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Some Results for Formal Local Cohomology Modules*

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Abstract: Let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, I an ideal of R and M a finitely generated R-module. Let $\varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$ be the ith formal local cohomology module of M with respect to I. In this paper, we discuss some properties of formal local cohomology modules $\varprojlim_n H^i_{\mathfrak{m}}(M/I^nM)$, which are analogous to the finiteness and Artinianness of local cohomology modules of a finitely generated module.

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1 Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a commutative Noetherian local ring, I an ideal of R and M a finitely generated R-module. For notations and terminologies not given in this paper, the reader is referred to [1] or [2] if necessary.

The role of formal cohomology in formal geometry is some how analogous to the role of cohomology of schemes. For more details on the notion of formal cohomology, we refer the reader to the interesting survey article by Illusie[3]. Let $\mathcal{U} = \mathrm{Spec}(R)\backslash\{\mathfrak{m}\}$ and $(\hat{\mathcal{U}}, \mathcal{O}_{\hat{\mathcal{U}}})$ denote the formal completion of $\mathcal{U}$ along $V(I)\backslash\{\mathfrak{m}\}$. Let $\hat{\mathcal{F}}$ denote the $\mathcal{O}_{\hat{\mathcal{U}}}$ sheaf associated to $\varprojlim_n M/I^nM$. Peskine and Szpiro have described the formal cohomology modules $H^i(\hat{\mathcal{U}}, \hat{\mathcal{F}})$ via the isomorphisms

$$H^i(\hat{\mathcal{U}}, \hat{\mathcal{F}}) \cong \varprojlim_n H^i_{\mathfrak{m}}(M/I^nM), \qquad i > 0$$

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(see [4], III, Proposition 2.2]). Recently, Schenzel[5] investigated the structure of the modules $\varprojlim_n H^i_{\mathfrak{m}}(M/I^n M)$ extensively, and gave the upper and lower vanishing and non-vanishing in terms of intrinsic data of M to these modules. For each $i \geq 0$, he called

$$\mathfrak{F}^i_I(M) := \varprojlim_n H^i_{\mathfrak{m}}(M/I^n M)$$

the ith formal local cohomology module of M with respect to I. In particular, he proved that

$$\dim M/IM = \mathrm{Sup}\{i \mid \varprojlim_n H^i_{\mathfrak{m}}(M/I^n M) \neq 0\}.$$

An R-module L is called a minimax module if there is a finitely generated submodule N such that L/N is Artinian (see [6]). The class of minimax modules thus includes all finitely generated and all Artinian modules. Zöschinger gave many equivalent conditions for a module to be a minimax module in [6] and [7]. It was shown by Zink[8] and Enochs[9] that a module over a complete local ring is minimax if and only if it is Matlis reflexive.

An R-module L is called an I-coartinian module if $\mathrm{Tor}^R_i(R/I, L)$ is Artinian for any i. Artinian modules are I-coartinian. Since the local cohomology modules of a finitely generated module enjoy many nice finiteness properties, it is natural to expect that analogues of some of these properties hold for formal local cohomology modules. In [10], Asgharzadeh and Divaani-Aazar discussed finiteness properties of formal local cohomology modules. In particular, in each of the cases i) $\dim R \leq 2$, ii) I is principal, and iii) $\dim R/I \leq 1$, they proved that $\mathfrak{F}^i_I(M)$ is I-coartinian for each i.

In this paper, we continue to study the structure and finiteness properties of formal local cohomology modules on the base of the work of Schenzel[5], Asgharzadeh and Divaani-Aazar[10].

Set

$$\mathfrak{F}^i_I(M) := \varprojlim_n H^i_{\mathfrak{m}}(M/I^n M),$$

$$q_I(M) = \sup\{i \mid \mathfrak{F}^i_I(M) \text{ is not minimax}\}$$

and

$$r_I(M) = \inf\{i \mid \mathfrak{F}^i_I(M) \text{ is not minimax}\}.$$

We prove that i) $\mathfrak{F}^{q_I(M)}_I(M)/I\mathfrak{F}^{q_I(M)}_I(M)$ is Artinian and $\mathfrak{F}^i_I(M)$ is I-coartinian for each $i > q_I(M)$; ii) $\mathrm{Hom}_R(R/\mathfrak{m}, \mathfrak{F}^{r_I(M)}_I(M))$ is finitely generated. We also consider the structure of the 0th formal cohomology modules.

2 The Results

Firstly we give some characterizations of I-coartinian modules.

Theorem 2.1 *Let I be an ideal of a ring R and L an R-module* (*not necessarily finitely generated*)*. Then the following conditions are equivalent:*

(1) *L is I-coartinian;*