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A Fixed Point Theorem and Some Generalized Ky Fan's Minimax Inequalities* **

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Abstract: In this paper, we establish a fixed point theorem for set-valued mapping on a topological vector space without "local convexity". And we also establish some generalized Ky Fan's minimax inequalities for set-value vector mappings, which are the generalization of some previous results.

Key words: fixed point, Ky Fan's minimax inequality, set-valued vector mapping, topological vector space
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1 Introduction

It is well known that fixed point theory plays a very important role in various fields of mathematics, such as variational inequality, game theory, mathematical economics, equilibrium problem, control theory and so on. The earliest extension of the topological theory of fixed points of continuous mappings to the case of multi-valued mappings was made by von Neumann in the connection with the proof of the fundamental theorem of game theory. From then on, Kakutani[1], Fan[2], and Glicksberg[3] have consecutively improved the result from finite-dimensional topological vector space to locally convex topological vector space, i.e., the famous Kakutani-Fan-Glicksberg fixed point theorem, which has had many extensions and has been applied to various fields of mathematics by many authors.

In this paper, we establish a fixed point theorem for set-valued mapping on a topological vector space without "local convexity" by using Kakutani-Fan-Glicksberg fixed point theorem. And we also establish some Ky Fan's minimax inequalities for set-value vector mappings. Throughout the paper, assume every space is of Hausdorff.

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2 Preliminaries

Now, we recall some definitions and preliminaries. Let X and Y be two nonempty sets. Let $T : X \to 2^Y$ be a nonempty set-valued mapping, such that $x \in T^{-1}(y)$ if and only if $y \in T(x)$, and $T(X) = \bigcup_{x\in X} T(x)$. We denote by 2^X and $\langle X\rangle$ the family of all subsets of X and the family of all nonempty finite subsets of X, respectively.

Definition 2.1[4] *For topological spaces X and Y, a mapping $T : X \to 2^Y$ is said to be*

(i) *upper semicontinuous* ($u.s.c$) *at $x_0 \in X$, if for each neighborhood $N(T(x_0))$, there exists a neighborhood $N(x_0)$ of x_0 such that for each $x \in N(x_0)$, $T(x) \subset N(T(x_0))$;*

(ii) *lower semicontinuous* ($l.s.c$) *at $x_0 \in X$, if for each net $\{x_\alpha\} \subset X$, $x_\alpha \to x_0$, $y_0 \in T(x_0)$ implies that there exists a net $y_\alpha \in T(x_\alpha)$ such that $y_\alpha \to y_0$;*

(iii) *continuous, if it is both* ($u.s.c.$) *and* ($l.s.c.$);

(iv) *compact-valued, if $T(x)$ is compact in Y for any $x \in X$.*

Lemma 2.1[5] *Let E be a topological vector space, X, Y be nonempty subsets of E, and $T : X \to 2^Y$ be a continuous set-valued mapping. Then the following facts hold*:

(i) *T has a closed graph, if T is upper semicontinuous with compact values. That is, for each $x_\alpha \to x_0$, $y_\alpha \in T(x_\alpha)$, if a subnet y_β of y_α is such that $y_\beta \to y_0$, then $y_0 \in T(x_0)$;*

(ii) *If X, Y are all nonempty compact, and $T : X \to 2^Y$ is a nonempty mapping with closed values, then T is upper semicontinuous if and only if T has a closed graph*;

(iii) *T is upper semicontinuous if and only if for each closed set B of Y, $\{x \in X : T(x)\bigcap B \neq \emptyset\}$ is a closed set of X; T is lower semicontinuous if and only if for each open set B of Y, $\{x \in X : T(x)\bigcap B \neq \emptyset\}$ is an open set of X;*

(iv) *Any compact subset A of E is totally bounded*;

(v) *If X is totally bounded, then for each neighborhood V of the zero point, there exist $x_1, x_2, \cdots, x_k \in X$ such that $X \subset \bigcup_{i=1}^{k} (x_i + V)$;*

(vi) *For any compact subset A in a finite-dimensional topological vector space, its convex hull $co(A)$ is compact*;

(vii) *If X is compact, and F is an upper semicontinuous mapping with compact values, then $F(X) = \bigcup_{x\in X} F(x)$ is compact.*

Lemma 2.2([3], Kakutani-Fan-Glicksberg fixed point theorem) *Let E be a locally convex topological vector space, and $X \subset E$ be a nonempty compact and convex set. If $T : X \to 2^X$ is an upper semicontinuous mapping with nonempty, closed and convex values, then T has a fixed point.*

3 A Fixed Point Theorem

In this section, we prove a fixed point theorem on a topological vector space without "local convexity", which is a generalization of the Kakutani-Fan-Glicksberg fixed point theorem. In the proof, we use those facts listed in Lemma 2.1.