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On Centralizer Subalgebras of Group Algebras*

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Abstract: Let G be a finite group, $H \le G$ and R be a commutative ring with an identity 1_R. Let $C_{RG}(H)$=$\{\alpha \in RG|\alpha h = h\alpha$ for all $h \in H\}$, which is called the centralizer subalgebra of H in RG. Obviously, if $H = G$ then $C_{RG}(H)$ is just the central subalgebra $Z(RG)$ of RG. In this note, we show that the set of all H-conjugacy class sums of G forms an R-basis of $C_{RG}(H)$. Furthermore, let N be a normal subgroup of G and γ the natural epimorphism from G to $\bar{G} = G/N$. Then γ induces an epimorphism from RG to $R\bar{G}$, also denoted by γ. We also show that if R is a field of characteristic zero, then γ induces an epimorphism from $C_{RG}(H)$ to $C_{R\bar{G}}(\bar{H})$, that is, $\gamma(C_{RG}(H)) = C_{R\bar{G}}(\bar{H})$.
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1 Introduction

Let G be a finite group and H a subgroup of G. We know that H acts on G by conjugation. With respect to this action, for any $g \in G$, the H-orbit $g^H = \{g^h \mid h \in H\}$ of G is called an H-conjugacy class of G containing g. We denote by $\mathrm{Cl}_H(G)$ the set of all H-conjugacy classes of G, namely,

$$\mathrm{Cl}_H(G) = \{g^H \mid g \in G\}.$$

It is clear that G is a disjoint union of all $g^H \in \mathrm{Cl}_H(G)$. Let R be a commutative ring with an identity 1_R. Denote by RG the group algebra of G over R. Let

$$C_{RG}(H) = \{\alpha \in RG \mid \alpha h = h\alpha \text{ for all } h \in H\},$$

which is obviously a subalgebra of RG and is called the centralizer subalgebra of H in RG. Note that if $H = G$ then $C_{RG}(H)$ is just the central subalgebra $Z(RG)$ of RG. For any

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$g^H \in \mathrm{Cl}_H(G)$, write

$$\widehat{g^H} := \sum_{x\in g^H} x = \sum_{h\in H} g^h \in RG,$$

the sum of all elements in g^H, which is called an H-conjugacy class sum of G. It is clear that if $H = G$ then the H-conjugacy class sums of G are the class sums of G in ordinary sense. For general applications of the class sums of G, see [1]–[4]. It is well-known that the set of all the class sums of G forms an R-basis of $Z(RG)$. As a generalization of this result, we show in this note that the set $\{\widehat{g^H} \mid g^H \in \mathrm{Cl}_H(G)\}$ of all the H-conjugacy class sums of G also forms an R-basis of $C_{RG}(H)$.

Let $N \trianglelefteq G$ and write $\bar{G} = G/N$. Let

$$\gamma : G \to \bar{G}(g \mapsto \bar{g})$$

be the natural epimorphism of groups. By extending γ linearly to the group ring RG, we obtain an epimorphism of algebras. By abuse of notation, it is also denoted by γ, i.e.,

$$\gamma : RG \to R\bar{G}\Big(\sum_{g\in G} r_g g \mapsto \sum_{g\in G} r_g \bar{g}\Big).$$

Write $\bar{H} = HN/N$ for any $H \le G$ and $\bar{g} = gN$ for any $g \in G$. Then $\bar{g}^{\bar{H}}$, $\widehat{\bar{g}^{\bar{H}}}$, $C_{R\bar{G}}(\bar{H})$ and $\mathrm{Cl}_{\bar{H}}(\bar{G})$ can be defined in the same way as g^H, $\widehat{g^H}$, $C_{RG}(H)$ and $\mathrm{Cl}_H(G)$, respectively. It is obvious that $\gamma(C_G(H)) \subseteq C_{\bar{G}}(\bar{H})$. Furthermore, $\gamma(C_G(H)) = C_{\bar{G}}(\bar{H})$ provided that H is a p-subgroup of G with $(|N|, p) = 1$. Similarly, we have $\gamma(C_{RG}(H)) \subseteq C_{R\bar{G}}(\bar{H})$. Then a question arising naturally is under what condition(s) can we ensure that $\gamma(C_{RG}(H)) = C_{R\bar{G}}(\bar{H})$. In this note, we show that $\gamma(C_{RG}(H)) = C_{R\bar{G}}(\bar{H})$ provided that R is a field of characteristic zero. This is an amazing result since it has nothing to do with the orders of H and G/N. Our main result is as follows:

Theorem 1.1 *Let G be a finite group, $H \le G$ and $N \trianglelefteq G$, and other notations be as above. Then the following hold*:

(1) *$C_{RG}(H)$ is a subalgebra of RG with the set $\{\widehat{g^H} \mid g^H \in Cl_H(G)\}$ of all H-conjugacy class sums as an R-basis*;

(2) *If R is a field of characteristic zero, then γ induces an epimorphism of algebras from $C_{RG}(H)$ to $C_{R\bar{G}}(\bar{H})$, that is, $\gamma(C_{RG}(H)) = C_{R\bar{G}}(\bar{H})$.*

Henceforth all groups under consideration are finite. Other notations without specified explicitly is standard (refer to [5] and [6]).

2 Proof of the Theorem 1.1

For convenience, we restate and fix some notations used throughout this section.

R: a commutative ring with an identity 1_R;

G: a finite group;

H: a subgroup of G;

N: a normal subgroup of G;