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The Factorization Method to Solve a Class of Inverse Potential Scattering Problems for Schrödinger Equations*

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Abstract: This paper is concerned with the inverse scattering problems for Schrödinger equations with compactly supported potentials. For purpose of reconstructing the support of the potential, we derive a factorization of the scattering amplitude operator A and prove that the ranges of $(A^*A)^{1/4}$ and G which maps more general incident fields than plane waves into the scattering amplitude coincide. As an application we characterize the support of the potential using only the spectral data of the operator A.

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1 Introduction

Inverse scattering problem for Schrödinger equation is an important class of problems in mathematical physics. It's more difficult and challenging than the case of Helmholtz equation for the complexity brought by the potential term. One way of stating this problem is to reconstruct the potential or at least its support given the scattering amplitude data. For one-dimensional and radial Schrödinger equations, the inverse scattering problems are fairly well understood (at least for certain classes of potentials) (see [1]). In higer dimensions, however, there are still many open problems, and the methods developed are also far from being complete. Some tipical methods developed in multidimensional inverse scattering problems can be found in [2]–[5]. A uniqueness theorem and some inversion formulas with both exact and noisy data and the stability results for the three-dimensional problems with compactly supported potentials were obtained by Ramm (see [6]–[7]).

For acoustic problems, the factorization method appeared in [8] for reconstructing an

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obstacle pointwise from knowledge of the normal far field integral operator F, and then was extended to reconstruct the support of $1 - n$ (n denotes the index of refraction) for inverse medium scattering problem (see [9]). For an operator G which maps more general incident fields than plane waves into the scattering amplitude and $r_z = \mathrm{e}^{-\mathrm{i}kz\cdot\hat{x}}$, the basic idea of the factorization method is to solve the equation $Gg = r_z$, and the test point z lies inside the obstacle or the support of $1 - n$ is equivalent that the equation is solvable. The far field operator F has a factorization related to the operator G, and thus one can get that $(F^*F)^{1/4}$ and G have the same range. This range and as an application the boundary of obstacle or the support of $1 - n$ can be characterized by the spectral data of the normal operator F. Grinberg extended this method to cover some kinds of boundary conditions which leads to non-normal operators for inverse obstacle scattering problems (see [10]).

In this paper we extend this method to the inverse scattering problem for Schrödinger equation to reconstruct the support of potential. Similar conclusions as the cases of inverse obstacle and medium scattering problems in acoustics can be obtained. We make the analysis carefully for the three-dimensioal case and state at last that this method can also be applied to the two-dimensional case.

Since the injectivity of the scattering amplitude opertor A is related to the existence of 'interior transmission eigenvalues', we have to consider that there are 'how many' such eigenvalues. We can follow the argument of inverse medium scattering problem under further assumption on potential, but it is not enough to make a simple parallel discussion for the difference of the operators in inverse medium and potential scattering cases. We resolve it by giving another assumption on potential so that the analytic Fredholm theorem can still be used to yield that there exists at most a countable number of eigenvalues k^2.

This paper is organized as follows. In section 2 we introduce the direct scattering problem for Schrödinger equation with compactly supported potential and sketch the proof of its existence and uniqueness and the equivalence with the Lippmann-Schwinger integral equation. Both the concept of weak solution and the inverse potential scattering problem are introduced in Section 3 and we derive a factorization of the scattering amplitude operator A. We discuss the interior transmission problem in Section 4 and prove that A is injective if k^2 is not an interior transmission eigenvalue. In section 5, the main theorem of this paper is given to describe those points inside the support of the potential. In Section 6, we first explain that all of the results also hold for the two-dimensional case with possibly different constants, and then give two numerical experiments to illustrate the theorem.

2 The Direct Potential Scattering Problem

Let $D \subset \mathbf{R}^3$ be a bounded domain with C^2 boundary ∂D and $\mathbf{R}^3 \setminus D$ connected, and

$$Q := \{q : q \geq 0,\ \mathrm{supp}\ q = \bar{D},\ \text{and there exists some } \alpha \in (0,1) \text{ such that } q \in C^{0,\alpha}(\mathbf{R}^3)\}, \tag{2.1}$$

where $C^{0,\alpha}(\mathbf{R}^3)$ denotes the Hölder continuous function space with Hölder index α. Let $u^{\mathrm{i}} = \mathrm{e}^{\mathrm{i}k\hat{\theta}\cdot x}$ denotes a plane wave in $\mathbf{R}^3$ of direction $\hat{\theta} \in S^2$ where S^2 denotes the unit sphere