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Critical Exponents for Fast Diffusion Equations with Nonlinear Boundary Sources*

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Abstract: In this paper, we study the large time behavior of solutions to a class of fast diffusion equations with nonlinear boundary sources on the exterior domain of the unit ball. We are interested in the critical global exponent q_0 and the critical Fujita exponent q_c for the problem considered, and show that $q_0 = q_c$ for the multi-dimensional Non-Newtonian polytropic filtration equation with nonlinear boundary sources, which is quite different from the known results that $q_0 < q_c$ for the one-dimensional case; moreover, the value is different from the slow case.
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1 Introduction

In this paper, we consider the following problem:

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m), \qquad (x,t) \in (\mathbf{R}^N \setminus B_1(0)) \times (0,T), \tag{1.1}$$

$$|\nabla u^m|^{p-2}\nabla u^m \cdot \vec{\nu}\Big|_{\partial B_1(0)} = u^q, \qquad t > 0, \tag{1.2}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbf{R}^N \setminus B_1(0), \tag{1.3}$$

where $0 < m(p-1) < 1$, $1 < p < 2$, $m > 0$, $N \geq 2$, $B_1(0)$ is the unit ball in $\mathbf{R}^N$ with boundary $\partial B_1(0)$ and $\vec{\nu}$ is the inner normal vector on $\partial B_1(0)$, $u_0(x)$ is a nonnegative, suitable smooth and bounded function with compact support, which satisfies some compatibility conditions.

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It is seen that the equation (1.1) is degenerate at the points where $\nabla u^m = 0$. When $p = 2$, it is the Newtonian filtration equation, and when $m = 1$ it becomes the Non-Newtonian filtration equation. This equation has drawn much attention (see [1] and [2] and the references therein); the local in time existence of the solutions and the comparison principle have already been established (see [3] and the references therein).

In the present paper, we mainly discuss the large time behavior of the solutions to the problem (1.1)–(1.3), including the global existence and blow-up in a finite time. A solution u is said to blow up in a finite time $0 < T < +\infty$ if

$$\|u(\cdot,t)\|_{L^\infty(\mathbf{R}^n\setminus B_1)} = \sup_{x\in\mathbf{R}^n\setminus B_1} u(x,t) \to +\infty \qquad \text{as } t \to T^-.$$

Specially, we are interested in the critical global exponent q_0 and the critical Fujita exponent q_c for the above problems. These two exponents have properties as follows:

(i) All solutions exist globally in time if $0 \le q < q_0$, while there exists at least one solution which blows up in a finite time if $q > q_0$.

(ii) Every nontrivial solution blows up in a finite time if $q \in (q_0, q_c)$, while there exist both global solutions and blow-up solutions with small initial data and large initial data respectively when $q > q_c$.

It was Fujita[4] who first considered the critical exponent for the Cauchy problem of the heat equation with interior source. From then on, many mathematicians have extended the results to numerous of partial differential equations and systems (such as quasilinear parabolic equation, higher-order diffusion equation, nonlinear wave equation and so on) in various of geometries (such as whole space, cones, sectorial domain in $\mathbf{R}^2$, orthant domain, complement of bounded domain and so on). We refer to the survey papers [5], [6] and the references therein and also the recent papers [7]–[14].

In the above works, the nonlinear source appears in the interior of the domain. When it appears on the boundary, there were also some results of Fujita type for the heat equation (see [15]–[17], etc.). However, when it came to the problem for the nonlinear diffusion equation, there was not many results. In 1996, Galaktionov and Levine firstly studied the critical exponents for the nonlinear diffusion equation with nonlinear boundary sources in [18], and proved that

$$q_0 = (m+1)/2, \qquad q_c = m+1$$

and

$$q_0 = 2(p-1)/p, \qquad q_c = 2(p-1)$$

for the equations

$$u_t = (u^m)_{xx}$$

and

$$u_t = (|u_x|^{p-2}u_x)_x$$

in the half space with boundary conditions

$$-(u^m)_x(0,t) = u^q$$