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Global and Blow-up Solutions to a p-Laplace Equation with Nonlocal Source and Nonlocal Boundary Condition*

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Abstract: This paper deals with an evolution p-Laplace equation with nonlocal source subject to weighted nonlocal Dirichlet boundary conditions. We give sufficient conditions for the existence of global and non-global solutions.

Key words: nonlocal boundary condition, evolution p-Laplace, nonlocal source, blow-up

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1 Introduction

In this paper, we consider the following problem:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p-2}\nabla u) = au^m \displaystyle\int_\Omega u^n(y,t)\mathrm{d}y, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \displaystyle\int_\Omega \varphi(x,y)u(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^N$ with smooth boundary $\partial\Omega$, $a > 0$, $p > 2$, $n > 0$, $m \geqslant 0$, while the weight function $\varphi(x,y)$ in the boundary conditions are continuous, nonnegative on $\partial\Omega \times \overline{\Omega}$ and not identically zero, and $\displaystyle\int_\Omega \varphi(x,y)\mathrm{d}y > 0$ on $\partial\Omega$. The initial data $u_0 \in C^{2+\alpha}(\overline{\Omega})$ with $0 < \alpha < 1$, $u_0 \geqslant 0$, and satisfies the compatibility condition

$$u_0 = \int_\Omega \varphi(x,y)u_0(y)\mathrm{d}y > 0, \qquad x \in \partial\Omega.$$

In the past several decades, many physical phenomena have been formulated as nonlocal mathematical models (see [1] and [2]). It has also been suggested that nonlocal growth terms present in a more realistic model in physics for compressible reactive gases. Problem (1.1) arises in the study of the flow of a fluid through a porous medium with an integral

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source (see [3] and [4]) and in the study of population dynamics (see [5] and [6]). There have been many articles which deal with properties of solutions to local semilinear parabolic equations with homogeneous Dirichlet boundary condition (see [1], [7] and the references therein). However, there are some important phenomena formulated as parabolic equations which are coupled with nonlocal boundary conditions in mathematical modelling such as thermoelasticity theory (see [8] and [9]). In this case, the solution $u(x,t)$ describes entropy per volume of the material. The problem of nonlocal boundary conditions for linear parabolic equations of the type

$$\begin{cases} u_t - Au = 0, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \displaystyle\int_\Omega \varphi(x,y)u(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.2}$$

with uniformly elliptic operator

$$A = \sum_{i,j=1}^{n} a_{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x)$$

was studied by Friedman[10]. It was proved that the unique solution of (1.2) tends to 0 monotonically and exponentially as $t \to +\infty$ provided

$$\int_\Omega |\varphi(x,y)|\mathrm{d}y \leqslant \rho < 1, \qquad x \in \partial\Omega.$$

As for more general discussions on the dynamics of parabolic problem with nonlocal boundary conditions, one can see, e.g., [11], where the problem

$$\begin{cases} u_t - Au = g(x,u), & (x,t) \in \Omega \times (0,T), \\ Bu(x,t) = \displaystyle\int_\Omega \varphi(x,y)u(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases}$$

with

$$Bu = \alpha_0 \frac{\partial u}{\partial \nu} + u$$

was considered and recently Pao[12] gave the numerical solutions for diffusion equations with nonlocal boundary conditions.

The scalar problems with both nonlocal sources and nonlocal boundary conditions have been studied as well. For example, a problem of the form

$$\begin{cases} u_t - \Delta u = \displaystyle\int_\Omega g(u)\mathrm{d}y, & (x,t) \in \Omega \times (0,T), \\ u(x,t) = \displaystyle\int_\Omega \varphi(x,y)u(y,t)\mathrm{d}y, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega \end{cases}$$

was studied by Lin and Liu[13]. They established local existence, global existence and nonexistence of solutions, and discussed the blow-up properties of solutions.