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A Pseudo-parabolic Type Equation with Nonlinear Sources*

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Abstract: This paper is concerned with the existence and uniqueness of nonnegative classical solutions to the initial-boundary value problems for the pseudo-parabolic equation with strongly nonlinear sources. Furthermore, we discuss the asymptotic behavior of solutions as the viscosity coefficient k tends to zero.

Key words: pseudo-parabolic equation, existence, uniqueness, asymptotic behavior

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1 Introduction

In this paper, we investigate the existence, uniqueness and asymptotic behavior of solutions to the following initial-boundary value problem for the pseudo-parabolic equation in one spatial dimension:

$$\frac{\partial u}{\partial t} - k\frac{\partial D^2 u}{\partial t} = D^2 u + m(x,t)u^q, \qquad (x,t)\in Q, \tag{1.1}$$

$$u(0,t) = u(1,t) = 0, \qquad t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x\in[0,1], \tag{1.3}$$

where $q > 1$, $Q \equiv (0,1)\times \mathbf{R}^+$, $D = \partial/\partial x$, $k > 0$ represents the viscosity coefficient, $m(x,t) \in C^{\alpha,\alpha/3}(\overline{Q})$ for some $\alpha \in (0,1)$ and satisfies $0 < \underline{m} \leq m(x,t) \leq \overline{m}$ for any $(x,t)\in Q$, $\underline{m}$ and $\overline{m}$ are positive constants.

The pseudo-parabolic equations are characterized by the occurrence of mixed third order derivatives, more precisely, second order in space and first order in time. Such equations are used to model heat conduction in two-temperature systems (see [1] and [2]), fluid flow in porous media (see [3] and [4]), two phase flow in porous media with dynamical capillary

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pressure (see [5]), and the populations with the tendency to form crowds (see [6] and [7]). Furthermore, according to experimental results, some researchers have recently proposed modifications to Cahn's model which incorporate out-of-equilibrium viscoelastic relaxation effects, and thus obtained this type of equations (see [8]). The pseudo-parabolic equations with strongly nonlinear sources we considered may arise in the study of nonstationary processes in semiconductors with sources of free-charge currents (see [9] and [10]). The processes can be described by the system that has the explicit form of the constitutive equations connecting the electric field strength E with the electric flux density G on the one hand, and the electric field strength E with the current density J in the semiconductor on the other hand, as follows:

$$\begin{aligned} &\mathrm{div} G = 4\pi e n, \qquad E = -\nabla u, \qquad G = E + 4\pi P, \\ &\frac{\partial n}{\partial t} = -\mathrm{div} J + Q, \quad J_i = \sigma_i E_i, \qquad i = 1, 2, \cdots, N, \end{aligned}$$

where P is the polarization vector and in some models there has the following phenomenological relation $\mathrm{div} P = k_1 u$, $k_1 > 0$, n is the free electron concentration, u is the electric potential, and σ_i is the conductivity tensor. Finally, assume that, in a semiconductor, there are sources of free-charge currents whose distribution in the self-consistent electric field of the semiconductor is of the form $Q = m(x, t)u^p$. By differentiating both sides of the first equation with respect to t and taking account of the second equation, the above system can be reduced to the equation (1.1).

The pseudo-parabolic equations have been extensively investigated. In [11]–[13], the authors investigated the initial-boundary value problem and the Cauchy problem for the linear pseudo-parabolic equation and established the existence and uniqueness of solutions. The nonlinear pseudo-parabolic type equations with undefined or uninvertible operator at the highest derivative with respect to time were studied in [14]. The degenerate and quasilinear degenerate pseudo-parabolic type equations were investigated in [15] and [16]. For the local solvability of the pseudo-parabolic type equations with variety nonlocal boundary conditions, see [17]–[19].

For pseudo-parabolic equations, classical maximum principle is invalid in general. For the nonnegativity of a solution, not only nonnegative initial data, but also an extra condition on the elliptic operator is needed (see [20]–[22]). Due to the special type of the problem (1.1)–(1.3) which is included in the studies of [22], we can prove the comparison principle of solutions, which enables us to obtain the existence of nonnegative solutions to the problem (1.1)–(1.3). For the asymptotic behavior of solutions, we know that in certain cases, the solution of a parabolic initial-boundary value problem can be obtained as a limit of solutions to the problem of the corresponding pseudo-parabolic equations, see [11]. In this paper, we show that the semilinear pseudo-parabolic equations still retain this character, namely, the solutions of the pseudo-parabolic equations converge to the solution of the parabolic equation as $k \to 0$.

This paper is organized as follows. In Section 2, we show the existence and uniqueness of nonnegative classical solutions to the initial-boundary value problem (1.1)–(1.3). Then,