)$ are constants to be determined later, $\xi = \xi(x, t)$ are arbitrary functions with the variables x and t, the parameter n can be determined by balancing the highest order derivative terms with the nonlinear terms in (2.1), and $E(\xi)$, $F(\xi)$, $G(\xi)$, $H(\xi)$ are arbitrary arrays of the four functions $e = e(\xi)$, $f = f(\xi)$, $g = g(\xi)$ and $h = h(\xi)$, whose selection obeys the principle which makes the calculation more simple. Here we ansatz

$$\begin{cases} e = \dfrac{1}{p + q\mathrm{sn}[\xi, m] + r\mathrm{cn}[\xi, m] + l\mathrm{dn}[\xi, m]}, \\ f = \dfrac{\mathrm{sn}[\xi, m]}{p + q\mathrm{sn}[\xi, m] + r\mathrm{cn}[\xi, m] + l\mathrm{dn}[\xi, m]}, \\ g = \dfrac{\mathrm{cn}[\xi, m]}{p + q\mathrm{sn}[\xi, m] + r\mathrm{cn}[\xi, m] + l\mathrm{dn}[\xi, m]}, \\ h = \dfrac{\mathrm{dn}[\xi, m]}{p + q\mathrm{sn}[\xi, m] + r\mathrm{cn}[\xi, m] + l\mathrm{dn}[\xi, m]}, \end{cases} \tag{2.3}$$

where p, q, r, l are arbitrary constants. The four functions e, f, g, h satisfy the following restricted relations:

$$\begin{cases} e' = -qgh + rfh + lm^2 fg, \\ f' = pgh + reh + leg, \\ g' = -pfh - qeh + l(m^2 - 1)ef, \\ h' = -m^2 pfg - r(m^2 - 1)ef - qeg, \\ g^2 = e^2 - f^2, \\ h^2 = e^2 - m^2 f^2, \end{cases} \tag{2.4}$$

$$pe + qf + rg + lh = 1, \tag{2.5}$$