COMMUNICATIONS IN MATHEMATICAL RESEARCH
26(4)(2010), 353–360

Uniqueness of Meromorphic Functions Sharing One Set*

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Communicated by Ji You-qing

Abstract: In this paper, we study uniqueness of meromorphic functions sharing one set, and obtain some results, which improve and extend the original results.

Key words: meromorphic function, Nevanlinna theory, uniqueness, sharing value
2000 MR subject classification: 30D35, 30D45
Document code: A
Article ID: 1674-5647(2010)04-0353-08

1 Introduction

In this work, by meromorphic function we always mean a meromorphic function in the complex plane $\mathbb{C}$. We assume that the reader is familiar with the standard notation of value distribution theory, and this can be found, for instance, in [1] or [2].

Let f be a nonconstant meromorphic function. We use $N_{(k}\left(r, \dfrac{1}{f}\right)$ to denote the counting function for the zeros of f with multiplicity $\geq k$ and counting multiplicities, while by $N_{k)}\left(r, \dfrac{1}{f}\right)$ the counting function of the zeros of f with multiplicities $\leq k$ and counting multiplicities. We denote by $S(r, f)$ any function satisfying

$$S(r,f) = o(T(r,f)), \qquad r \to \infty,$$

possibly outside a set of finite measure. Define

$$\Theta(\infty, f) = 1 - \lim_{r\to\infty} \sup \frac{\bar{N}(r,f)}{T(r,f)},$$

$$\delta_2(0,f) = 1 - \lim_{r\to\infty} \sup \frac{\bar{N}\left(r, \dfrac{1}{f}\right) + \bar{N}_{(2}\left(r, \dfrac{1}{f}\right)}{T(r,f)}.$$

Let S be a set of complex numbers. Define

$$E(S,f) = \bigcup_{a\in S} \{z \mid f(z) - a = 0\},$$

*Received date: May 18, 2009.
Foundation item: The NSF (10771121) of China, the NSF (Z2008A01) of Shandong and the RFDP (20060422049).

where each zero of $f(z)-a$ with multiplicity m is counted m times in $E(S,f)$. The notation $\bar{E}(S,f)$ expresses the set which contains the same points as $E(S,f)$ but without counting multiplicities. We denote by $E_{(k}(S,f)$ the counting function of the zeros of $f-a$ with multiplicities $\geq k$ and counting multiplicities, $E_{k)}(S,f)$ the counting function of the zeros of $f-a$ with multiplicities $\leq k$ and counting multiplicities. The notations $\bar{E}_{(k}(S,f)$ and $\bar{E}_{k)}(S,f)$ express the sets which contain the same points as $E_{k)}(S,f)$ and $E_{k)}(S,f)$ respectively, but ignoring multiplicities.

In 1976, Gross[3] posed the following question.

Question A *Does there exist a finite set S such that, for any pair of nonconstant entire functions f and g, $E(S,f)=E(S,g)$ implies $f\equiv g$?*

If such a finite set exists, a natural question is the following.

Question B *What is the smallest cardinality for such a finite set?*

Yi[4] first gave an affirmative answer to Question A. So far, the best answer to Question B was obtained by Yi[5], as follows.

Theorem A *There exists a set S with 7 elements such that $E(S,f)=E(S,g)$ implies $f\equiv g$ for any pair of nonconstant entire functions f and g.*

Later, many authors studied these questions for meromorphic functions. The present best answer to Question B for meromorphic functions was obtained by Frank and Reinders[6]. They proved the following result.

Theorem B *There exists a set S with 11 elements such that $E(S,f)=E(S,g)$ implies $f\equiv g$ for any pair of nonconstant meromorphic functions f and g.*

A natural problem arises: What can we say if nonconstant meromorphic functions f and g have "few" poles? In 1999, Fang *et al.*[7],[8] proved the following theorem.

Theorem C *Let $S=\{z:\ z^7-z^6=1\}$. Suppose that f, g are two nonconstant meromorphic functions satisfying $\Theta(\infty,\ f)>\dfrac{11}{12}$, $\Theta(\infty,\ g)>\dfrac{11}{12}$. If $E(S,f)=E(S,g)$ and $E(\infty,f)=E(\infty,g)$, then $f\equiv g$.*

Recently, Zhang and Xu[9] proved the following results.

Theorem D *Let $S=\{z:z^7-z^6=1\}$. Suppose that f, g are two nonconstant meromorphic functions satisfying $\Theta(\infty,f)+\Theta(\infty,g)>1$. If $E(S,f)=E(S,g)$ and $E(\infty,f)=E(\infty,g)$, then $f\equiv g$.*

Theorem E *Let $S=\{z:z^7-z^6=1\}$. Suppose that f, g are two nonconstant meromorphic functions satisfying $\Theta(\infty,f)+\Theta(\infty,g)>\dfrac{4}{3}$. If $E(S,f)=E(S,g)$ and*