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Locally Transitive Graphs Admitting a Group with Cyclic Sylow Subgroups*

CHEN SHANG-DI AND FENG QING-LIN
(*College of Science, Civil Aviation University of China, Tianjin, 300300*)

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Abstract: All graphs are finite simple undirected and of no isolated vertices in this paper. Using the theory of coset graphs and permutation groups, it is completed that a classification of locally transitive graphs admitting a non-Abelian group with cyclic Sylow subgroups. They are either the union of the family of arc-transitive graphs, or the union of the family of bipartite edge-transitive graphs.
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1 Introduction

All graphs are finite, simple, undirected and of no isolated vertices in this paper. The reader is referred to [1]–[3], respectively, for notations and terminologies on permutation groups and graphs.

For a graph Γ, we denote the vertex set, edge set and arc set of Γ by $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, respectively, and for each $u \in V(\Gamma)$, denote the set of vertices adjacent to u by $\Gamma(u)$. If there is an edge in Γ connecting the vertices u and v, this edge is denoted by $\{u, v\}$. $d(\Gamma)$ is used to denote the degree of a regular graph Γ. A cycle of length k and a complete bipartite graph of type $(m,\ n)$ are denoted by C_k and $K_{m,n}$, respectively. Z_n is used to denote the residue class of rings of module n, and Z_n^* is used to denote the multiplicative group which is made up of the co-prime residue class of n in Z_n.

If a group G is transitive on $V(\Gamma)$, $E(\Gamma)$ and $A(\Gamma)$, respectivly, then Γ is said to be G-vertex-transitive, G-edge-transitive and G-arc-transitive, respectively. For $u \in V(\Gamma)$, if G_u acts transitively on $\Gamma(u)$, then Γ is said to be G-locally-transitive. A group G is said to act on a graph Γ, if an action of G on $V(\Gamma)$ induces an action on $E(\Gamma)$. In particular, if G acts faithfully on $V(\Gamma)$, then the group G is said to be an automorphism group of Γ. A

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non-abelian group G is called an SC-group if all sylow subgroups of G are cyclic.

A significant method for studying graphs and groups is using some transitive properties of automorphism groups of graphs. For instance, there are a lot of rich results of edge-transitive and locally quasiprimitive graphs, respectively, in [4]–[9]. We continue to study in this respect, and complete the classification of locally transitive graphs admitting a non-abelian group with cyclic sylow subgroups. As a byproduct, we obtain the classification of edge-transitive-graphs admitting a group with cyclic sylow subgroups. The main results of this paper is the following:

Theorem 1.1 *Let $G = \langle a, b \mid a^m = b^n = 1,\ a^b = a^r\rangle$ be an automorphism group of a graph Γ with $((r-1)n,\ m) = 1$, $r^n \equiv 1 \pmod m$. Let n_0 be the order of $r \pmod m$. Then Γ is G-locally-transitive if and if only Γ is one of the following graphs*:

(1) *$\Gamma \in \mathrm{I}(m, n)$, and n is even*;

(2) *$\Gamma \in \mathrm{IV}(m, n)$*;

(3) *$X_1 = \bigcup\limits_{i=1}^{p} \Gamma_i(s_i, t_i)$, where $s_i > 1$,*

$$[s_1, s_2, \cdots, s_p] = m, \qquad [t_1, t_2, \cdots, t_p] = n, \qquad [\alpha_1, \alpha_2, \cdots, \alpha_p] = n_0,$$

α_i is the order of $r \pmod{s_i}$, and $\alpha_i \mid t_i$; when t_i is even, $\Gamma_i(s_i, t_i) \in \mathrm{I}(s_i, t_i)$ or $\mathrm{IV}(s_i, t_i)$; and when t_i is odd, $\Gamma_i(s_i, t_i) \in \mathrm{IV}(s_i, t_i)$;

(4) *$X_1 \bigcup X_2$, $X_1 = \bigcup\limits_{i=1}^{q} \Gamma_i(s_i, t_i)$, $X_2 = \bigcup\limits_{j=1}^{p-q} \Gamma_{q+j}(1, t_{q+j})$, where $1 \le q < p$; in the case of $1 \le i \le q$, $s_i > 1$, $[s_1, s_2, \cdots, s_q] = m$, $[t_1, t_2, \cdots, t_p] = n$, and $[\alpha_1, \alpha_2, \cdots, \alpha_q] = n_0$, where α_i is the order of $r \pmod{s_i}$, and $\alpha_i \mid t_i$; in the case of t_i being even, $\Gamma_i(s_i, t_i) \in \mathrm{I}(s_i, t_i)$ or $\mathrm{IV}(s_i, t_i)$; in the case of t_i being odd, $\Gamma_i(s_i, t_i) \in \mathrm{IV}(s_i, t_i)$; in the case of t_{q+j} being even, $\Gamma_{q+j}(1, t_{q+j}) \in \mathrm{III}(t_{q+j})$; in the case of t_{q+j} being odd, $\Gamma_{q+j}(1, t_{q+j}) \in \mathrm{II}(t_{q+j})$.*

Here $\mathrm{I}(m, n)$, $\mathrm{II}(t)$, $\mathrm{III}(t)$ and $\mathrm{IV}(m, n)$ are the graph families stated in the following Lemma 5.1, Lemma 5.3, Corollary 5.1, and Theorem 4.2, respectively.

For the convenience of narrative, below in this paper, the group $\langle a, b \mid a^m = b^n = 1,\ a^b = a^r\rangle$ is denoted by $\mathrm{SC}(a, b;\ m, n, r)$, where $((r-1)n, m) = 1$, $r^n \equiv 1 \pmod m$.

2 Main Lemmas

In this section, we give some lemmas that we need.

Lemma 2.1 ([10], Theorem 1.6A) *Let a finite group G act transitively on a set Ω and H be a normal subgroup of G. Then*

(1) *The orbits of H form a fixed block system for G*;

(2) *If Δ and Δ' are two H-orbits then H^{Δ} and $H^{\Delta'}$ are permutation isomorphic*; *and*

(3) *The number of the orbits of H divides $|G : H|$.*

Lemma 2.2 ([11], Lemma 3.2.1) *Let Γ be a G-edge-transitive graph. If Γ is not G-vertex-transitive, then G has exactly two orbits, and these two orbits are a bipartition of Γ.*