Communications
in
Mathematical
Research
26(4)(2010), 313–320

On π-regularity of General Rings*

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Communicated by Du Xian-kun

Abstract: A general ring means an associative ring with or without identity. An idempotent e in a general ring I is called left (right) semicentral if for every $x \in I$, $xe = exe$ ($ex = exe$). And I is called semiabelian if every idempotent in I is left or right semicentral. It is proved that a semiabelian general ring I is π-regular if and only if the set $N(I)$ of nilpotent elements in I is an ideal of I and $I/N(I)$ is regular. It follows that if I is a semiabelian general ring and K is an ideal of I, then I is π-regular if and only if both K and I/K are π-regular. Based on this we prove that every semiabelian GVNL-ring is an SGVNL-ring. These generalize several known results on the relevant subject. Furthermore we give a characterization of a semiabelian GVNL-ring.

Key words: semiabelian ring, π-regular ring, GVNL-ring, exchange ring
2000 MR subject classification: 16D25, 16E50
Document code: A
Article ID: 1674-5647(2010)04-0313-08

1 Introduction

The term ring means an associative ring with identity and a general ring means an associative ring with or without identity. Let I be a general ring. Call I abelian if all idempotents in I are central, and I reduced if it has no nonzero nilpotent elements. It is well known that if I is reduced then it is abelian. An element a in I is π-regular if there exist a positive integer n and $b \in I$ such that $a^n = a^n b a^n$. And I is π-regular if every element in I is π-regular. An element a in I is strongly π-regular if there exist a positive integer n and $b \in I$ such that $a^n = a^{n+1}b$ with $ab = ba$. And I is strongly π-regular if every element in I is strongly π-regular. Clearly a strongly π-regular element (general ring) is π-regular. An element a in I is strongly regular if there exist $x, y \in I$ such that $a = xa^2 = a^2y$. For a ring R, this is equivalent to saying that there exist an idempotent $e \in R$ and a unit $u \in R$ such that $a = eu = ue$ (cf. [1]). A general ring I is an exchange ring if for each $x \in I$,

***Received date:** Nov. 25, 2008.
Foundation item: The NSF (Y2008A04) of Shandong Province of China.

there exist $r, s \in I$ and $e^2 = e \in I$ such that $e = rx = s + x - sx$. It is known by [2] that a π-regular general ring is an exchange general ring. A general ring I is semipotent if each left (equivalently right) ideal not contained in Jacobson radical contains a nonzero idempotent, and every exchange general ring is semipotent (see [2]). A ring R is a GVNL-ring if for each $a \in R$, either a or $1 - a$ is π-regular. And R is an SGVNL-ring if for every nonempty subset $S \subseteq R$ whenever $(S)_r = R$ there exists an element in S which is π-regular in R, where $(S)_r$ is the right ideal generated by S (see [3]).

According to [4], an element in a general ring I is called left (right) semicentral if for each $x \in I$, $xe = exe$ ($ex = exe$). And I is called semiabelian if every idempotent is left or right semicentral (see [5]). We prove that a semiabelian general ring I is π-regular if and only if the set $N(I)$ of nilpotent elements in I is an ideal of I and $I/N(I)$ is regular, which generalizes the main result in [5]. It follows that if I is a semiabelian general ring and K is an ideal of I then I is π-regular if and only if both K and I/K are π-regular, generalizing the main result in [6]. Moreover we prove that every semiabelian GVNL-ring is an SGVNL-ring, extending one of the main results in [7]. At last we give a characterialization of a semiabelian GVNL-ring.

Throughout this note, we use the symbol $S(I)$ to denote the set of idempotents in a general ring I, and $S_l(I)$ ($S_r(I)$) to denote the set of left (right) semicentral idempotents in I. The set of nilpotent elements in I is denoted by $N(I)$. As usual, we use $J(I)$ to denote the Jacobson radical of a general ring I. Let I be a general ring, we write $\mathbb{E}(\mathbb{Z}, I)$ for the standard unitization of the general ring I ([cf. [2]). For a ring R, we use the symbol $U(R)$ to denote its unit group.

2 Semiabelian π-Regular Rings

We start this section with the following observation.

Proposition 2.1 *Let I be a semiabelian general ring and $R = \mathbb{E}(\mathbb{Z}, I)$. Then $a \in I$ is regular in I if and only if a is strongly regular in I, and if and only if $(0, a) \in R$ is strongly regular in R.*

Proof. Assume that $a \in I$ is regular in I. Then there exists $b \in I$ such that $a = aba$. Clearly, both ab and ba are in $S(I)$. If $ba \in S_l(I)$, then

$$a = aba = ababa = (ab)a(ba) = (ab)(ba)a(ba) = (ab)(ba)(aba) = abbaa = ab^2a^2.$$

If $ba \in S_r(I)$, then

$$a = aba = ababa = a(ba)ba = a(ba)b(ba)a = ab^2a^2.$$

Similarly, if $ab \in S_l(I)$, then

$$a = aba = ababa = ab(ab)a = a(ab)b(ab)a = a^2b^2aba = a^2b^2a.$$

If $ab \in S_r(I)$, then

$$a = aba = ababa = (ab)a(ba) = (ab)a(ab)(ba) = (aba)abba = a^2b^2a.$$