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Solutions to Boundary Value Problem of Nonlinear Impulsive Differential Equation of Fractional Order*

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Abstract: This paper is devoted to study the existence and uniqueness of solutions to a boundary value problem of nonlinear fractional differential equation with impulsive effects. The arguments are based upon Schauder and Banach fixed-point theorems. We improve and generalize the results presented in [B. Ahmad, S. Sivasundaram, Existence results for nonlinear impulsive hybrid boundary value problems involving fractional differential equations, Nonlinear Analysis: Hybrid Systems, 3(2009), 251–258].

Key words: impulsive differential equation, boundary value problem, fractional derivative, fixed-point theorem
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1 Introduction

Fractional differential equations have gained considerable popularity and importance during the past three decades or so, due mainly to their varied applications in many fields of science and engineering. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see, for instance, the monographs of Kilbas *et al.*[1], Miller and Ross[2], Podlubny[3] and the papers [4]–[19] and the references therein.

The importance of impulse effects in many areas, such as biology, physics, medicine, control theory, etc. makes it necessary to investigate the behavior of impulsive differential equations as models for many real situations. We mention, for instance, the books [20] and

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[21], dealing with impulsive differential equations.

Very recently, some authors have initiated the study on the impulsive differential equations of fractional order (see [22]–[25]). Ahmad and Sivasundaram[26] discussed the nonlinear impulsive hybrid boundary value problem involving the Caputo fractional derivative as follows:

$$\begin{cases} {}^cD^q x(t) = f(t, x(t)), \qquad 1 < q \le 2, \quad t \in J = [0,1] \setminus \{t_1, t_2, \cdots, t_p\}, \\ \triangle x(t_k) = I_k(x(t_k^-)), \quad \triangle x'(t_k) = J_k(x(t_k^-)), \qquad t_k \in (0,1), \quad k = 1, 2, \cdots, p, \\ x(0) + x'(0) = 0, \qquad x(1) + x'(1) = 0, \end{cases} \tag{$*$}$$

where $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is a continuous function, $I_k, J_k : \mathbf{R} \to \mathbf{R}$,

$$\triangle x(t_k) = x(t_k^+) - x(t_k^-)$$

with

$$x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$$

and

$$x(t_k^-) = \lim_{h \to 0^-} x(t_k + h), \qquad k = 1, 2, \cdots, p$$

for $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$. The existence and uniqueness results are obtained by applying contraction mapping principle and Krasnoselskii's fixed-point theorem.

Motivated by [22]–[26], we consider in the present paper the following problem:

$$\begin{cases} {}^cD^q_{t_k^+} x(t) = f(t, x(t)), \qquad 1 < q \le 2, \quad t \in J = [0,1] \setminus \{t_1, t_2, \cdots, t_p\}, \\ \triangle x(t_k) = I_k(x(t_k^-)), \quad \triangle x'(t_k) = J_k(x(t_k^-)), \qquad t_k \in (0,1), \quad k = 1, 2, \cdots, p, \\ x(0) = x(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0,1] \times \mathbf{R} \to \mathbf{R}$ is a continuous function, $I_k, J_k : \mathbf{R} \to \mathbf{R}$,

$$\triangle x(t_k) = x(t_k^+) - x(t_k^-)$$

with

$$x(t_k^+) = \lim_{h \to 0^+} x(t_k + h)$$

and

$$x(t_k^-) = \lim_{h \to 0^-} x(t_k + h), \qquad k = 1, 2, \cdots, p$$

for $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = 1$ and

$${}^cD^q_{t_k^+} x(t) = \frac{1}{\Gamma(2-q)} \int_{t_k}^t (t-s)^{1-q} x''(s) \mathrm{d}s, \qquad t > t_k$$

is the Caputo fractional derivative of $x(t)$ on the interval $[t_k, t_{k+1}]$, $k = 0, 1, \cdots, p$. We shall apply the Schauder and Banach fixed-point theorems to prove existence and uniqueness results to the problem (1.1) with some weaker conditions than in [26].

It is worth mentioning that, for the sake of simplicity, we are concerned with the simplest two-point boundary value conditions in (1.1). Otherwise, the expressions of solution are too lengthy and here we do not elaborate.

For the readers' convenience, we point out at the end of this section that ${}^cD^q x(t)$, $D^q x(t)$ and $I^q x(t)$ often denote Caputo fractional derivative ${}^cD^q_{0^+} x(t)$, Riemann-Liouville fractional derivative $D^q_{0^+} x(t)$ and Riemann-Liouville fractional integral $I^q_{0_+} x(t)$ of the function $x(t)$ for $t > 0$, respectively.