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Generalized PP and Zip Subrings of Matrix Rings*

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Abstract: Let R be an abelian ring. We consider a special subring A_n, relative to $\alpha_2, \cdots, \alpha_n \in R\mathrm{End}(R)$, of the matrix ring $M_n(R)$ over a ring R. It is shown that the ring A_n is a generalized right PP-ring (right zip ring) if and only if the ring R is a generalized right PP-ring (right zip ring). Our results yield more examples of generalized right PP-rings and right zip rings.

Key words: generalized right PP-ring, PP-ring, right zip ring
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1 Introduction

All rings considered here are associative with identity.

A ring R is called a generalized right PP-ring if for any $x \in R$ the right ideal x^nR is projective for some positive integer n, depending on x, or equivalently, if for any $x \in R$ the right annihilator of x^n is generated by an idempotent for some positive integer n, depending on x. Left cases may be defined analogously. A ring R is called a generalized PP-ring if it is both generalized right and left PP-ring. Right PP-rings are generalized right PP obviously. But the converse is not true (see [1]).

Let R be a ring. Denote by $R\mathrm{End}(R)$ the set of all ring homomorphisms from R to R. Define a subring of matrix ring $M_n(R)$ over R as

$$S_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & a & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \mid a, a_{ij} \in R \right\}.$$

By Proposition 3 of [1], if R is an abelian ring, then the ring S_n is a generalized right PP-ring if and only if the ring R is a generalized right PP-ring. In Theorem 5 of [2], it was shown

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that for a ring R, S_n is a right (left) zip ring if and only if R is a right (left) zip ring. In this paper we consider the subring

$$A_n = \left\{ \begin{pmatrix} a & a_{12} & \cdots & a_{1n} \\ 0 & \alpha_2(a) & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n(a) \end{pmatrix} \mid a, a_{ij} \in R \right\},$$

where $\alpha_2, \cdots, \alpha_n \in R\mathrm{End}(R)$ are compatible ring homomorphisms from R to R. We show that if R is an abelian ring, then the ring A_n is a generalized right PP-ring if and only if the ring R is a generalized right PP-ring, and for any ring R, A_n is a right (left) zip ring if and only if R is a right (left) zip ring. Our results yield more examples of generalized right PP-rings and zip rings.

2 Main Results

Let R be a ring, and α be a ring endomorphism of R. According to [3] or [4], α is called compatible if for each $a, b \in R$, we have

$$ab = 0 \Longleftrightarrow a\alpha(b) = 0.$$

For more information, examples, and other applications of this concept, we refer the reader to [3] and [4].

According to [5], α is called a rigid endomorphism if $r\alpha(r) = 0$ implies $r = 0$ for $r \in R$. A ring R is said to be α-rigid if there exists a rigid endomorphism α of R. Clearly, any rigid endomorphism is a monomorphism and any α-rigid ring is reduced. Rigid endomorphisms were discussed in [5–7]. In [8], a ring endomorphism α of R is called a weakly rigid endomorphism if α is a monomorphism and if there exist $a, b \in R$ such that $ab = 0$ then $a\alpha(b) = \alpha(a)b = 0$. Clearly the identity map of R is weakly rigid. Every monomorphism of rings without non-zero zero-divisors is weakly rigid.

Let α be a rigid endomorphism of R. It was shown in [6] that if $ab = 0$ then $a\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer n. Thus any rigid endomorphism is weakly rigid. But the converse is not true (see, for example, [8]). By Proposition 3 of [8], α is rigid if and only if α is weakly rigid and R is reduced. Further examples of weakly rigid endomorphisms of rings can be found in [8] and [9].

Clearly, all weakly rigid endomorphisms and so, all rigid endomorphisms are compatible. Thus there exist many examples of compatible endomorphisms as shown in [5]–[9].

Lemma 2.1 *Let α be compatible. Then*

(1) *α is a monomorphism, and*

(2) *for any $e^2 = e \in R$, $\alpha(e) = e$.*

Proof. Clearly, α is a monomorphism. Suppose that $e^2 = e \in R$. Then

$$(1 - e)e = 0.$$