COMMUNICATIONS
IN
MATHEMATICAL
RESEARCH
27(3)(2011), 243–252

Stable Border Bases for Ideals of Numerical Cartesian Sets*

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Communicated by Du Xian-kun

Abstract: In this paper, we discuss a special class of sets of bivariate empirical points, namely, numerical cartesian sets. We find that the stable quotient bases for numerical cartesian sets are unique if they exist. Furthermore, the corresponding border bases are the unique stable border bases for the vanishing ideals of numerical cartesian sets.

Key words: empirical point, numerical cartesian set, vanishing ideal, border base
2000 MR subject classification: 65H10, 14Q15, 13P10
Document code: A
Article ID: 1674-5647(2011)03-0243-10

1 Introduction

Let $P = R[x_1, \ldots, x_n]$ be the polynomial ring in n variables over $\mathbf{R}$. Given a set of finite distinct points $\mathbb{X} \subset \mathbf{R}^n$, it is well-known that the set of all polynomials vanishing at $\mathbb{X}$ constitutes a radical zero-dimensional ideal, denoted by $\ell(\mathbb{X})$, which is called the vanishing ideal of $\mathbb{X}$.

If $\mathbb{X}$ is a set of empirical points, representing real-world measurements, then the coordinates are known only imprecisely. In brief, if $\widetilde{\mathbb{X}}$ is another set of points, each point of $\widetilde{\mathbb{X}}$ is different by less than the uncertainty from the corresponding element of $\mathbb{X}$, then the two sets can be regarded as equivalent. In order to emphasize the numerical equivalence of $\mathbb{X}$ and its perturbation $\widetilde{\mathbb{X}}$, we look for a set of polynomials to simultaneously characterize $\ell(\mathbb{X})$ and $\ell(\widetilde{\mathbb{X}})$.

In recent years, authors have great interest in the problem about describing vanishing ideals of sets of empirical points. Sauer[1] considered a method, suitable for numerical computations, which computes a low-degree algebraic variety containing the input points. Heldt *et al.*[2] introduced a numerically stable AVI-algorithm which computes a set of polynomials

*Received date: Sept. 2, 2010.
Foundation item: Partially supported by the National Grand Fundamental Research 973 Program (2004CB318000) of China.

that almost vanish at the given points and almost form a border basis. Abbott *et al.*[3] presented SOI-algorithm which computes a stable order ideal $\mathcal{O}$, and if $\mathcal{O}$ contains enough elements to form a basis of $P/\ell(\mathbb{X})$, the corresponding stable border basis is also computed. It should be noticed that SOI-algorithm is based on a priori error estimate. Fassino[4] gave us NBM-algorithm, which computes an order ideal $\mathcal{O}$, and a set $\mathcal{G}$ of polynomials such that each element of $\mathcal{G}$ is almost vanishing at $\mathbb{X}$ and at all its equivalent sets $\widetilde{\mathbb{X}}$. Even though NBM-algorithm can not guarantee the stability of $\mathcal{O}$, there is a high probability with the stable order ideal $\mathcal{O}$.

Given a set $\mathbb{X}^\varepsilon$ of empirical points, where ε is a tolerance on the data error, the study of computing the stable quotient basis for $\mathbb{X}^\varepsilon$ is a rather complicated topic. So far, there is no definite algorithm without any priori error estimate about the above problem. In the theory of finite element, we just interpolate at cartesian point site. Since the interpolate data almost come from real-world measurements, we essentially interpolate at numerical cartesian point site in the theory of finite element. Hence, the study of the stability of numerical cartesian set is necessary.

In this paper, we discuss a special type of sets of bivariate empirical points, namely, numerical cartesian sets. For a numeric cartesian set $\mathbb{X}^\varepsilon$, if the stable quotient basis for $\mathbb{X}^\varepsilon$ exists, then it must be unique.

The remainder of our paper is organized as follows. The next section is devoted as a preparation for the paper. In Section 3, we introduce the definition of numerical cartesian sets, and present test algorithm of numerical cartesian sets. In Section 4, we discuss the stable quotient bases for numerical cartesian sets, and stable border bases for the vanishing ideals of numerical cartesian sets. Finally, in Section 5, we give some examples to illustrate our conclusions.

2 Preliminaries

First of all, we recall some concepts and facts about bivariate polynomial ring and cartesian set. For more details, we refer the reader to [5]–[7].

Suppose that $R[x, y]$ denotes the polynomial ring in two variables. If $\mathbb{X} = \{p_1, \cdots, p_s\} \subset \mathbf{R}^2$ and $f \in R[x, y]$, then

$$f(\mathbb{X}) = (f(p_1), \cdots, f(p_s))^T$$

is the evaluation column vector of f at $\mathbb{X}$. For a non-empty finite polynomial subset

$$G = \{g_1, \cdots, g_k\} \subset R[x, y],$$

the evaluation matrix of G associated to $\mathbb{X}$, written as $M_G(\mathbb{X})$, is the $s \times k$ matrix whose jth column is $g_j(\mathbb{X})$.

A finite subset $\mathcal{A} \subset \mathbb{N}_0^2$ is called lower if for any $(\alpha_1, \alpha_2) \in \mathcal{A}$, we always have

$$R(\alpha_1, \alpha_2) := \{(\alpha_1', \alpha_2') : \ 0 \le \alpha_1' \le \alpha_1, \ 0 \le \alpha_2' \le \alpha_2\} \subseteq \mathcal{A}.$$

Let $\mathbb{T}(x, y)$ denote the set of all monomials in x, y. A monomial set $\mathcal{O} \subseteq \mathbb{T}(x, y)$ is called an order ideal if it is closed under monomial division partial order. We can easily check that if